I. INTRODUCTION

Different analytical expressions for the lift force on a particle in a shear flow can be found in the literature. Auton gave a formula for the lift on a particle in an inviscid fluid in which uniform motion is perturbed by a weak shear. Bretherton found an expression for the lift per unit length on a cylinder in an unbounded linear shear flow at small values of Reynolds number. Saffman gave an expression for the lift on a sphere in an unbounded linear shear flow at small values of the particle’s equilibrium. We observe stability regions with the same pattern in flows of shear thinning fluids and study the effects of shear thinning properties on the distribution of the stability regions. Joseph and Ocando [J. Fluid Mech. 454, 263 (2002)] analyzed the role of the slip velocity $U_s = U_f - U_p$ and the angular slip velocity $\Omega_s = \Omega_p - \Omega_f$ on migration and lift in plane Poiseuille flow of Newtonian fluids. They concluded that the discrepancy $\Omega_s - \Omega_{se}$, where $\Omega_{se}$ is the angular slip velocity at equilibrium, changes sign across the equilibrium position. In this paper we verify that this conclusion holds in shear thinning fluids. Correlations for lift forces may be constructed by analogy with the classical lift formula $L = C \frac{V^2}{2}$ of aerodynamics and the proper analogs of $U$ and $V$ in the present context are $U_s$ and $\Omega_s - \Omega_{se}$. Using dimensionless parameters, the correlation is a power law near the wall and a linear relation (which can be taken as a power law with the power of one) near the centerline. The correlations are compared to analytical expressions for lift forces in the literature and we believe that the correlations capture the essence of the mechanism of the lift force. Our correlations for lift forces can be made completely explicit provided that the correlations relating $U_s$ and $\Omega_s$ to prescribed parameters are obtained.

Lift forces on a cylindrical particle in plane Poiseuille flow of shear thinning fluids

J. Wang and D. D. Joseph
Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, Minnesota 55455

(Received 25 November 2002; accepted 13 May 2003; published 30 June 2003)

Lift forces on a cylindrical particle in plane Poiseuille flow of shear thinning fluids are investigated by direct numerical simulation. Previous works on this topic for Newtonian fluids show that the two-dimensional channel can be divided into alternating regions defined by the stability of the particle’s equilibrium. We observe stability regions with the same pattern in flows of shear thinning fluids and study the effects of shear thinning properties on the distribution of the stability regions. Joseph and Ocando [J. Fluid Mech. 454, 263 (2002)] analyzed the role of the slip velocity $U_s = U_f - U_p$ and the angular slip velocity $\Omega_s = \Omega_p - \Omega_f$ on migration and lift in plane Poiseuille flow of Newtonian fluids. They concluded that the discrepancy $\Omega_s - \Omega_{se}$, where $\Omega_{se}$ is the angular slip velocity at equilibrium, changes sign across the equilibrium position. In this paper we verify that this conclusion holds in shear thinning fluids. Correlations for lift forces may be constructed by analogy with the classical lift formula $L = C \frac{V^2}{2}$ of aerodynamics and the proper analogs of $U$ and $V$ in the present context are $U_s$ and $\Omega_s - \Omega_{se}$. Using dimensionless parameters, the correlation is a power law near the wall and a linear relation (which can be taken as a power law with the power of one) near the centerline. The correlations are compared to analytical expressions for lift forces in the literature and we believe that the correlations capture the essence of the mechanism of the lift force. Our correlations for lift forces can be made completely explicit provided that the correlations relating $U_s$ and $\Omega_s$ to prescribed parameters are obtained. © 2003 American Institute of Physics.
ies of solid-liquid flows. A famous example is the Richardson–Zaki correlation, which is obtained by processing the data of fluidization experiments. The Richardson–Zaki correlation describes the complicated dynamics of fluidization by drag and is widely used for modeling the drag force on particles in solid-liquid mixtures. Correlations can also be drawn from numerical data; for example, power law correlations for single particle lift and for the bed expansion of many particles in slurries were obtained by processing simulation data (Patankar et al.,; Choi and Joseph; Patankar, Ko, Choi, and Joseph). The prediction of power laws from numerical data suggests that the same type correlations could be obtained from experimental data as was done by Patankar, Joseph, Wang, Barree, Conway, and Asadi and Wang, Joseph, Patankar, Conway, and Barree. The existence of such power laws is an expression of self-similarity, which has not yet been predicted from analysis or physics. The flow of dispersed matter appears to obey those self-similar rules to a large degree (Barenblatt).

