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Virtual Nikuradse

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In this paper we derive an accurate composite friction factor versus Reynolds number correlation formula for laminar, transition and turbulent flow in smooth and rough pipes. The correlation is given as a rational fraction of rational fractions of power laws which is systematically generated by smoothly connecting linear splines in log-log coordinates with a logistic dose function algorithm. We convert Nikuradse's (1933) (J. Nikuradse, 1933 *Stromungsgesetz in rauhren rohren*, vDI Forschungshefte **361**. (*English translation: Laws of flow in rough pipes*). Technical report, *NACA* Technical Memorandum 1292. *National Advisory Commission for Aeronautics* (1950), Washington, DC.) data for six values of roughness into a single correlation formula relating the friction factor to the Reynolds number for all values of roughness. Correlation formulas differ from curve fitting in that they predict as well as describe. Our correlation formula describes the experimental data of Nikuradse's (1932, 1933) (J. Nikuradse, *Laws of turbulent flow in smooth pipes (English translation)*, NASA (1932) TT F-10: 359 (1966).) and McKeon et al. (2004) (B.J. Mckeon, C.J. Swanson, M.V. Zaragola, R.J. Donnelly, and J.A. Smits, *Friction factors for smooth pipe flow*, J. Fluid Mech. 511 (2004), 41–44.) but it also predicts the values of friction factor versus Reynolds number for the continuum of sand-grain roughness between and beyond those given in experiments. Of particular interest is the connection of Nikuradse's (1933) data for flow in artificial rough pipes to the data for flow in smooth pipes presented by Nikuradse (1932) and McKeon et al. (2004) and for flow in smooth and effectively smooth pipes. This kind of correlation seeks the most accurate representation of the data independent of any input from theories arising from the researchers ideas about the underlying fluid mechanics. As such, these correlations provide an objective metric against which observations and other theoretical correlations may be applied. Our main hypothesis is that the data for flow in rough pipes terminates on the data for smooth and effectively smooth pipes at a definite Reynolds number $R_\sigma(\sigma)$; if $\lambda = f(Re, \sigma)$ is the friction factor in a pipe of roughness parameter σ then $\lambda = f(R_\sigma(\sigma), \sigma)$ is the friction factor at the connection point. An analytic formula giving $R_\sigma(\sigma)$ is obtained here for the first time.

Keywords: friction factor; sand-grain roughness; artificial rough pipe; Nikuradse's data; smooth and effectively smooth pipes

1. Introduction

Here we convert Nikuradse's data into explicit analytic correlation formulas by smoothly connecting different power laws with logistic dose functions. The complicated data set with

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multiple pieces of spline-like distribution is correlated by the roughness parameter σ in the full range of Reynolds number Re. The correlation formulas are rational fractions of rational fractions of power laws. The method leads to a tree-like structure with many branches that we call a correlation tree. Curves relating friction factors to the Reynolds number for a fixed value of the roughness ratio can be found from formulas on the correlation tree. Formulas predicting the values of the Reynolds number and friction factor for which the effects of roughness first appear are derived here for the first time. Many obscure features of turbulent flow in rough pipes are embedded in the correlation tree. The flow of fluids in rough pipes has been a topic of great interest to engineers for over a century. The landmark experiments of Nikuradse [11] are the gold standard for work on this topic even today. Understanding the fluid mechanics of turbulent flow in rough pipes is still subject to controversy because mathematically rigorous approaches are not known and theoretical ideas must rest on the interpretations of the data. The problem discussed in this paper is related to how flows in a rough pipe connect to flows which are effectively smooth in the same rough pipe. We call this *the connection problem*. Virtual Nikuradse is a consequence of our hypothesis that the transition from rough flow to effectively smooth flow in the same rough pipe occurs at a definite Reynolds number located on the bottom envelope of rough pipe data given in the famous plot of experiments in six pipes with different values of sand-grain roughness given by Nikuradse [11]. Other ideas about the nature of the connection are also discussed in this paper.

Our particular interest is the connection of Nikuradse's [11] data for flow in rough pipes to the data for flow in smooth pipes presented by Nikuradse [10] and McKeon

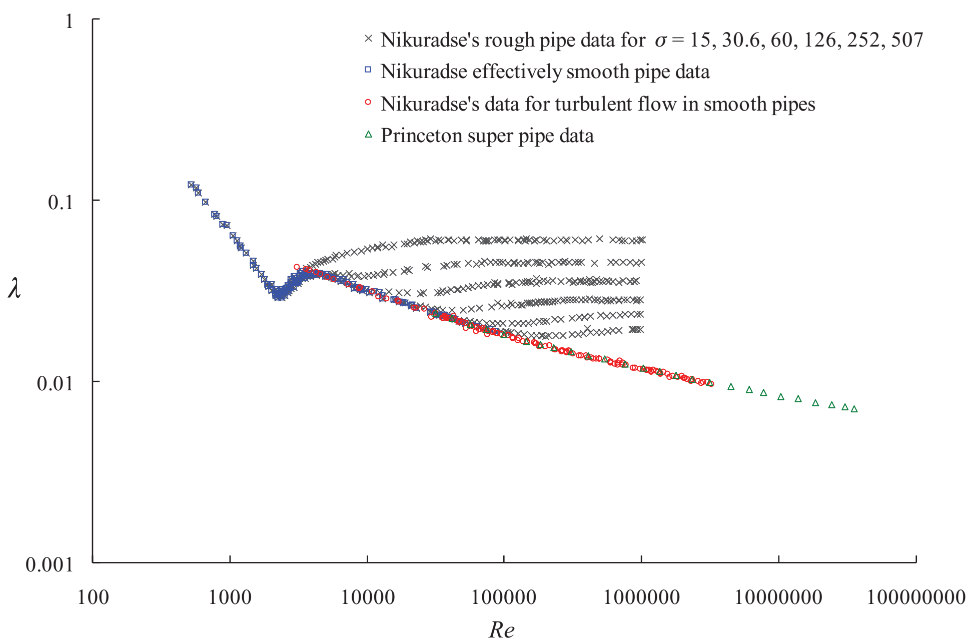

Figure 1. Friction factor vs. Reynolds number. The data from Nikuradse [10] for flow in smooth pipes, the data from Nikuradse [11] for effectively smooth turbulent flow in rough pipes and data the from the Princeton super pipe coincide. Data in the transition region is influenced strongly by instabilities that depend on many parameters and is better described as data cloud rather than a data curve.

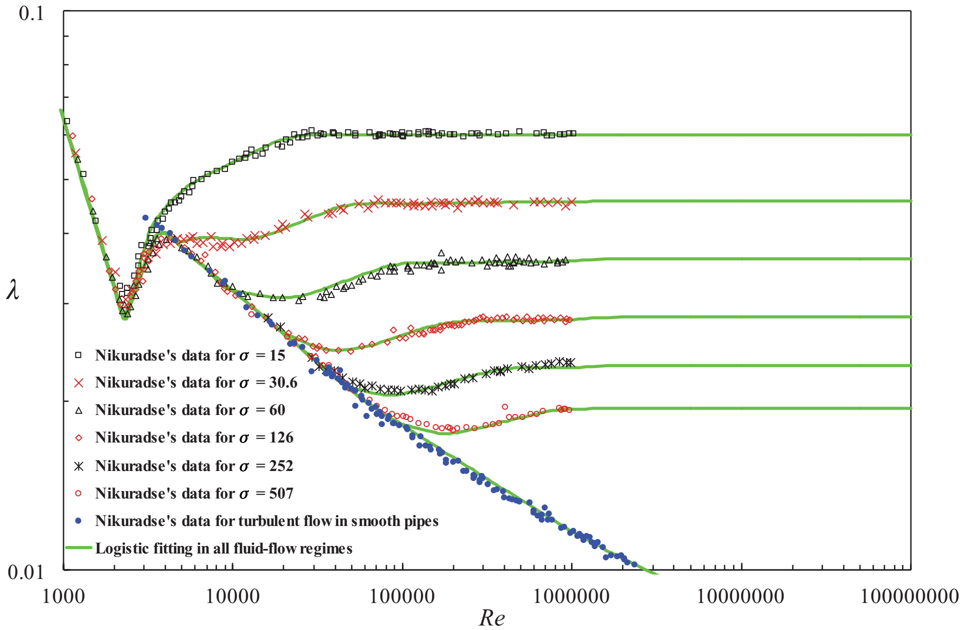

Figure 2. Data fit $\lambda = f(Re, \sigma)$ for laminar, transition and turbulent regimes in smooth and rough pipes. The solid lines are calculated from Equation (1). When $\sigma = \sigma_j$, $\lambda = f(Re, \sigma)$ describes the fit for Nikuradse's data with six values of roughness. For nominally smooth pipe, the roughness approaches to zero and therefore $\lambda = f(Re, \sigma)$ reduces to $\lambda = f(Re)$ in which the roughness parameter does not play a role.

et al. [8] and for effectively smooth flow in the same rough pipes. The data sets from real experiments are shown in Figure 1. In Figure 2 we have fit this data to linear splines smoothly connected by the five-point rule of logistic dose curves (see Joseph and Yang [8]). In this paper, we extend this method to convert Nikuradse's [10, 11] data for smooth pipe and artificial rough pipes with six values of roughness into a single formula (see Equation (5)) relating the friction factor to the Reynolds number for all values of roughness. The graph of Equation (5) is shown in Figure 3. We call this figure "Virtual Nikuradse" because it predicts experiments for all values of the roughness parameter which have not yet been done from the six values which have been done. Equation (5) extrapolates between and beyond experimental data. The data between is well supported by Nikuradse's six curves but the data beyond has no experimental support as yet. The kind of correlation achieved in Equation (5) seeks the most accurate representation of the data independent of any input from theories arising from the researchers ideas about the underlying fluid mechanics. As such, these correlations provide an objective metric against which observations and other theoretical correlations may be applied. Our main hypothesis is that the data for flow in rough pipes terminates on the data for smooth and effectively smooth pipes at a definite Reynolds number function $R_\sigma(\sigma)$ where $\sigma = a/k$ is the roughness ratio, a is the pipe radius and k is the average depth of roughness. If $\lambda = f(Re, \sigma)$ is the friction factor in a pipe of roughness σ then $\lambda = f(R_\sigma(\sigma), \sigma)$ is the friction factor at the connection point. Nikuradse [11] presented his data for six values of roughness $\sigma_j (j = 1, 2, 3, 4, 5, 6) = (15, 30.6, 60, 126, 252, 507)$. A formula giving $R_\sigma(\sigma)$ is obtained here for the first time.

