Let G be a finite group

$$G = \{g_1, g_2, \ldots, g_p\}, \quad |G| = p, \quad g_1 \text{ the identity}$$

g_a and g_b are <u>conjugate</u> (in G) if $\exists\, h \in G$

$$\ni \; g_a = h^{-1} g_b h$$

There are k classes, $\mathcal{K}_i$ $i=1,2,\ldots,k$, of conjugate elements.

Let $\mathcal{V}$ be a finite N-dimensional <u>Complex</u> Vector space with inner prod $\langle \underline{a}, \underline{b} \rangle$, $\underline{a}, \underline{b} \in \mathcal{V}$

A <u>representation</u> of G on $\mathcal{V}$ is a homomorphism

$$T : G \to GL(\mathcal{V}) \qquad g \mapsto \underline{T}_g$$

$$\ni \; \underline{T}_g \underline{T}_h = \underline{T}_{gh} \quad \text{(preserves the group product)}$$

The kernel of T is $\ker(T) = \{ g \in G \mid \underline{T}_g = \underline{I} \}$

Two representations (reps) T and S are <u>equivalent</u> if $\exists\, \underline{Q} \ni GL(\mathcal{V}) \ni \underline{Q}^{-1} \underline{S}_g \underline{Q} = \underline{T}_g \;\; \forall g \in G$

Theorem: Every rep T of G on $\mathcal{V}$ is equivalent to a unitary rep.

So, we will assume, from now on, T is unitary. That is,

$$\underline{T}_g^{\dagger} = \underline{T}_g^{*T} = \underline{T}_g^{-1} = \underline{T}_{g^{-1}}$$

Eg: Elasticity Tensors $\underline{\underline{C}} = C_{ijkl}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l \in \mathcal{V}^4$

$\mathcal{V} \equiv \mathbb{R}^3$; $Q : G \supseteq SO(3) \rightarrow GL(\mathcal{V})$ is a 3D rep on $\mathcal{V}$

$$\underline{w} = \underline{Q}\, \underline{v}$$

For $\mathcal{V}^N$ tensor product rep $\underline{T}_Q^{[N]} = \underline{Q} \otimes \underline{Q} \otimes \cdots \otimes \underline{Q}$

$\mathcal{V}^2$: consider $T^{[2]}$ and $\underline{C} \in GL(\mathcal{V}^2)$

symmetry of $\underline{C}$ $\Rightarrow$ $\underline{T}_Q^{[2]} \underline{C} = \underline{C}\, \underline{T}_Q^{[2]} \quad \forall Q \in G$

Alternatively

$\mathcal{V}^4$: $T^{[4]}$, $\underline{C}$ is invariant:

$$\underline{T}_Q^{[4]} \underline{C} = \underline{C} \quad \forall Q \in G$$

$\underline{W} \subseteq \mathcal{V}$ is an <u>invariant subspace</u> (w.r.t. T) if

$$\underline{w} \in \underline{W} \Rightarrow \underline{T}_g \underline{w} \in \underline{W} \quad \forall g \in G, \ \forall w \in \underline{W}$$

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We can define a rep T' on $\underline{W}$, $T' = T|\underline{W}$, called the restriction of T on $\underline{W}$ by

$$\underline{T}'_g \underline{w} = \underline{T}_g \underline{w}, \quad \underline{w} \in \underline{W}$$

<u>Definition</u>: A rep is <u>reducible</u> if $\exists$ a proper subspace $\underline{W}$ of $\mathcal{V}$ which is invariant under T. Otherwise T is <u>irreducible</u>.

So, if T is reducible and $\underline{W}$ is invariant, then we have (in an appropriate basis)

$$\mathcal{V} = \underline{W} \oplus \underline{W}^{\perp} \quad \text{and} \quad \underline{T}_g = \left[\begin{array}{c|c} \underline{T}'_g & \underline{0} \\ \hline \underline{0} & \underline{T}''_g \end{array}\right] = T' \oplus T''$$

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with $T' = T|\underline{W}$, $T'' = T|\underline{W}^{\perp}$

Eg: rotations about a fixed axis: $SO(2) \subset SO(3)$

$$\underline{R}_\theta = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{array}\right]$$

If T' is reducible, the process is repeated and eventually ends, since $\mathcal{V}$ is finite dimensional.