Most of studies on migration and lift are for Newtonian fluids. However, in many of the applications the fluids used are not Newtonian and shear thinning is one of the most important non-Newtonian properties. Papers treating migration of particles in shear flows of shear thinning fluids were done by Huang, Feng, Hu, and Joseph, Huang, Hu, and Joseph and especially by Huang and Joseph. The numerical methods used by these authors are used in this work and will not be described here. Suffice to say that the method is based on unstructured body-fitted moving grids (ALE method). All these authors use the Carreau–Bird viscosity function (1) but only Huang and Joseph study the case when there is shear thinning but no normal stresses.

In the present paper, we extend previous studies of lift on a single cylindrical particle in plane Poiseuille flows of Newtonian fluids to shear thinning fluids. We show that the pattern of the stability regions in shear thinning fluids is the same as that in Newtonian fluids. The effects of shear thinning on the distribution of the stability regions are discussed. We verify that the angular slip velocity discrepancy changes sign across the equilibrium position for both neutrally buoyant and heavy particles. We derive power law correlations for the lift force in terms of the slip velocity and angular slip velocity discrepancy and demonstrate that these correlations can be made completely explicit.

II. GOVERNING EQUATIONS AND NUMERICAL METHODS

The 2D computational domain is shown in Fig. 1. \( l \) and \( W \) are the length and width of the channel, respectively, and \( d \) is the diameter of the particle. The simulation is performed with a periodic boundary condition in the \( x \) direction. The geometric parameters are \( W/d = 12 \) and \( l/d = 22 \). The values of these parameters are taken from Patankar et al. where they justified that the channel length \( l \) is sufficiently large so that the solutions are essentially independent of \( l \).

A constant pressure gradient \( -\bar{p} \) gives rise to the Poiseuille flow and the direction of the gravity force is perpendicular to the flow direction. In simulations in periodic do-

\[
P = p + \rho_f g x - \bar{p} e_x x = - \nabla P = - \nabla p - \rho_f g + \bar{p} e_x.
\]

where \( e_x \) is the unit vector in \( x \) direction, \( x \) is the position vector of any point in the domain and \( g \) is the gravitational acceleration. \( p \) is periodic and solved in simulations.

We use the Carreau–Bird model for the shear thinning effects:

\[
\frac{\eta - \eta_c}{\eta_0 - \eta_c} = \left[ 1 + (\lambda_3 \dot{\gamma})^{n-1} \right]^{\lambda_3},
\]

where \( \dot{\gamma} \) is the shear rate defined in terms of the second invariant of the rate of strain tensor \( D \). The shear thinning index \( n \) is in the range of 0–1 and \( \eta_0 \), \( \eta_c \), \( \lambda_3 \) are prescribed parameters.

We consider cylindrical particles of diameter \( d \) with the mass per unit length \( m = \rho_p \pi d^2 l/4 \) and the moment of inertia per unit length \( I = \rho_p \pi d^4 /32 \). A dimensionless description of the governing equations can be constructed by introducing scales: the particle size \( d \) for length, \( V \) for velocity, \( d/V \) for time, \( V/d \) for angular velocity and \( \eta_0 V/d \) for stress and pressure. We choose \( V = \bar{p} W^2/(12 \eta_0) \), which is the average velocity of the undisturbed Poiseuille flow in Newtonian fluids. \( V \) can be related to the shear rate at the wall \( \dot{\gamma}_w = \bar{p} W/(2 \eta_0) \):

\[
V = 2 \frac{\dot{\gamma}_w W}{12} = 2 \dot{\gamma}_w d.
\]