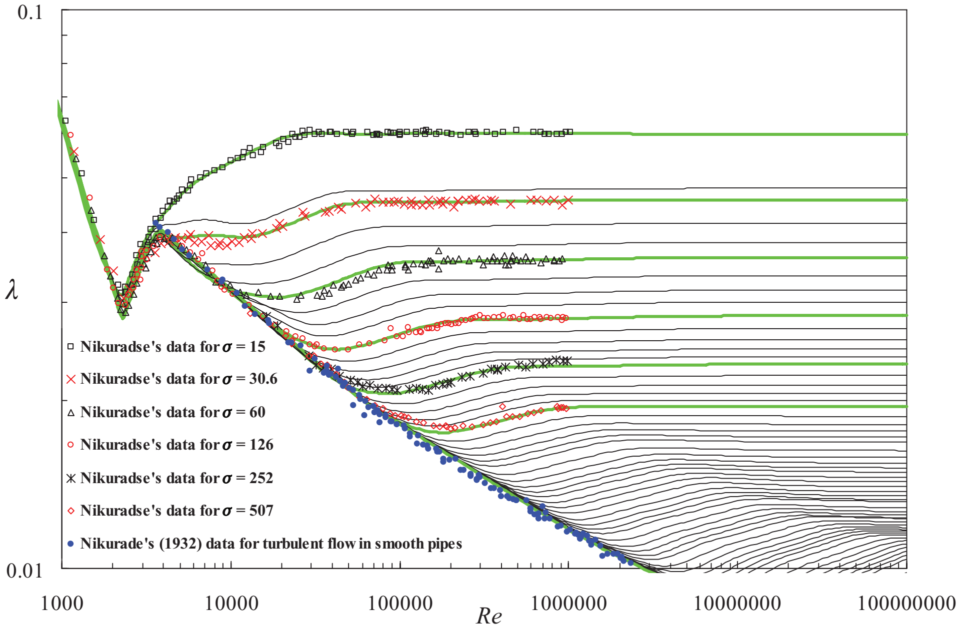

Figure 3. Virtual Nikuradse. Correlations $\lambda = f(Re, \sigma)$ for values of σ are up to 10^5, computed from Equation (5). Nikuradse's [11] data for rough pipes and (1932) data for smooth pipe are included for comparison. The solid green lines are the data fits shown in Figure 2. The dotted curves between and beyond the data fits are virtual curves obtained from Equation (5) predicting the results of experiments not yet done.

2. Turbulent flow in smooth and effectively smooth pipes

Joseph and Yang [8] showed that data for the friction factor versus Reynolds number in turbulent flow in the smooth pipes studied by Nikuradse [10] coincide with data for effectively smooth flows in rough pipes studied by Nikuradse [11] (see Figure 1).

Figure 1 suggests that the connection between rough and effectively smooth pipe data occurs at definite Reynolds number. Unfortunately deductions from data involve value judgments; it is not a science. Borrowing from mathematics where we can have a high degree of comfort, we imagine that the data for rough pipes connects with the smooth pipe data smoothly with a continuous first and discontinuous second derivative. It is not possible to read the coordinates of these connections from experimental data. We admit that our estimates of the connection values are not accurate but these estimates are all the better because they involve a progression of six values.

In analyzing the effect of surface roughness on flow in pipes, the ratio of the roughness dimension to the thickness of the viscous sublayer has long been accepted as the governing factor. Thus, if the roughness elements are so small that the viscous sublayer enclosing them is stable against the perturbation, the roughness will have no drag increasing effect. This is called the "effectively smooth" case. On the other hand, if the size of the roughness is so large as to disrupt the viscous sublayer completely, the surface resistance will then be independent of the viscosity. This is called the case of fully developed roughness action. Between these two extremes there exists an intermediate region in which only a fraction of the roughness elements disturbs the viscous sublayer. Consequently, the resistance law in this intermediate region depends upon both the roughness magnitude and the thickness of the viscous sublayer. In Figure 1, to the right, each pipe with a unique roughness has a constant friction factor indicating that completely rough conditions have been reached, whereas to the left all curves converge towards that for smooth or effectively smooth surfaces.

In Figure 2 we have displayed a data fit for the data displayed in Figure 1. These fits are generated by fitting the data with logistic dose curves following procedures described in the appendix. In the present case, the complicated multiple spline-like data sets are correlated by the roughness parameter σ in the full ranges of Re for both smooth and rough pipes. All the solid curves in Figure 2 are calculated from the single Equation (5). The virtual curves generated from correlating the fits in Figure 2 are shown in Figure 3 and the methods used to obtain the correlations are described in Section 5 and the appendix.

3. Colebrook and Moody

Colebrook [2] used the data from Colebrook and White [3] to develop a function which gives a practical form for the transition curve between rough and smooth pipes which agrees with the two extremes of roughness and gives values in very satisfactory agreement with actual measurements on most forms of commercial piping and usual pipe surfaces. The Colebrook correlations were used by Moody [9] to create the Moody diagram to be used in computing the loss of head in clean new pipes and in closed conduits running full with steady flow. Moody diagrams are collections of friction factor versus Reynolds number curves for different values of commercial roughness. The variations of the fits of λ versus Re in the Moody diagrams is monotonic; they do not exhibit dips and bellies of Nikuradse data shown, for example in Figures 4 and 5. This difference is due to the fact that commercial or natural roughness is different from controlled sand-grain roughness.

Shockling, Allen and Smits [13] studied roughness effects in turbulent pipe flows with honed roughness. They showed that in the transitionally rough regime where the friction factor depends on roughness height and Reynolds number $\lambda = f(Re, \sigma)$, the friction factor

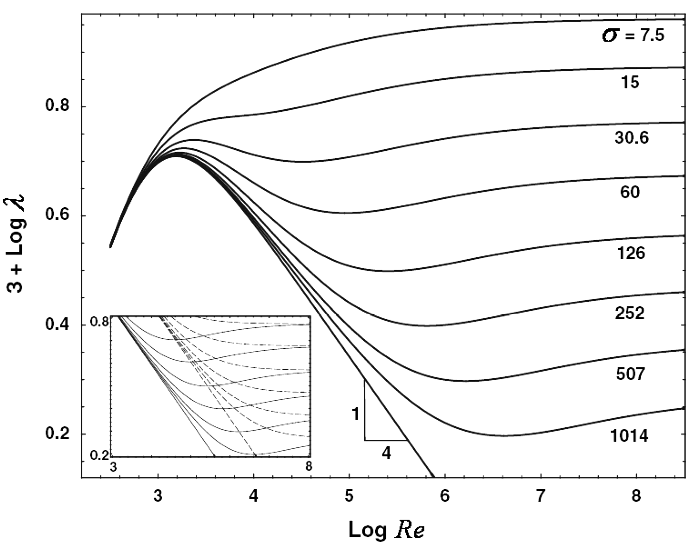

Figure 4. Friction factor curves produced by the analytic model of Gioia and Chakraborty [4] (see Figure 5).

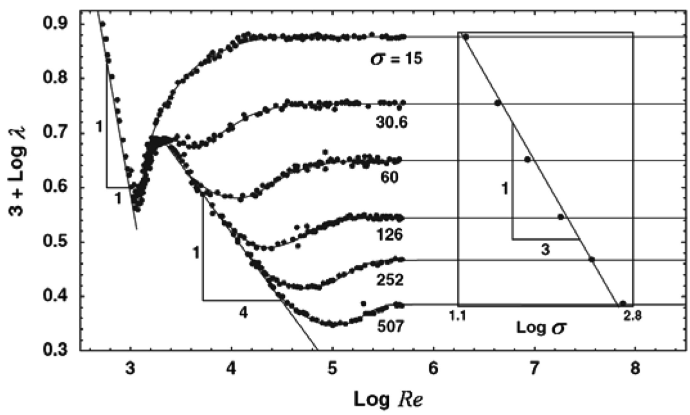

Figure 5. [After Gioia and Chakraborty (2006)] Nikuradse's [11] data for the friction factor vs. Reynolds number emphasizing the Blausius' 1/4 law and Strickler's $\sigma^{-1/3}$ correlation at large Reynolds numbers. Our processing of the Nikuradse's data does not lead to Strickler's correlation (see Equation (5)) and it does not lead to the Blasius law (see Equation (4)).

for honed surfaces follows the Nikuradse [11] form with dips and bellies rather than the monotonic relations seen in the Moody diagram.