So, all reps can be reduced to a direct sum of irreps.

$$T = T^{(1)} \oplus T^{(2)} \oplus \cdots \oplus T^{(\ell)}$$

some of these might be equivalent:

$$T = a_1 T^{(1)} \oplus a_2 T^{(2)} \oplus \cdots \oplus a_m T^{(m)} = \sum_{\mu=1}^{m} \oplus\, a_\mu T^{(\mu)}$$

where all the $T^{(\mu)}$ are inequivalent irreps of dimension n_μ and a_μ is the number of times $T^{(\mu)}$ occurs in T.

That is, for every T $\exists\ \underline{Q} \in GL(\mathcal{V}) \ni$

$$\left[\underline{Q}^{-1}\, \underline{T}_g\, \underline{Q}\right] = \left[\begin{array}{c|c|c|c}
\begin{matrix} \underline{T}_g^{(1)} & & \\ & \underline{T}_g^{(1)} & \\ & & \ddots \end{matrix} & \underline{0} & \cdots & \underline{0} \\ \hline
\underline{0} & \begin{matrix} \underline{T}_g^{(2)} & & \\ & \underline{T}_g^{(2)} & \\ & & \ddots \end{matrix} & \cdots & \underline{0} \\ \hline
\underline{0} & \vdots & \ddots & \underline{0} \\ \hline
\underline{0} & \underline{0} & \cdots & \begin{matrix} \underline{T}_g^{(m)} & & \\ & \underline{T}_g^{(m)} & \\ & & \ddots \end{matrix}
\end{array}\right]
\begin{array}{l} \} \ a_1 \text{ times} \\ \\ \} \ a_2 \text{ times} \\ \vdots \\ \} \ a_m \text{ times} \end{array}$$

$\forall g \in G$

So, irreps are fundamental & finding all irreps of G is crucial.

Knowing the above allows solving $\underline{T}_g \underline{C} = \underline{C}\, \underline{T}_g$ and $\underline{T}_g \underline{c} = \underline{c}$ systematically

We start with <u>Shur's lemma</u>

Let T and T' be $p \times p$ and $q \times q$ complex matrix irreps of G, and let A be a $q \times p$ matrix $\ni$

$$\underline{T}'_g \underline{A} = \underline{A}\,\underline{T}_g \quad \forall g \in G$$

Then,

if T & T' are nonequivalent, $\underline{A} = \underline{0}$.

otherwise, $\underline{A} = \lambda \underline{I}$, $\lambda \in \mathbb{C}$.

(Note, these only depend on irreducibility and finite-dim of T & T', not the structure of G)

This leads to the <u>orthogonality relations</u>

$$\sum_{g \in G} T^{(\mu)}_{ij}(g) T^{(\nu)*}_{k\ell}(g) = \left(\frac{p}{n_\mu}\right) \delta_{ik}\, \delta_{j\ell}\, \delta_{\mu\nu}$$

$i, j = 1, 2, \ldots, n_\mu \qquad n_\mu - \dim T^{(\mu)}$

$\begin{pmatrix} \text{Recall} \\ p = |G| \end{pmatrix}$ $\qquad k, \ell = 1, 2, \ldots, n_\nu \qquad n_\nu - \dim T^{(\nu)}$