Hat variables are dimensionless in the following part. The dimensionless governing equations are

\[
\hat{\nabla} \cdot \hat{\mathbf{u}} = 0,
\]

\[
R \left( \frac{\partial \hat{\mathbf{u}}}{\partial t} + (\hat{\mathbf{u}} \cdot \hat{\nabla}) \hat{\mathbf{u}} \right) = - \hat{\nabla} \hat{p} + \frac{d}{W} e_x + \hat{\nabla} \left[ \Theta (\hat{\nabla} \hat{\mathbf{u}} + \hat{\nabla} \hat{\mathbf{u}}^T) \right]
\]

for the velocity \( \hat{\mathbf{u}} \) and pressure \( \hat{p} \) of the fluid and

\[
\frac{\rho_p R}{\rho_f} \frac{d \hat{\mathbf{u}}}{d t} = - Ge_x + \frac{d}{W} e_x
\]

\[
+ \frac{4}{\pi} \oint \{ - \hat{p} + \Theta (\hat{\nabla} \hat{\mathbf{u}} + \hat{\nabla} \hat{\mathbf{u}}^T) \} \cdot n d \hat{F},
\]

where \( \Theta \) is the shear rate at the wall, \( \dot{\gamma}_w = \bar{p} W/(2 \eta_0) \), which is the average velocity of the undisturbed Poiseuille flow in Newtonian fluids. \( V \) can be related to the shear rate at the wall \( \dot{\gamma}_w = \bar{p} W/(2 \eta_0) \):

\[
V = 2 \frac{\dot{\gamma}_w W}{12} = 2 \dot{\gamma}_w d.
\]
increases significantly with $X^R_d$ undisturbed Poiseuille flow. In Table I, we list the average and

$$
\frac{\rho_p}{\rho_f} \frac{d \vec{v}_p}{dt} = \frac{32}{\pi} \int \left( \mathbf{\hat{x}} - \mathbf{\hat{X}} \right) \times \left( \mathbf{\hat{F}} \hat{u} + \mathbf{\hat{V}} \hat{u}^\top \right) \cdot \mathbf{n} \, d \hat{r} 
$$

(5)

for the velocity $\mathbf{\hat{u}}_p$ and angular velocity $\mathbf{\hat{w}}_p$ of the particle whose center of mass has the coordinate $\mathbf{\hat{X}}$. In Eqs. (3)–(5) we use

$$
R = \frac{\rho_f Vd}{\eta_0},
$$

$$
G = \frac{(\rho_p - \rho_f)gd^2}{\eta_0 V},
$$

and

$$
\Theta = \frac{\eta_s}{\eta_0} + \left( \frac{1}{\eta_s} - \frac{1}{\eta_0} \right) \left[ 1 + (2\lambda_3 \gamma_w)^2 \gamma^2 \right]^{(n-1)/2}.
$$

The no-slip condition is imposed on the particle boundaries

$$
\mathbf{\hat{u}} = \mathbf{\hat{u}}_p + \mathbf{\hat{w}}_p \times (\mathbf{\hat{x}} - \mathbf{\hat{X}}).
$$

(6)

Following is a list of the dimensionless parameters:

- $\rho_p/\rho_f$, density ratio;
- $W/d$, aspect ratio;
- $\eta_s/\eta_0$, viscosity ratio;
- $\Lambda^2 = (2\lambda_3 \gamma_w)^2$, shear rate parameter;
- $n$, shear thinning index;
- $R = \frac{\rho_f Vd}{\eta_0} = \frac{2\rho_f \gamma_w d^2}{\eta_0}$, Reynolds number;
- $G = \frac{(\rho_p - \rho_f)gd^2}{\eta_0 V} = \frac{d}{W} \frac{(\rho_p - \rho_f)g}{\rho}$, gravity parameter.