Nikuradse's [11] experiments measured the flow through uniformly roughened pipes and found comparatively abrupt transition from "smooth" law at slow speeds to "rough" law at high speeds. Other experimenters using natural surfaces, obtained results which can only be explained by a much more gradual transition between the two resistance laws. Colebrook and White [3] carried out systematical experiments for artificial pipes with five different types of roughness, which were formed from various combinations of two sizes of sand grain (0.035 cm and 0.35 cm diameters). They found that with non-uniform roughness, the transition between two resistance laws is gradual, and in extreme cases so gradual that the whole working range lies within the transition zone. The experiments of Colebrook and White [3] closed the gap between Nikuradse's artificial roughness, and roughness normally found in natural pipes. Their results demonstrated that the nature of the effect of surface roughness in the intermediate region depends as well on the geometrical characteristics of the roughness pattern; i.e. the spacing between sand grains and the composition of grain sizes. Bradshaw [1] noted that "... an unrigorous but plausible analysis suggests that the concept of a critical roughness height, below which roughness does not affect a turbulent wall flow, is erroneous." They use the Oseen approximation to construct their ad hoc argument. Their conclusion apparently is not applicable to sand-grain roughness in Nikuradse's experiments where the concept of effectively smooth flows in rough pipes is completely supported by experiments (see Figure 1).

4. The work of Gioia and Chakraborty [4]

An impressive theoretical study of turbulent flow in rough pipes by Gioia and Chakraborty [4] gives rise to curves with bellies and valleys (Figure 4) which resemble the shape of the Nikuradse's data (Figure 5). They use the phenomenological theory of Kolmogorov to model the shear that a turbulent eddy imparts to a rough surface. However, their model does not resemble the way that the friction factor for flow in rough pipes connects with the data for effectively smooth flow in rough pipes (Figure 1); in fact, their model does not connect flow in rough pipes to effectively smooth flows in the same rough pipes. Their roughness curves start in a cluster at one and the same point in the region of transition from laminar to turbulent flow and then separate into curves with different roughness values which do not connect to smooth flow or each other.

Goldenfeld [5] discussed the scaling of turbulent flow in rough pipes in the frame of a theory of critical phenomenon. He constructs the form of a formula $\lambda = Re^{-1/4} g(Re^{3/4}(r/D))$ with g undetermined but such that the correlation reduces to Strickler's on the left and Blausius' on the right. When plotted in the reduced variables, the spread of the six curves for turbulent flow in rough pipes are greatly reduced and a partial collapse of the data is achieved.

Mehrafarin and Pourtolami [7] modified Goldenfeld's [5] formula to take into account effects associated with intermittency. They achieve a better collapse of data but their correlation no longer reduces to Strickler's on the left or Blausius' on the right.

5. Construction of friction factor correlation for Nikuradse's [10, 11] data for flow in smooth and rough pipes

Joseph and Yang [8] have illustrated a simple sequential construction procedure for correlating friction factor to Reynolds number in smooth pipes using logistic dose functions. In this section, we introduce a much more complicated sequential construction procedure for processing Nikuradse's [10, 11] data for smooth and rough pipes. A correlation tree is used in this procedure (Figure 6), which includes one chain on the left for flow in smooth pipes and six chains on the right for flow in rough pipes with six values of roughness.

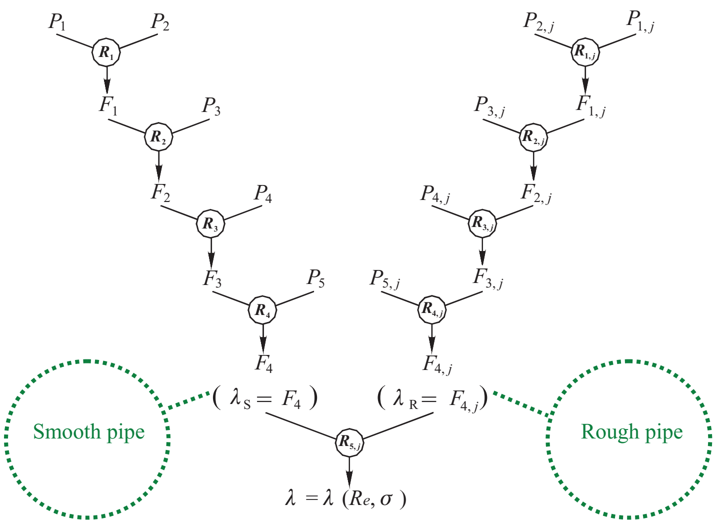

Figure 6. Correlation tree for smooth and rough pipes. P_i and $P_{i,j}$ are power laws, F_i and $F_{i,j}$ are rational fractions of power laws. R_i are branch points for smooth pipe and $R_{i,j}$ are branch points for rough pipes. At each branch point, two assembly member functions are merged into a rational fraction of power laws by processing data with the logistic dose function algorithm. The chain on the left is for smooth pipes and leads to a rational fraction correlation F_4. The six chains on the right are for rough pipes and lead to six rough pipe correlations $F_{4,j}(j = 1, 2, 3, 4, 5, 6)$. In the correlation tree, $\lambda_S = \lambda_S(Re) = F_4(Re)$ and $\lambda_R = \lambda_R(Re, \sigma) = F_{4,j}(Re, \sigma_j)$. These correlations are merged into a single composite correlation $\lambda = f(Re, \sigma)$.

In Figure 6, the power laws for rough pipes are $P_{i,j} = a_{i,j} Re^{b_{i,j}}$ ($i = 1, 2, 3, 4, 5$; $j = 1, 2, 3, 4, 5, 6$). In the construction of our correlations, the prefactors $a_{i,j}$ and exponents $b_{i,j}$ of power laws and the branch points $R_{i,j}$ in the correlation tree for rough pipes are all correlated by power law functions or rational fractions of power laws of the roughness ratio σ and do not depend on j. The power law formulas obtained here by processing the data for straight-line segments in log-log coordinates converts the six data points in Nikuradse's data into continuous functions of σ. These functions reduce to the original data at six values of σ. We may imagine that the range of these functions extend well beyond the range of the six data points. These correlations allow us to introduce the explicit dependence of the final correlation on the roughness ratio σ.

The correlation formula obtained from the correlation tree for smooth and rough pipes is

$$\lambda = f(Re, \sigma) = \lambda_S + \frac{\lambda_R - \lambda_S}{\left[1 + \left(\frac{Re}{R_\sigma}\right)^{m_5}\right]^{n_5}} = F_4 + \frac{F_{4,j} - F_4}{\left[1 + \left(\frac{Re}{R_{5,j}}\right)^{m_5}\right]^{n_5}}, \quad (j = 1, 2, \ldots, 6), \quad (1)$$

where $\lambda_S(Re) = F_4(Re)$ is the friction factor correlation for smooth and effectively smooth pipes and $\lambda_R(Re, \sigma) = F_{4,j}(Re, \sigma_j)$ is the correlation for rough pipes. This formula is generated in the following sequence

$$F_i = F_{i-1} + \frac{P_{i+1} - F_{i-1}}{\left[1 + \left(\frac{Re}{R_i}\right)^{s_i}\right]^{t_i}} \quad (i = 1, 2, \ldots, 4), \quad F_0 = P_1, \quad (2)$$

$$F_{i,j} = P_{i+1,j} + \frac{F_{i-1,j} - P_{i+1,j}}{\left[1 + \left(\frac{Re}{R_{i,j}}\right)^{m_i}\right]^{n_i}} \quad (i = 1, 2, \ldots, 4;\ j = 1, 2, \ldots, 6), \quad F_{0,j} = P_{1,j}. \quad (3)$$

where R_i, s_i, t_i, m_i, n_i ($i = 1, 2, 3, 4$), a_i, b_i ($i = 1, 2, 3, 4, 5$) are all constants and $a_{i,j}, b_{i,j}$ and $R_{i,j}$ are all power law functions or rational fractions of power laws of the roughness ratio σ (see Tables A1 and A2 in Appendix).

The correlation

$$R_{5,j} = 45.196502\sigma^{1.2369807} + 1891 = R_\sigma(\sigma) \quad (4)$$

is very important. It correlates the six branch points where the flow in smooth pipes are joined to six points for flow in rough pipes into a continuous power law function of σ with a constant correlation term. This function predicts the Reynolds numbers on the smooth pipe curve at which the effects of roughness commence between and beyond the six values given in Nikuradse's experiments. That is to say, $R_\sigma(\sigma)$ identifies the minimum Reynolds number at which the roughness σ first appears. For any pipe flow with a given equivalent sand-grain roughness σ and Reynolds number Re, the friction factor can be calculated explicitly by Equation (1) for a wide and extended range of roughness and for all fluid-flow regimes including laminar, transition and turbulent flows.

The final composite data fit (1) is shown by the heavier solid lines in Figure 2. This formula gives the friction factor as a function of the Reynolds number and roughness ratio for Nikuradse's [10, 11] data for smooth and rough pipes and the Princeton data for smooth pipes. Equation (1) is valid for continuous σ and $R_{5,j}$ does not depend on j. The solid lines in Figure 2 show the fits of λ versus Re for flows in smooth pipes and in rough pipes with six different values of roughness. Given a smooth pipe or one of roughness parameter σ, the friction factor can be explicitly calculated from Equation (1). The friction factor λ reduces to λ_S for smooth pipe and λ_R for rough pipes. For continuous roughness $\sigma > 15$, Equation (1) can sweep the huge area between the curve for $\sigma = 15$ and the one for smooth pipe.

6. Conversion of Nikuradse's and Princeton experimental data to a continuous family of virtual curves between and beyond the original data as described by one explicit formula

Our main results are presented in the previous section and this section. Figure 3 shows the curves for virtual experiments that arise from correlation of data leading to the long but explicit Equation (5) which miraculously describes Nikuradse's real experiments and the virtual extension.