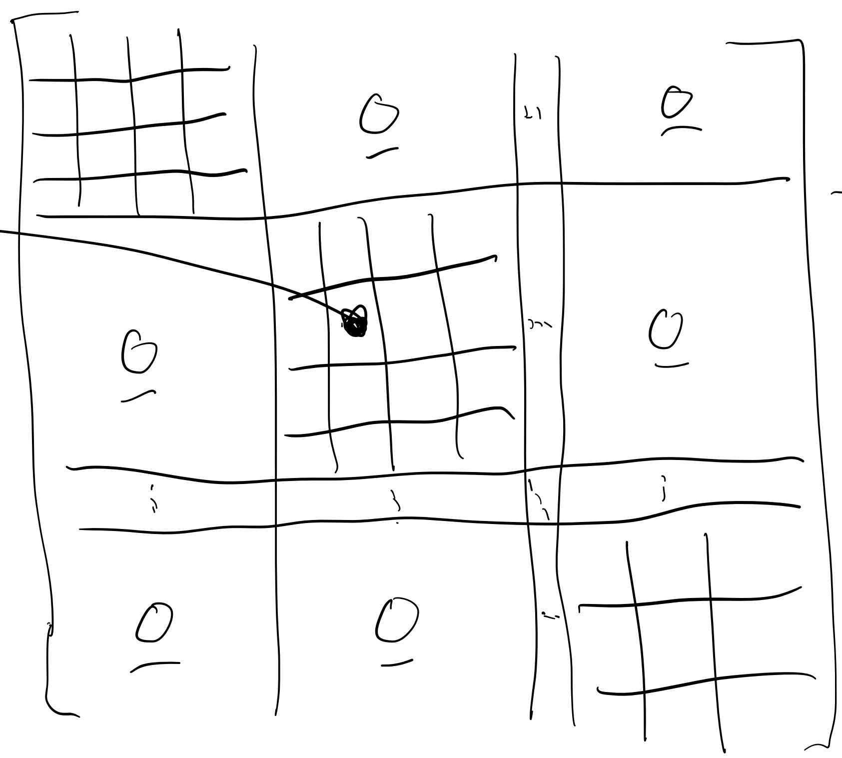
$\mu, \nu = 1, 2, \ldots, m \qquad m - \#$ irreps of G

We use these orthogonality relations to find the irreps.

Shur also implies properties for linear operators that commute with the rep:

$$\underset{\sim}{T}_g \underline{A} = \underline{A} \underset{\sim}{T}_g \; ; \quad T = \sum \oplus q_\mu T^{(\mu)}$$

$\Rightarrow \quad \underline{A} =$



$a_1 \times a_1$ $n_1 \times n_1$ blocks

$a_2 \times a_2$ $n_2 \times n_2$ blocks

$a_m \times a_m$ $n_m \times n_m$ blocks

Typical Entry:

$$\lambda \underline{I}_{n_\mu \times n_\mu}$$

Eg: $\underline{\sigma}$ for $SO(2)$ state: $\underline{Q}_\theta \underline{\sigma} = \underline{\sigma} \underline{Q}_\theta$; $\underline{Q}_\theta =$

$$\Rightarrow \underline{\sigma} = \left[\begin{array}{c|cc} \sigma_{11} & 0 & 0 \\ \hline 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{22} \end{array}\right] \begin{array}{l} \} \, 1 \; 1\times1 \\ \} \, 1 \; 2\times2 \end{array}$$

Eg:

$$\underset{\sim}{T}_g^{[2]} \underline{C} = \underline{C} \underset{\sim}{T}_g^{[2]}$$

$g \in$ cubic group

roughly $\simeq T^{[2]} = 1\, T^{(1)} + 1\, T^{(2)} + 1\, T^{(3)}$

$n_1 = 1, n_2 = 2, n_3 = 3$

Voigt notation

$$\Rightarrow \left[\begin{array}{ccc|ccc} A & B & B & & & \\ B & A & B & & \underline{0} & \\ B & B & A & & & \\ \hline & & & G & 0 & 0 \\ & \underline{0} & & 0 & G & 0 \\ & & & 0 & 0 & G \end{array}\right] \Rightarrow \left[\begin{array}{c|c|c} (A+2B) & \underline{0} & \underline{0} \\ \hline \underline{0} & (A-B)\, \underline{I}_{2\times2} & \underline{0} \\ \hline \underline{0} & \underline{0} & G\, \underline{I}_{3\times3} \end{array}\right]$$

Now, we learn more about <u>irreps</u>!

Define the p-dimensional complex vector space $\mathcal{V}_G$ consisting of all elements of the form

$$\underline{x} = \sum_{g \in G} x(g) \cdot g \quad , \quad x(g) \in \mathbb{C}$$

with innerproduct

$$\langle \underline{x}, \underline{y} \rangle = \frac{1}{p} \sum_{g \in G} x(g)\, y^*(g)$$

in essence the $g_i \in G$ form an Ortho basis of $\mathcal{V}_G$

Notice that, for fixed μ, i, j, $T^{(\mu)}_{ij}(g)$ can be viewed as the components of a vector in $\mathcal{V}_G$!