Instead of $G$, we use the gravity Reynolds number

$$
R_G = R \cdot G = \frac{\rho_f (\rho_p - \rho_f)gd^3}{\eta_0^2}.
$$

$W/d = 12$ and $\eta_s/\eta_0 = 0.1$ are constant throughout our simulations; the parameters for fluid properties and $d$ are also constant and lead to $\Lambda = R$ in our simulations, so $\Lambda$ does not provide more information. Thus $\rho_p/\rho_f$, $R$, $n$, and $R_G$ are the four dimensionless parameters at play. The Reynolds number $R$ and shear thinning index $n$ together, characterize an undisturbed Poiseuille flow. We define an average Reynolds number $\bar{R} = \rho_p u_0 d / \eta_0$ where $u_0$ is the average velocity of the undisturbed Poiseuille flow. In Table I, we list the average Reynolds numbers $\bar{R}$ for flows characterized by $(n, R)$ pairs. $\bar{R}$ increases significantly with $n$ decreasing at a fixed $R$.

### III. UNDISTURBED FLOW

We refer to Poiseuille flow without particles as undisturbed flow. The dimensionless momentum equation in the $x$ direction for the undisturbed flow is

$$
\frac{\rho_p}{\rho_f} \frac{d \vec{v}_p}{dt} = \frac{32}{\pi} \int \left( \mathbf{\hat{x}} - \mathbf{\hat{X}} \right) \times \left( \mathbf{\hat{F}} \hat{u} + \mathbf{\hat{V}} \hat{u}^\top \right) \cdot \mathbf{n} \, d \hat{r} 
$$

An analytical solution for the Poiseuille flow of a Carreau–Bird fluid is not known. However, a numerical solution can be achieved by an iterative method. First $\bar{\gamma}(\gamma)$ is assumed to be the shear rate of the Poiseuille flow of a Newtonian fluid and $\Theta(\bar{\gamma}(\gamma))$ is obtained. A new shear rate profile $\bar{\gamma}(\gamma)$ is then computed and the steps are repeated until $\bar{\gamma}(\gamma)$ converges. The velocity $\hat{u}(\gamma)$ is obtained by integrating the shear rate.

The velocity profiles of the Poiseuille flows of shear thinning fluids are qualitatively similar to the parabolic profiles seen in flows of Newtonian fluids. However, the maximum velocity in the channel increases significantly as $n$ decreases at a fixed $R$. The viscosity profiles have their minimums at the wall (corresponding to the maximum $\bar{\gamma}$), and their maximums at the centerline (corresponding to zero $\bar{\gamma}$).

### IV. STABLE AND UNSTABLE EQUILIBRIUM REGIONS

An equilibrium is achieved for a freely moving and rotating cylindrical particle with a given density in a Poiseuille flow when the particle migrates to a position $y_e$ of steady rectilinear motion in which the acceleration and angular acceleration vanish and the hydrodynamic lift just balances the buoyant weight. Two types of simulations are performed, unconstrained simulation and constrained simulation. In unconstrained simulations, a particle is allowed to move and rotate freely to migrate to its equilibrium position. The initial translational and angular velocities of the particle are prescribed and initial-value problems are solved to obtain the equilibrium state. In constrained simulations, the position of the particle in the $y$ direction $y_p$ is fixed and the particle is allowed to move in the $x$ direction and rotate. The solution of the flow evolves dynamically to a steady state at which the lift force per unit length $L$ on the particle is computed. Such a steady state will be an equilibrium at $y = y_p$ if the density of the particle is selected so that $L$ just balances the buoyant weight per unit length, satisfying