Substituting all the data in Tables A1 and A2 into Equations (2), (3) first and then Equation (1), we can obtain the explicit composite correlation for λ as a function of σ and Re for laminar, transition and turbulent flow in smooth and sand-grain rough pipes.

$$\lambda = f(Re, \sigma) = \lambda_S + \frac{\lambda_R - \lambda_S}{\left[1 + \left(\frac{Re}{R_\sigma}\right)^{m_5}\right]^{n_5}} = \lambda_S + \frac{\lambda_R - \lambda_S}{\left[1 + \left(\frac{Re}{45.196502\sigma^{1.2369807} + 1891}\right)^{-5}\right]^{0.5}}, \quad (5)$$

where λ_S and λ_R are given by Equations (6) and (7). The Reynolds numbers $R_{5,j}$ are the six branch points where λ_S and λ_R are joined.

We can also write out λ_S and λ_R explicitly. For flow in smooth pipes, $\lambda_S = \lambda_S(Re) = F_4(Re)$ is an explicit rational power law function of Re given by

$$\lambda_S = F_4(Re) = F_3 + \frac{P_5 - F_3}{\left[1 + \left(\frac{Re}{R_4}\right)^{s_4}\right]^{t_4}} = F_3 + \frac{0.0753Re^{-0.136} - F_3}{\left[1 + \left(\frac{Re}{2000000}\right)^{-2}\right]^{0.5}}, \quad (6)$$

where $F_3 = F_2 + \dfrac{P_4 - F_2}{\left[1 + \left(\frac{Re}{R_3}\right)^{s_3}\right]^{t_3}} = F_2 + \dfrac{0.1537Re^{-0.185} - F_2}{\left[1 + \left(\frac{Re}{70000}\right)^{-5}\right]^{0.5}}$,

$$F_2 = F_1 + \frac{P_3 - F_1}{\left[1 + \left(\frac{Re}{R_2}\right)^{s_2}\right]^{t_2}} = F_1 + \frac{0.3164Re^{-0.25} - F_1}{\left[1 + \left(\frac{Re}{3810}\right)^{-15}\right]^{0.5}},$$

$$F_1 = F_0 + \frac{P_2 - F_0}{\left[1 + \left(\frac{Re}{R_1}\right)^{s_1}\right]^{t_1}} = F_0 + \frac{0.000083Re^{0.75} - F_0}{\left[1 + \left(\frac{Re}{2320}\right)^{-50}\right]^{0.5}},$$

$$F_0 = P_1 = \frac{64}{Re}.$$

For flow in rough pipes, we have

$$\lambda_R = F_{4,j}(Re, \sigma_j) = P_{5,j} + \frac{F_{3,j} - P_{5,j}}{\left[1 + \left(\frac{Re}{R_{4,j}}\right)^{m_4}\right]^{n_4}} = P_{5,j} + \frac{F_{3,j} - P_{5,j}}{\left[1 + \left(\frac{Re}{783.39696\sigma^{0.75245644}}\right)^{-5}\right]^{0.5}}, \quad (7)$$

where
$$P_{5,j} = \left(\frac{(0.92820419\sigma^{0.03569244} - 1) - (0.00255391\sigma^{0.8353877} - 0.022)}{\left[1 + \left(\frac{\sigma}{93}\right)^{-50}\right]^{0.5}} + (0.00255391\sigma^{0.8353877} - 0.022) \right) \cdot Re^{\left(7.3482780\sigma^{-0.96433953} - 0.2032\right)},$$

$$F_{3,j} = P_{4,j} + \frac{F_{2,j} - P_{4,j}}{\left[1 + \left(\frac{Re}{R_{3,j}}\right)^{m_3}\right]^{n_3}} = P_{4,j} + \frac{F_{2,j} - P_{4,j}}{\left[1 + \left(\frac{Re}{406.33954\sigma^{0.99543306}}\right)^{-5}\right]^{0.5}},$$

$$P_{4,j} = \left(0.01105244\sigma^{0.23275646}\right) Re^{\left(0.62935712\sigma^{-0.28022284} - 0.191\right)},$$

$$F_{2,j} = P_{3,j} + \frac{F_{1,j} - P_{3,j}}{\left[1 + \left(\frac{Re}{R_{2,j}}\right)^{m_2}\right]^{n_2}} = P_{3,j} + \frac{F_{1,j} - P_{3,j}}{\left[1 + \left(\frac{Re}{1451.4594\sigma^{1.0337774}}\right)^{-5}\right]^{0.5}},$$

$$P_{3,j} = \left(0.02166401\sigma^{-0.30702955} + 0.0053\right) Re^{\left(0.26827956\sigma^{-0.28852025} + 0.015\right)},$$

$$F_{1,j} = P_{2,j} + \frac{F_{0,j} - P_{2,j}}{\left[1 + \left(\frac{Re}{R_{1,j}}\right)^{m_1}\right]^{n_1}} = P_{2,j} + \frac{F_{0,j} - P_{2,j}}{\left[1 + \left(\frac{Re}{295530.05\sigma^{0.45435343}}\right)^{-2}\right]^{0.5}},$$

$$P_{2,j} = \left(0.18954211\sigma^{-0.51003100} + 0.011\right) Re^{0.002},$$

$$F_{0,j} = P_{1,j} = \left(0.17805185\sigma^{-0.46785053} + 0.0098\right) Re^{0}$$
$$= 0.17805185\sigma^{-0.46785053} + 0.0098.$$

7. Comparison with Strickler's correlation in completely rough pipe flow

Strickler's correlation for λ at high Re is given by $\lambda \sim \sigma^{-1/3}$ (see Figure 5). We have noted that the points in the region of high Re, where the friction factor is independent of Re, do not fit Strickler's 1/3 law. The coordinates in Figure 5 are logarithmic and a very small deviation from Strickler's straight line can cause huge differences in the value of λ. Our correlation (5) reduces to $\lambda \approx 0.17805185\sigma^{-0.46785053} + 0.0098$ when $Re > 1000000$.

8. Discussion and prediction

The correlations derived in this paper allows one to analyze and predict the properties of friction factor in all fluid-flow regimes. Equation (5) shows that for any roughness σ, λ depends on Re alone when Re is smaller than its threshold value $R_\sigma = 45.196502\sigma^{1.2369807} + 1891$ but it depends on both Re and σ when Re is greater than R_σ. R_σ is the locus of points where the curves for smooth and rough pipes are joined by a logistic dose function. $R_\sigma\,(\sigma)$ identifies the minimum Reynolds number at which the roughness σ first appears. Joseph and Yang [8] have noted that the transition from smooth to rough pipe flow in their data fit seems to occur near a value of 13.6×10^6 in agreement with a similar, earlier and independent analysis of McKeon et al. [8]. From Equation (4) we may compute that when

$$Re = 13.6 \times 106,\ \sigma = a/k = 2.68 \times 10^4. \tag{8}$$

For any pipe flow with a given roughness σ and Reynolds number Re, the friction factor can be calculated explicitly by Equation (5) for an extended range of roughness and for all fluid-flow regimes including laminar, transition and turbulent flows.

9. Summary and conclusion

Power law representations of physical data are ubiquitous in science and in fluid mechanics. Very complicated data may be represented by piecewise power law coverings supplemented by fitting transition regions with logistic dose function algorithms. In this way we go from data to formulas.

Discrete data is converted by correlations into formulas, which allow one to fill gaps in the data and to greatly extend the range of data for which prediction can be made. In the case of Nikuradse's data for laminar, transition and turbulent flow in pipes, we have produced formulas from the data which track the data, fill in the gaps and greatly extend the range of conditions to which friction factor predictions can be given. For example the roughness inception function predicts the Reynolds number in very smooth pipes at which the effects of roughness first appear.

Our method has produced formulas which track, interpolate and extend the data. In the case of flow in pipes, we found formulas generated sequentially in branches with a tree-like structure that we called a correlation tree. The formulas that we obtained are algebraic and easily programmed. These formulas, produced from data, could never be derived by mathematical analysis and could not now be produced by numerical analysis.

It is necessary to stress that some uncertainties cannot be avoided during the construction of our correlation formula and the formula (Equation (5)) is not optimized. The quality of our correlation formula is judged by the fitting error of the original data obtained in experiments. In the case of flow in smooth pipes considered by Joseph and Yang [8], the fitting errors were evaluated and compared with the errors generated by other formulas. In the present case, there are no other papers which attempt to do what we have done, so comparisons are not possible. Here we judge the quality of the fits from the graphical displays; a more precise evaluation of the errors can be done but it would be very time consuming and of limited value.

In principle, this method is easy to use, and the computation is only related to two forms of basic functions, logistic dose function and power law. In practice, good fits are obtained by trial and error in adjusting the number of intervals covered by the splines and by adjusting the five parameters of the logistic dose curve. It is not possible to get perfect fits even to perfect unscattered data if for no other reason than five points cannot give a perfectly accurate representation of the continuous data connecting power laws.

We have developed correlations of λ versus Re for flows in smooth and rough pipes from Nikuradse's [10, 11] data for smooth and rough pipes and the Princeton data for smooth pipes. We found one formula, Equation (5), as a composition of power laws, which give the friction factor versus Reynolds number as a family of curves with a continuous dependence on the roughness ratio σ in all flow regimes.

For the fully rough wall turbulence at high Reynolds numbers, we have shown that Strickler's one-third scaling does not accurately describe Nikuradse's data. Instead, our equation $\lambda = 0.17805185\sigma^{-0.46785053} + 0.0098$ predicts the friction factor as a function of roughness parameter σ in the high Reynolds number region where λ is independent of Re. Our formula (4) for smooth flow in nominally smooth and rough pipes does not fit the Blasius 1/4 law or the high Reynolds number Prandtl law over the whole range of data.

We must remember, the roughness presented in this paper is the equivalent sand-grain roughness, and the natural roughness must be expressed in terms of the sand-grain roughness which would result in the same friction factor. This is not easily achieved; in fact, the only way it can be done is by comparison of the behavior of a naturally rough pipe with a sand-roughened pipe. Moody [9] has made such comparisons, and his widely used chart (Figure 2 of Moody [9]) gives the absolute and relative sand-grain roughness of a variety of pipe-wall materials and can be used for reference.