The innerproduct of $\mathcal{V}_G$ shows these form an orthogonal set in $\mathcal{V}_G$:

$$\langle T^{(\mu)}_{ij}, T^{(\nu)}_{kl} \rangle = \frac{1}{p} \left\{ \sum_{g \in G} T^{(\mu)}_{ij}(g)\, T^{(\nu)*}_{kl}(g) \right\}$$

$$= \frac{1}{p} \left\{ \left(\frac{p}{n_\mu} \right) \delta_{ik}\, \delta_{jl}\, \delta_{\mu\nu} \right\} \text{ by Shur.}$$

$$= \frac{1}{n_\mu} \delta_{ik}\, \delta_{jl}\, \delta_{\mu\nu}$$

Thus:

$$\sum_{\mu=1}^{m} n_\mu^2 \le P, \quad \text{in fact } = P$$

This is a major constraint on the # of irreps of G and the n_μ

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To prove equality (although we will not do so here) we need:
(this does depend on the structure of G)

On $\mathcal{V}_G$ we construct a rep, the <u>right regular rep</u>, R

$$\underline{R}_h \underline{x} = \sum_{g\in G} [\underline{R}_h \underline{x}](g)\cdot g = \sum_{g\in G} x(gh)\cdot g$$

Check

$$\underline{R}_{h_1}\underline{R}_{h_2}\underline{x} = \underline{R}_{h_1}\left(\underline{R}_{h_2}\underline{x}\right) = \underline{R}_{h_1}\left(\underbrace{\underline{R}_{h_2}\sum_{g\in G} x(g)\cdot g}\right)$$

$$y = \sum_{g\in G} y(g) g = \sum_{g\in G} x(gh_2) g$$

$$y(g) = x(gh_2)$$

$$= \underline{R}_{h_1} y = \underbrace{\underline{R}_{h_1}\sum_{g\in G} y(g)\cdot g}$$

$$z = \sum_{g\in G} z(g) g = \sum_{g\in G} y(gh_1) g \Rightarrow z(g) = y(gh_1)$$

$$\Rightarrow z(g) = y(gh_1) = x(gh_1h_2)$$

$$\Rightarrow \underline{R}_{h_1}\underline{R}_{h_2} = \underline{R}_{h_1h_2}$$

To further constrain the irreps, we introduce

Group Characters

$$\chi^{(\mu)}(g) = \mathrm{tr}\left(\underline{T}_g^{(\mu)}\right) \in \mathcal{V}_G$$

since $\mathrm{tr}(\underline{A}\,\underline{B}) = \mathrm{tr}(\underline{B}\,\underline{A})$, we get characters are constant on conjugacy classes of G:

$$g_j = h^{-1} g_i h \;;\; \chi^{(\mu)}(g_j) = \chi^{(\mu)}_{h^{-1} g_i h} = \chi^{(\mu)}_{g_i h^{-1} h} = \chi^{(\mu)}_{g_i}$$

thus, each $\chi^{(\mu)}$ is fully characterized by k numbers the number of conjugacy classes of G.

It follows that the $\chi^{(\mu)}$ span a k-dimensional subspace of $\mathcal{V}_G$. Indeed,

$$\langle \chi^{(\mu)}, \chi^{(\nu)} \rangle = \frac{1}{p} \sum_{g \in G} \chi^{(\mu)}(g)\, \chi^{(\nu)}(g)^*$$

$$= \frac{1}{p} \sum_{g \in G} \left(\sum_i T^{(\mu)}_{iig} \right) \left(\sum_j T^{(\nu)}_{jjg} \right)^*$$

$$= \frac{1}{p} \sum_i \sum_j \left(\sum_{g \in G} T^{(\mu)}_{iig} \, T^{(\nu)*}_{jjg} \right)$$

$$= \frac{1}{p} \sum_i \sum_j \left(\frac{p}{n_\mu} \delta_{ij} \delta_{ij} \delta_{\mu\nu} \right)$$

$$= \frac{1}{n_\mu} \delta_{\mu\nu} \underbrace{\left(\sum_i \sum_j \delta_{ij} \delta_{ij} \right)}_{n_\mu}$$

$$= \delta_{\mu\nu}$$

It can be shown that every vector in this k-dim space is a linear combination of the $\chi^{(\mu)}$.