$$
\frac{d}{dy} \left( \frac{\Theta \hat{u}}{d\gamma} \right).
$$

(7)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$R$</th>
<th>$\bar{R}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>20</td>
<td>20.00</td>
</tr>
<tr>
<td>0.9</td>
<td>20</td>
<td>24.28</td>
</tr>
<tr>
<td>0.8</td>
<td>20</td>
<td>30.48</td>
</tr>
<tr>
<td>0.7</td>
<td>20</td>
<td>39.70</td>
</tr>
<tr>
<td>1.0</td>
<td>40</td>
<td>40.00</td>
</tr>
<tr>
<td>0.9</td>
<td>40</td>
<td>51.84</td>
</tr>
<tr>
<td>0.8</td>
<td>40</td>
<td>69.97</td>
</tr>
<tr>
<td>0.7</td>
<td>40</td>
<td>97.89</td>
</tr>
<tr>
<td>1.0</td>
<td>80</td>
<td>80.00</td>
</tr>
<tr>
<td>0.9</td>
<td>80</td>
<td>110.72</td>
</tr>
<tr>
<td>0.8</td>
<td>80</td>
<td>160.06</td>
</tr>
<tr>
<td>0.7</td>
<td>80</td>
<td>237.60</td>
</tr>
</tbody>
</table>
where \( \hat{L} \) is a dimensionless lift force and represents the ratio between the hydrodynamic lift force \( L \) and the buoyant force \( \rho_f g \pi d^2/4 \).

From the steady state values which evolve in constrained simulations, we are able to obtain \( \hat{L} \) on the particle at any position \( y/d \) in the channel. We can divide the curve of \( \hat{L} \) vs \( y/d \) from the wall to the centerline into four branches by three "turning points" (see Fig. 2). The "turning point" is defined as the position where the slope of the \( \hat{L} \) vs \( y/d \) curve is zero. On the first and third branches of steady solutions, the slope of \( \hat{L} \) vs \( y/d \) curve is negative, and the equilibrium points on these branches are stable. On the second and fourth branches of steady solutions, the slope of \( \hat{L} \) vs \( y/d \) curve is positive, and the equilibrium points are unstable. We will indicate the unstable branches by dotted lines in the figures.

From the \( \hat{L} \) vs \( y/d \) curve, the equilibrium position for a particle with a certain \( \rho_p/\rho_f \) can be determined. The lift force required to balance the buoyant weight of a particle can be computed from (8). If we draw a line on which \( \hat{L} \) equals to this required lift force, the points of intersections between this line and the \( \hat{L} \) vs \( y/d \) curve are the equilibrium points for this particle. For heavier-than-fluid particles with intermediate densities, there exist multiple stable equilibrium positions from the wall to the centerline (see Fig. 2 where two stable equilibrium points for a particle with \( \rho_p/\rho_f = 1.01 \) are shown).

![FIG. 2. A plot of \( \hat{L} \) vs \( y/d \) for a flow with \( n=0.8 \) and \( R=20 \) from constrained simulations. The stable and unstable branches and three turning points are illustrated. Unstable branches are indicated by dotted lines. Two stable equilibrium points for a particle with \( \rho_p/\rho_f = 1.01 \) are shown.](image1)

![FIG. 3. Lateral force as a function of lateral position, both in dimensionless form. ---, simple shear flow; -- -- --, 2D Poiseuille flow. (Adapted from Ho and Leal (Ref. 7).)](image2)

![FIG. 4. Near-the-wall part of \( \hat{L} \) vs \( y/d \) curves of the Poiseuille flows with \( R=20 \) and \( n=0.7, 0.8, 0.9, \) and 1.0 (Newtonian fluid). The unstable branches are indicated by dotted lines and their starting and ending points are marked by pairs of short vertical lines. With the shear index \( n \) decreasing, the stable branch near the wall decreases in size and the unstable branch near the wall moves closer to the wall.](image3)

![FIG. 5. Near-the-centerline part of \( \hat{L} \) vs \( y/d \) curves of the Poiseuille flows with \( R=80 \) and \( n=0.7, 0.8, 0.9, \) and 1.0 (Newtonian fluid). The unstable branches are indicated by dotted lines and short vertical lines are used to mark the starting points of these unstable branches. With the shear index \( n \) decreasing, the unstable branch near the centerline decreases in size.](image4)
shown). However, for a neutrally buoyant particle \((\hat{L} = 0)\), only one stable equilibrium point exists from the wall to the centerline.