The procedure described in this paper may be easily implemented in the age of computers and it may find many applications in science and engineering.

Acknowledgement

This work was supported by the NSF/CTS under grant 0076648. We have profited from conversations with Grigory I. Barenblatt, Nigel Goldenfeld, Ivan Marusic, Beverley McKeon and Alex Smits about topics discussed in this paper. The research reported in this paper is taken from the Ph.D. thesis of Bobby (Haoping) Yang.

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Appendix: Processing of Nikuradse's [10, 11] data for constructing friction factor correlations for flow in smooth and artificial rough pipes

(1) Construction of correlation trees using the logistic dose function algorithm (LDFA)

The logistic function is one of the oldest growth functions and a best candidate for fitting sigmoidal (also known as "logistic") curves. In life sciences, logistic dose-response curves are widely used to fit forward or backward S-shaped data sets with two plateau regions and a transition region. In a companion paper of Joseph and Yang [8], we have showed how this method could be generalized to the case in which a power law and a rational fraction of power laws separated by a transition region could be assembled into a smooth function. To construct these functions, we first identify the transition region from one to the other. Then, we lay down the tangent of each function at the points of transition; there is a tangent to the function on the left and a tangent to the function on the right side. We are working this for the cases in which the two tangents intersect; in this the data in the transition region can be processed in the wedge formed by the two tangents. When we work in log-log planes, as is the case here, the tangents are power laws and can be fit smoothly as logistic dose curves.

We now shall show how to create a logistic dose function $f(x)$ for two arbitrary functions $f_L(x)$ and $f_R(x)$

$$y = f(x) = f_L(x) + \frac{f_R(x) - f_L(x)}{G(x)} = f_L(x) + \frac{f_R(x) - f_L(x)}{[1 + (x/x_c)^{-m}]^n}, \tag{A.1}$$

where m and n are positive constants, x_c is the critical value of the independent variable (x_c is also called "branch point" or "connection point" in a correlation tree), $f_L(x)$ and $f_R(x)$

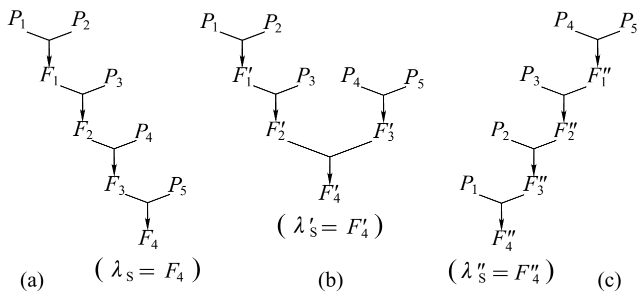

Figure A1. Three typical correlation trees leading to a rational fraction of five power laws $P_i (i = 1, 2, 3, 4, 5)$. The construction of correlation starts from the top to the bottom.

can be power laws, rational fractions of power laws or other types of continuous functions. The logistic function $f(x)$ in Equation (A.1) describes a smooth connection of $f_L(x)$ and $f_R(x)$. The shape of $f(x)$ is related to the three constants m, n and x_c.

There are three key steps for assembling logistic dose functions; these are (1) the selection of two appropriate assembly members $f_L(x)$ and $f_R(x)$, the identification of the transition region and the tangent extension of the assembly members, (2) the estimate of the threshold value x_c which identifies the point of intersection of $f_L(x)$ and $f_R(x)$, and (3) the five-point sharpness control for fitting the transition between the two assembly members. A main idea in the logistic dose-curve fitting is to force the denominator function $G(x)$ in Equation (2) to move towards $+\infty$ or 1 rapidly on different sides of the threshold value x_c once the independent variable x deviates x_c, so that the logistic dose function can approach to $f_L(x)$ on the left side and $f_R(x)$ on the right side of x_c.

In the present application, a logistic dose curve is always a rational fraction of power laws. If the number of power laws is M, then the number of rational fractions is $M - 1$. The logistic dose-fitting curve of two power laws gives rise to a rational function of power laws. The logistic-fitting curve of a power law and a rational fraction of power laws leads to a rational fraction of a rational fraction of power laws and so on. To simplify the writing, all orders of rational fractions are called rational fractions. In this appendix, we use five power laws and four rational fractions for smooth pipes and each of the six rough pipes used in Nikuradse's [10, 11] data. These elements are assembled sequentially as is shown in Figure A1. The construction of a correlation tree is within a hierarchy system and starts from branches to the trunk of the tree. In this system, element power laws may enter at different levels for the assembly. The construction of the tree could be unidirectional, from left to right or from right to left, or more complicated and not unidirectional. Three typical tree structures are shown in Figure A1.

Branches of the correlation tree

Chords or tangents can be used to approximate any curve as in the construction of a circle as a limit of interior or exterior polygons. The chords and tangents are straight lines in log-log coordinates and power laws in regular coordinates. The application of these spline-like approximations is especially powerful for the representation of physical phenomena where log laws are so ubiquitous. Straight lines approximate the response curves in log-log coordinates piecewise, and each straight line represents a power law. The points of intersections of these straight lines are the locations where the branches of the tree are

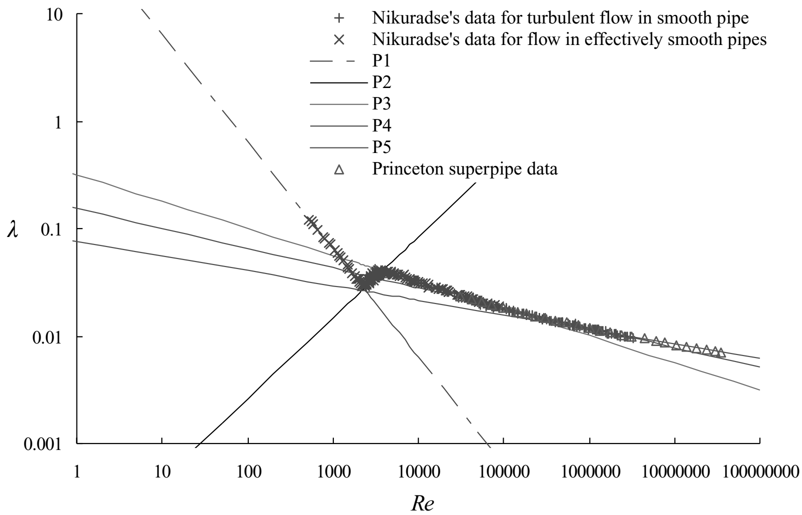

Figure A2. Nikuradse's [10] and Princeton data for flow in smooth and effectively smooth pipes. Five branches of power laws are identified in the graph. They are P_i ($i = 1, 2, 3, 4, 5$) in Equation (A.2).

created. Each point of intersection is a branch point of the tree. The transition of the data from one branch to another lies in the wedge defined at the branch point. Each branch point identifies two adjacent power laws or rational fractions of power laws.

The two positive constants m and n in Equation (A.1) can be tuned to fit transition data near the branch points. When m is large, the transition is sharp. When m is small, the transition is smooth There is certain flexibility in selection of the points where the transition

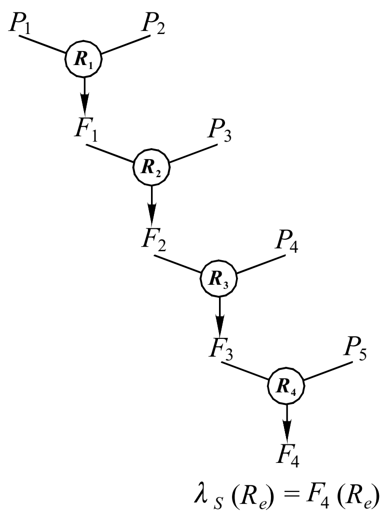

Figure A3. Correlation tree describing Nikuradse's and Princeton data for flow in smooth and effectively smooth pipes. The tree leads to the friction factor correlation $\lambda_S = F_4$. The prefactors a_i, exponents b_i of five power laws, the branch points R_i and the sharpness control parameters s_i and t_i are listed in Table A1.

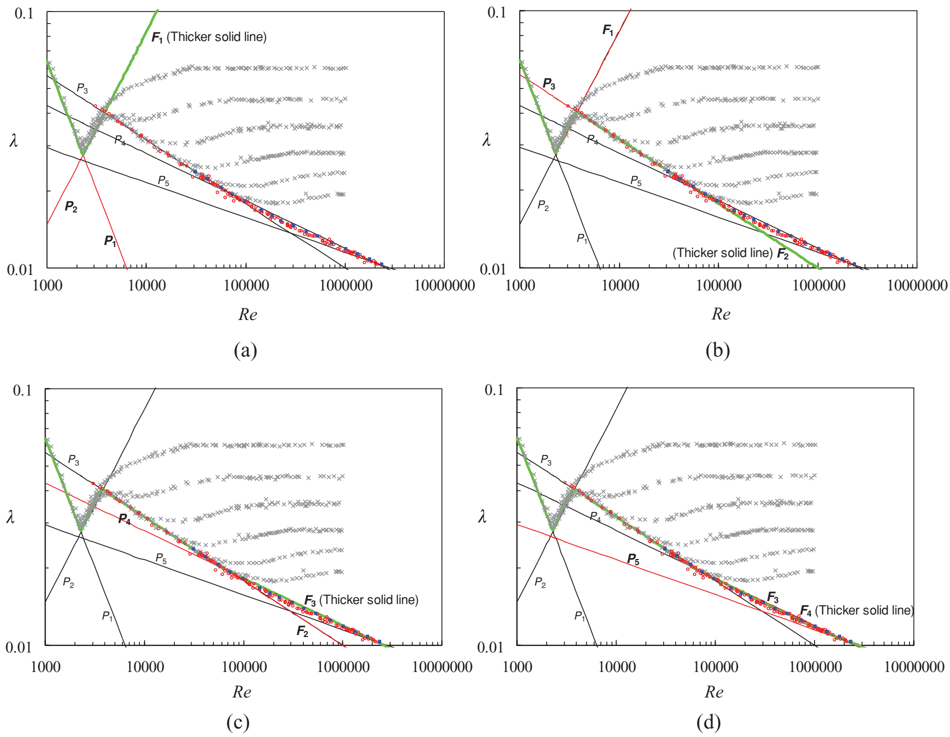

Figure A4. Sequential construction of the correlation tree for smooth pipes. In each of the four panels the straight lines are power laws fit to the data at five places. In panel (a), two power laws P_1 and P_2 are composed into a rational fraction F_1. In panel (b), P_3 is composed with F_1 to get F_2. In panel (c), P_4 is composed with F_2 to get F_3. In panel (d), P_5 is composed with F_3 to get F_4 which gives the final formula giving friction factors versus Reynolds numbers in smooth pipes (i.e. $\lambda_S = F_4$).