So the $\chi^{(\mu)}$ are an O.N. basis for k-dim space

$\Rightarrow$ the number of inequivalent irreps is k.

$$\sum_{\mu=1}^{k} n_\mu^2 = p \quad ; \quad p = |G| \; ; \; n_\mu = \dim(T^{(\mu)})$$

$$k = \#\text{ classes}$$

$\langle \chi, \chi \rangle = 1$ iff T is <u>irred</u>.

A <u>second orthogonality relation</u>:

Define m_i $i = 1, 2, \dots, k$ to be the size of the i^{th} conjugacy class, $\mathcal{K}_i$, and $\chi_i^{(\mu)}$ its character, it can be shown:

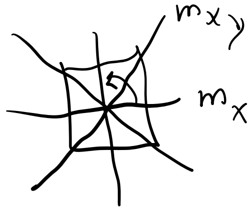
$$\sum_{\mu=1}^{k} \chi_i^{(\mu)} \overline{\chi}_j^{(\mu)} = \frac{p}{m_i} \delta_{ij}$$

Character tables

	$1\mathcal{K}_1$	$m_2\mathcal{K}_2$		$m_k\mathcal{K}_k$
$\chi^{(1)}$	$\chi_1^{(1)}$	$\chi_2^{(1)}$		$\chi_k^{(1)}$
$\chi^{(2)}$	$\chi_1^{(2)}$	$\chi_2^{(2)}$	$\cdots$	$\chi_k^{(2)}$
$\vdots$				
$\chi^{(k)}$	$\chi_1^{(k)}$	$\chi_2^{(k)}$		$\chi_k^{(k)}$

- $\sum_{\mu=1}^{k} n_\mu^2 = P$
- $\chi_1^{(\mu)} = \chi^{(\mu)}(\overset{id}{g_1}) = n_\mu \quad , \mu = 1, \dots, k$
- $\chi_i^{(1)} = 1 \quad , i = 1, \dots, k$
- $\sum_{\mu=1}^{k} \chi_i^{(\mu)} \bar{\chi}_j^{(\mu)} = \frac{P}{m_i}\delta_{ij}$
- $\langle \chi^{(\mu)}, \chi^{(\nu)} \rangle = \sum_{i=1}^{k} \chi_i^{(\mu)} \bar{\chi}_j^{(\nu)} m_i = \delta_{\mu\nu}$

Eg: $G = C_{4v} = \{e, R_{90}, R_{180}, R_{270}, m_x, m_y, m_{xy}, m_{\bar{x}y}\}$



$P = 8$ $\quad m_1 = 1$ $\quad m_2 = 2$ $\quad m_3 = 2$

$k = 5$ $\quad \mathcal{K}_1 = \{e\}$ $\quad \mathcal{K}_2 = \{R_{180}\}$ $\quad \mathcal{K}_3 = \{R_{90}, R_{180}\}$

$m_4 = 2$ $\quad m_5 = 2$

$\mathcal{K}_4 = \{m_x, m_y\}$ $\quad \mathcal{K}_5 = \{m_{xy}, m_{\bar{x}y}\}$

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 8 \Rightarrow 1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8$$

C_{4v}	$1\mathcal{K}_1$	$1\mathcal{K}_2$	$2\mathcal{K}_3$	$2\mathcal{K}_4$	$2\mathcal{K}_5$
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	1	-1	-1
$\chi^{(3)}$	1	1	-1	1	-1
$\chi^{(4)}$	1	1	-1	-1	1
$\chi^{(5)}$	2	-2	0	0	0

Additional Theorems :

Fixed point space : $\underline{T}_g \underline{v} = \underline{v} \quad \forall g \in G$

Projection:

$$\underline{P} = \frac{1}{p} \sum_{g \in G} \overbrace{\chi^{(1)}(g)}^{1} \underline{T}_g = \frac{1}{p} \sum_{g \in G} \underline{T}_g$$

$$\underline{T}_g^{[4]} \underline{C} = \underline{C} \Rightarrow \underline{C} \text{ is in fixed point space}$$

$$T = \sum_{\mu=1}^{k} \oplus a_\mu T^{(\mu)}$$

$$a_\mu = \frac{n_\mu}{p} \sum_{g \in G} \chi(g) \chi^{(\mu)*}(g) \quad ; \quad \chi(g) = tr(\underline{T}_g)$$

References:

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