from one power law or rational fraction of power laws begins. If you change the position of these points you will change the slope of the tangent there. The parameter m may be used to move these points. This type of tuning is needed when m is relatively small. The coefficient n has only a weak influence on sharpness and it is often kept constant in the construction. The sharpness control parameters m and n are bundled together with the position of the branch point x_c.

Table A1. Coefficients of power laws $P_i = a_i Re^{b_i}$ $(i = 1, 2, 3, 4, 5)$ for fitting λ vs. Re correlations and branch points in the correlation tree for smooth pipes. R_i $(i = 1, 2, 3, 4)$ are the Reynolds numbers at the points of intersection of the power laws at the branch points shown in Figures 6 and A3. s_i and t_i $(i = 1, 2, 3, 4)$ are sharpness control parameters for flow in smooth pipes.

I	1	2	3	4	5
a_i	64	0.000083	0.3164	0.1537	0.0753
b_i	−1	0.75	−0.25	−0.185	−0.136
s_i	−50	−15	−5	−2	—
t_i	0.5	0.5	0.5	0.5	—
R_i	2320	3810	70,000	2,000,000	—

Rules for constructing the correlation tree

(1) Two adjacent power laws can be assembled into a rational fraction of power laws. (2) A rational fraction of power laws assembled with an adjacent power law gives rise to a new rational fraction of power laws, and the number of power laws is increased by one. (3) The direction of the assembly of adjacent members, from left to right, from right to left, or from side to middle, is not important. The direction of assembling members does give rise to different trees as shown in Figure A1 but the final fitting curves are almost the same, even though the final expression of fitting curve may look very different.

The construction of the trees shown in Figure A1 starts from branches to the trunk of the tree: (1) left to right, (2) side to middle and (3) right to left. λ_S, λ'_S and λ''_S are correlation formulas for the friction factor in laminar, transition and turbulent flow in smooth pipes. P_i $(i = 1, 2, \ldots, 5)$ are power laws. F_i, $F_i^{'}$ and $F_i^{''}$ $(i = 1, 2, \ldots, 4)$ are rational fractions of power laws. The sharpness control parameters m and n are the same on each point of intersection in all the three tree structures.

The logistic dose function $f(x)$ cannot pass through any points exactly on the two assembly member functions $f_L(x)$ and $f_R(x)$ except the point of intersection. In most cases of smooth transitions, modifications of assembly members may be necessary so that the logistic dose function of the modified assembly member functions can best fit the data points on the transition segment. When the assembly member functions are power laws, the prefactors and exponents can be easily modified. The point of intersection of the two power laws must be located on the trend of the smooth transition region, so that the logistic dose curve can automatically pass through that point. The details of modifications depend upon the distribution of data points in the whole domain. An example of constructing a logistic dose curve for two power laws is illustrated in Joseph and Yang [8].

(2) Correlation of data for friction factors vs. Reynolds number in smooth and rough pipes

Smooth pipes

The Princeton data presented by McKeon et al. [8] includes a wide range of Reynolds numbers from 3.131×10^4 to 3.554×10^7 and agrees well with Nikuradse's [10, 11] data

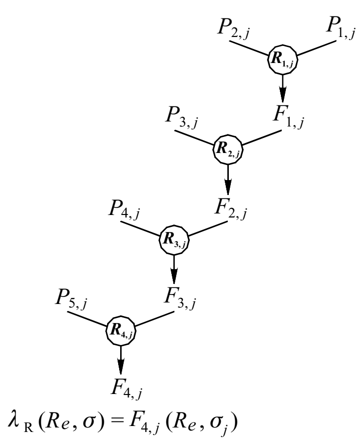

Figure A5. The correlation trees for λ vs. Re in each of the six rough pipes $[j = 1, 2, 3, 4, 5, 6]$ in Nikuradse's experiments. There are six final correlations $F_{4,j}$, 24 interim rational fractions and 30 power laws for fitting the data and listed in Table A2.

Table A2. Coefficients of power laws $P_{i,j} = a_{i,j} Re^{b_{i,j}}$ ($i = 1, 2, 3, 4, 5$; $j = 1, 2, 3, 4, 5, 6$) for fitting Nikuradse's data and branch points in the correlation tree for rough pipes. $R_{i,j}$ are the Reynolds numbers at the point of intersection of the power laws at the branch points in Figures 6 and A5; there are 5 branch points for each of 6 roughness values, 30 in all. m_i and n_i are sharpness-control parameters for flow in rough pipes. $\tilde{a}_{i,j}$, $\tilde{b}_{i,j}$ and $\tilde{R}_{i,j}$ are corrections of the prefactors $a_{i,j}$, the exponents $b_{i,j}$ and the branch points $R_{i,j}$, respectively (also see Figures A7, A8 and A9).

			j	1	2	3	4	5	6
i	m_i	n_i	σ_j	15	30.6	60	126	252	507
1	2	0.5	$a_{i,j}$	0.05996	0.04579	0.03595	0.02831	0.02324	0.01945
			$\tilde{a}_{i,j} = a_{i,j} - 0.0098$	0.05016	0.03599	0.02615	0.01851	0.01344	0.00965
			$b_{i,j}$	0	0	0	0	0	0
			$R_{i,j}$	1,010,000	1,400,000	1,900,000	2,660,000	3,650,000	5,000,000
2	5	0.5	$a_{i,j}$	0.0586	0.0441	0.0345	0.0271	0.0223	0.0189
			$\tilde{a}_{i,j} = a_{i,j} - 0.011$	0.0476	0.0331	0.0235	0.0161	0.0113	0.0079
			$b_{i,j}$	0.002	0.002	0.002	0.002	0.002	0.002
			$R_{i,j}$	23,900	49,800	100,100	214,500	441,000	910,000
3	5	0.5	$a_{i,j}$	0.01474	0.01288	0.01145	0.01021	0.00927	0.00850
			$\tilde{a}_{i,j} = a_{i,j} - 0.0053$	0.00944	0.00758	0.00615	0.00491	0.00397	0.00320
			$b_{i,j}$	0.1379	0.115	0.0972	0.0815	0.0694	0.0595
			$\tilde{b}_{i,j} = b_{i,j} - 0.015$	0.1229	0.1	0.0822	0.0665	0.0544	0.0445
			$R_{i,j}$	6000	12,300	23,900	50,100	99,900	200,000
4	5	0.5	$a_{i,j}$	0.02076	0.02448	0.02869	0.03410	0.04000	0.04710
			$b_{i,j}$	0.1035	0.0503	0.0093	0.0291	0.0573	0.0811
			$\tilde{b}_{i,j} = b_{i,j} + 0.191$	0.2945	0.2413	0.2003	0.1619	0.1337	0.1099
			$R_{i,j}$	6000	10280	17100	29900	50000	85070
5	5	0.5	$a_{i,j}$	0.00253	0.0225	0.0561	0.1031	0.1307	0.1593
			$b_{i,j}$	0.3403	0.0655	−0.0615	−0.1339	−0.1676	−0.1851
			$\tilde{b}_{i,j} = b_{i,j} + 0.2032$	0.5435	0.2687	0.1417	0.0693	0.0356	0.0181
			$R_{i,j}$	3180	5000	9000	20,000	44,000	102,000
			$\tilde{R}_{i,j} = R_{i,j} - 1891$	1289	3109	7109	18,109	42,109	100,109

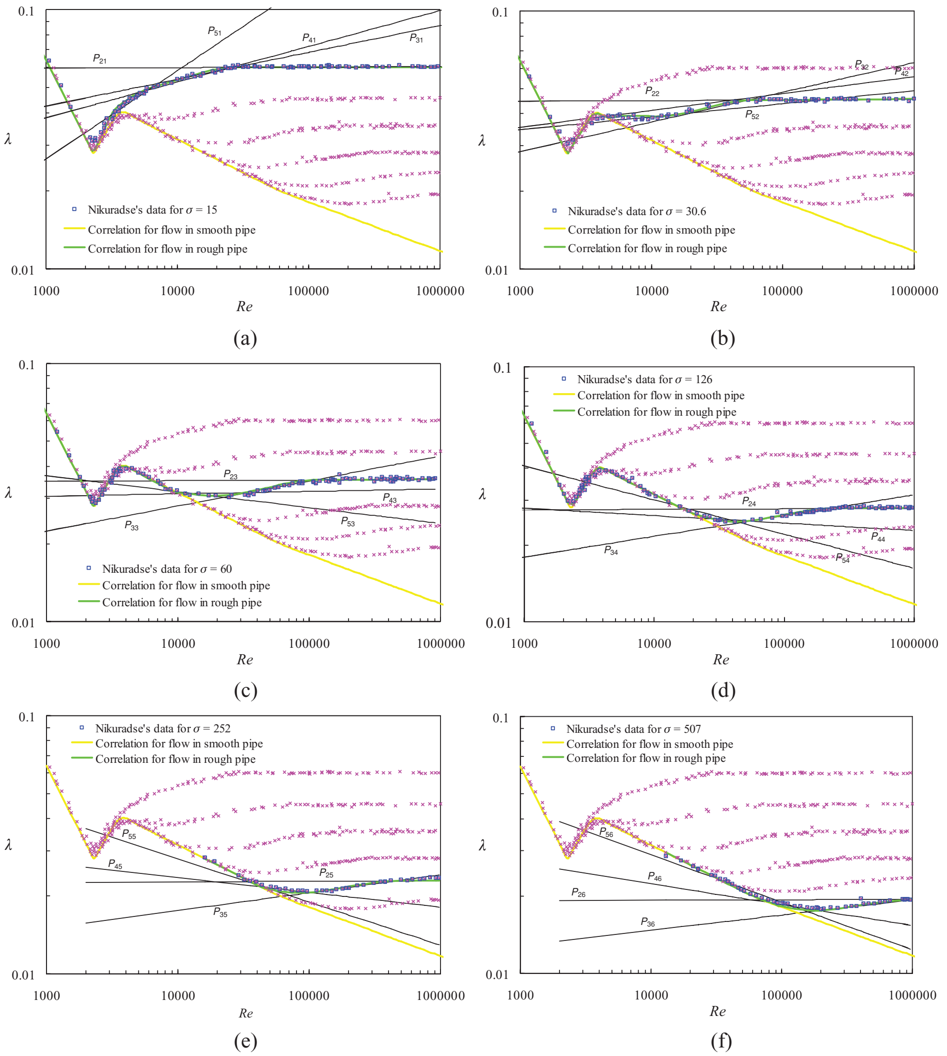

Figure A6. The construction of λ vs. Re correlations in rough pipes using Nikuradse's data for six values of the roughness ratio σ. The dotted line is the correlation for smooth pipe, and the heavier solid lines are the correlations for rough pipes.

for smooth and effectively smooth pipes. Since the largest Reynolds number in Nikuradse's data is only 3.23×10^6, the Princeton data may be considered as an excellent extension of Nikuradse's [10] data for flow in smooth pipes.

The smooth pipe data is enormously important for the description of turbulent flow in rough pipes. The idea pursued here is that the smooth pipe data is an envelope for the initiation of effects of roughness. The effects of roughness for the friction factor in a pipe of fixed roughness is not felt for Reynolds numbers smaller than those in a smooth pipe and they begin to be felt at a critical Reynolds number at a point on the friction factor curve for smooth pipes.

Five element power laws P_i were chosen for fitting the λ vs. Re correlations of Nikuradse's and Princeton data for smooth and effectively smooth pipes. We use one power law

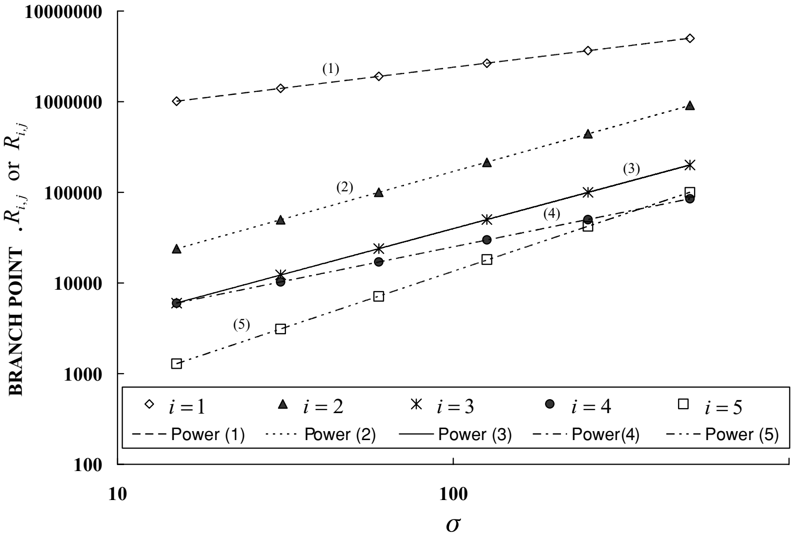

Figure A7. Power law functions in the roughness ratio σ for Reynolds number for each of 5 branch points (see Table A2). The five power law functions are (1) $R_{1,j} = 295530.05\sigma^{0.45435343}$; (2) $R_{2,j} = 1451.4594\sigma^{1.0337774}$; (3) $R_{3,j} = 406.33954\sigma^{0.99543306}$; (4) $R_{4,j} = 783.39696\sigma^{0.75245644}$; (5) $\tilde{R}_{5,j} = 45.196502\sigma^{1.2369807}$.

for fitting the data in laminar regime and another for transition regime. To best represent the data in turbulent regime, in which roughnesses start to be effective, we choose three power laws for Reynolds number ranging from 3.81×10^3 to 3.55×10^7. The five power laws, which were chosen to construct the λ vs. Re correlation for flow in smooth pipes, are

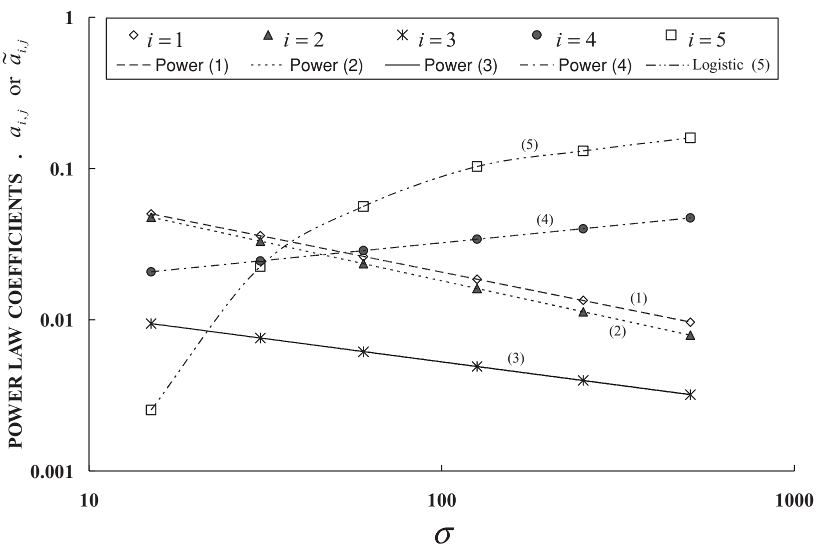

Figure A8. Power law functions or rational fraction of power laws in the roughness ratio σ for the power law coefficients $a_{i,j}$ (see Table A2). The five power law functions are: (1) $\tilde{a}_{1,j} = 0.17805185\sigma^{-0.46785053}$; (2) $\tilde{a}_{2,j} = 0.18954211\sigma^{-0.51003100}$; (3) $\tilde{a}_{3,j} = 0.02166401\sigma^{-0.30702955}$; (4) $a_{4,j} = 0.01105244\sigma^{0.23275646}$; (5) Logistic dose function.

$$
\begin{aligned}
&P_1: \quad \lambda = 64/Re,\\
&P_2: \quad \lambda = 8.3 \times 10^{-5} Re^{0.75},\\
&P_3: \quad \lambda = 0.3164 Re^{-0.25},\\
&P_4: \quad \lambda = 0.1537 Re^{-0.185},\\
&P_5: \quad \lambda = 0.0753 Re^{-0.136},
\end{aligned}
\tag{A.2}
$$

respectively (see Figure A2). The chain for the sequential construction of λ vs. Re correlation for smooth pipes is shown in Figure A3. The curve which emerges after processing power laws with the logistic dose function algorithm is shown in panel (d) of Figure A4.

We have compared the three rational fractions F_4, F_4' and F_4'' shown in Figure 1, which are corresponding to the power laws $\lambda = 64/Re$, $\lambda = 8.3 \times 10^{-5} Re^{0.75}$, $\lambda = 0.3164 Re^{-0.25}$, $\lambda = 0.1537 Re^{-0.185}$ and $\lambda = 0.0753 Re^{-0.136}$. The results indicate that the correlation tree for flow in smooth pipes exhibited in Figure A3 is largely independent of the way that the branches of the tree are assembled. The power laws coefficients, branch points and sharpness-control parameters for the smooth pipe correlation are shown in Table A1.

Rough pipes

Nikuradse [11] is responsible for the most comprehensive studies of turbulent flow in pipes of well-defined roughness, prepared by cementing sand grains to the inside of the walls. The relative roughness is defined as $r = k/a$, where k is the average depth of roughness and a is the radius of the pipe. The reciprocal of the relative roughness, $\sigma = 1/r$, is often used as the dimensionless parameter to represent roughness. Nikuradse (1933) presented

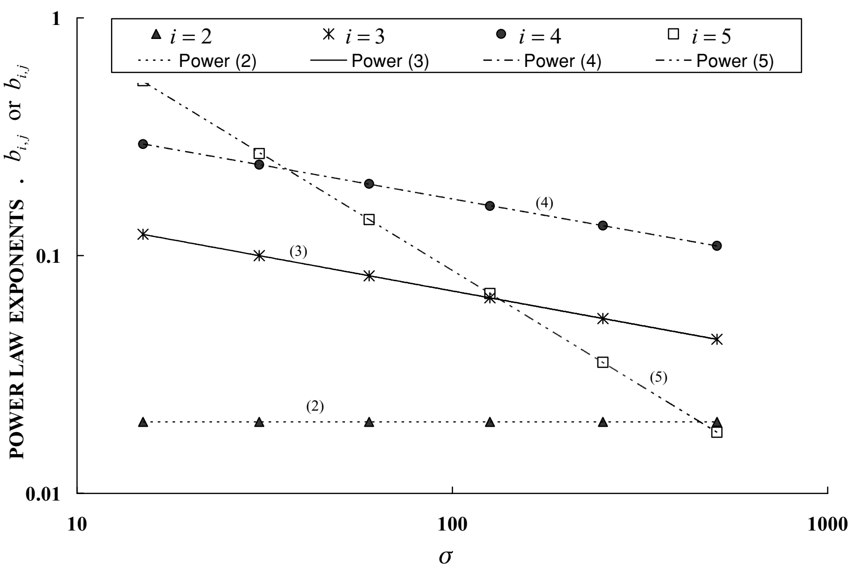

Figure A9. Power law functions in the roughness ratio σ for the power law exponents $b_{i,j}$ (see Table A2). $b_{1,j} = 0$ The other four power law functions are: (2) $b_{2,j} = 0.02$; (3) $\tilde{b}_{3,j} = 0.26827956\sigma^{-0.28852025}$; (4) $\tilde{b}_{4,j} = 0.62935712\sigma^{-0.28022284}$; (5) $\tilde{b}_{5,j} = 7.3482780\sigma^{-0.96433953}$.

his rough pipe data for six values of the roughness

$$\sigma_j[j = 1, 2, \ldots, 6] = [15, 30.6, 60, 126, 252, 507]. \tag{A.4}$$

The structure of the correlation tree for rough pipes is shown in Figure A5.

Figures A7, A8 and A9 show that $a_{i,j}$, $b_{i,j}$, and $R_{i,j}$ are power law functions or rational fractions of power laws of the roughness ratio σ defined in Table A2 and do not depend on

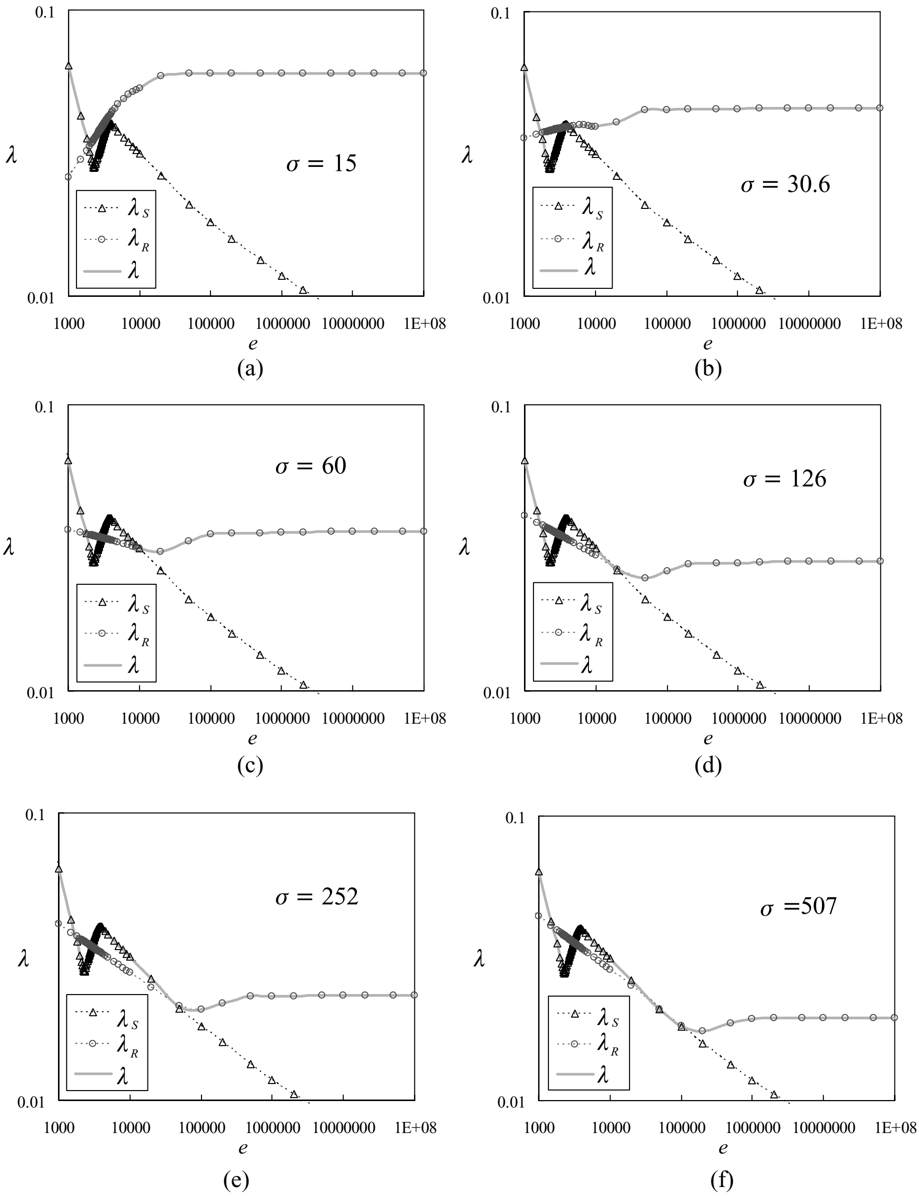

Figure A10. The merge of the final correlations for smooth pipe and rough pipes with six values of roughnesses. λ_S and λ_R are merged to be $\lambda = f(Re, \sigma)$ for "Virtual Nikuradse".

Table A3. Power law functions and rational fractions of power laws for the prefactors, exponents and joining Reynolds numbers.

$a_{1,j} = 0.17805185\sigma^{-0.46785053} + 0.0098,$	$b_{1,j} = 0,$	$R_{1,j} = 295530.05\sigma^{0.45435343};$
$a_{2,j} = 0.18954211\sigma^{-0.51003100} + 0.011,$	$b_{2,j} = 0.002,$	$R_{2,j} = 1451.4594\sigma^{1.0337774};$
$a_{3,j} = 0.02166401\sigma^{-0.30702955} + 0.0053,$	$b_{3,j} = 0.26827956\sigma^{-0.28852025} + 0.015,$	$R_{3,j} = 406.33954\sigma^{0.99543306};$
$a_{4,j} = 0.01105244\sigma^{0.23275646},$	$b_{4,j} = 0.62935712\sigma^{-0.28022284} - 0.191,$	$R_{4,j} = 783.39696\sigma^{0.75245644};$
$a_{5,j} = \left(0.00255391\sigma^{0.8353877} - 0.022\right) + \left[\left(0.92820419\sigma^{0.03569244} - 1\right) - \left(0.00255391\sigma^{0.8353877} - 0.022\right)\right] / \left[1 + (\sigma/93)^{-50}\right]^{0.5},$		
	$b_{5,j} = 7.3482780\sigma^{-0.96433953} - 0.2032,$	$R_{5,j} = 45.196502\sigma^{1.2369807} + 1891.$

j. The power law formulas obtained here by processing the data for straight lines in log-log coordinates converts the six data points in Nikuradse's [11] data into continuous functions of σ. These functions reduce to the original data at six discrete points. We may imagine that the range of these functions extend well beyond the range of the six data points. These correlations allow us to introduce the explicit dependence of the final correlation on the roughness ratio σ. These power law based functions are listed in Table A3.

Joining correlation curves for flows in smooth and rough pipes

Using the correlations derived in above sections, we can merge the two final correlations for smooth and rough pipes and join them together at $Re = R_{5,j}$ to get the final formula $\lambda = f(Re, \sigma)$ for λ vs. Re correlations in all fluid flow regimes. The correlation tree for this merge is shown in Figure 6. The merge procedures for the rough pipes with six values of roughnesses are plotted in Figure A10.

The correlation formula for rough pipes is

$$\lambda = f(Re, \sigma) = \lambda_S + \frac{\lambda_R - \lambda_S}{\left[1 + \left(\frac{Re}{R_\sigma(\sigma)}\right)^{m_5}\right]^{n_5}} = F_4 + \frac{F_{4,j} - F_4}{\left[1 + \left(\frac{Re}{R_{5,j}}\right)^{m_5}\right]^{n_5}}, (j = 1, 2, \ldots, 6), \tag{A.5}$$

This formula is generated in the following sequence

$$F_i = F_{i-1} + \frac{P_{i+1} - F_{i-1}}{\left[1 + \left(\frac{Re}{R_i}\right)^{s_i}\right]^{t_i}} \;\; (i = 1, 2, \ldots, 4), \;\; F_0 = P_1,$$

$$F_{i,j} = P_{i+1,j} + \frac{F_{i-1,j} - P_{i+1,j}}{\left[1 + \left(\frac{Re}{R_{i,j}}\right)^{m_i}\right]^{n_i}} \;\; (i = 1, 2, \ldots, 4; j = 1, 2, \ldots, 6), \;\; F_{0,j} = P_{1,j}.$$

where R_i, s_i, t_i, m_i, n_i ($i = 1, 2, 3, 4$), a_i, b_i ($i = 1, 2, 3, 4, 5$) are all constants and $a_{i,j}$, $b_{i,j}$ and $R_{i,j}$ are all power law functions or rational fraction of power laws of the roughness ratio σ (see Table A3).

The final composite correlation (A.5) is shown by the heavier solid lines in Figure 2. This formula gives the friction factor as a function of the Reynolds number and roughness ratio for Nikuradse's data for smooth and rough pipes and the Princeton data for smooth pipes.