Ho and Leal\(^7\) studied the equilibrium position of a neutrally buoyant freely moving and rotating sphere between plane bounding walls. They assumed that the walls were so closely spaced that the lift could be obtained by perturbing Stokes flow with inertia. They calculated dimensionless lateral force vs lateral position curves (equivalent to our \(\hat{L}\) vs \(y/d\) curve) for simple shear flow and two-dimensional (2D) Poiseuille flow which are shown in Fig. 3. Comparing the dashed line in Fig. 3 which is for (2D) Poiseuille flow and the \(\hat{L}\) vs \(y/d\) curve in Fig. 2, one can see that both of the two plots imply the centerline is an unstable equilibrium position. However, the dashed line in Fig. 3 indicates that there are two branches from the wall to the centerline: wall-stable-unstable-centerline, whereas four branches exist according to Fig. 2. Ho and Leal only considered neutrally buoyant particle and did not include the gravity term in the governing equation used in their calculation. The frame of their work did not enable them to study the multi-equilibrium positions of heavier-than-fluid particles. The results shown in Figs. 2 and 3 are not strictly comparable; Ho and Leal studied 3D spheres between plane walls at indefinitely small \(R\) whereas our calculation is for 2D cylinders at much higher Reynolds numbers.

The distribution of the equilibrium branches is affected by the shear thinning effects. The \(\hat{L}\) vs \(y/d\) curves are computed for the flows with \(R = 20, 40,\) and \(80\) and \(n = 0.7, 0.8,\) \(0.9,\) and \(1.0\) (Newtonian fluid). Two groups of typical curves are plotted in Figs. 4 and 5.

We find that when the shear thinning effects become stronger, the stable branch near the wall decreases in size; the unstable branch near the wall moves closer to the wall; the stable branch near the centerline increases in size; the unstable branch at the centerline decreases in size. The shrinkage of the unstable branch at the centerline implies that a particle could be lifted to an equilibrium position closer to the centerline if shear thinning effects are stronger. A closer equilibrium position to the centerline could also be achieved when the pressure gradient is higher, as shown first in Patankar et al.\(^9\) and confirmed in our simulations. It seems that a higher pressure gradient and stronger shear thinning both lead to stronger inertia effects and could lift a particle closer to the centerline. In the range of the Reynolds number and shear thinning index we simulated, the unstable branch at the centerline never vanishes. Patankar et al.\(^9\) reported that in 2D Poiseuille flows of an Oldroyd-B fluid at high Deborah numbers, the centerline can be a stable equilibrium position and the Segrè and Silberberg effect does not occur. We did not observe the same phenomenon in shear thinning fluids.

V. ANGULAR SLIP VELOCITY DISCREPANCY AND NET LIFT FORCE

Joseph and Ocando\(^10\) studied slip velocities and particle lift in 2D Poiseuille flows of Newtonian fluids. The slip velocity is \(U_s = U_p - \hat{U}_f\) and the angular slip velocity is \(\Omega_s = \Omega_p - \hat{\Omega}_f\), where \(U_p\) and \(\Omega_p = \dot{\gamma}/2\) are the translational velocity and angular velocity of the undisturbed Poiseuille flow at the position of the particle and \(\dot{\gamma}\) is the local shear rate. The net lift force is

\[
L_n = L - (\rho_p - \rho_f) \pi d^2 g / 4 \Rightarrow \text{L}_n = \hat{L} - \left( \frac{\rho_p}{\rho_f} - 1 \right).
\]

Joseph and Ocando found that the angular slip velocity discrepancy \(\Omega_s - \Omega_{se}\), where \(\Omega_{se}\) is the angular slip velocity at equilibrium, changes sign across the equilibrium position. Furthermore, they showed that across a stable equilibrium position, the net lift force \(L_n\) has the same sign as the discrepancy \(\Omega_s - \Omega_{se}\); whereas across an unstable equilibrium position, the net lift force \(L_n\) has the opposite sign as the discrepancy \(\Omega_s - \Omega_{se}\). In this section, we verify that these conclusions hold in shear thinning fluids.

We fix a particle at positions slightly above \((y_p > y_c)\) and below \((y_p < y_c)\) its equilibrium positions and compute the steady state lift force and angular slip velocity \(\Omega_s\). For a neutrally buoyant particle, both stable and unstable equilibrium positions are investigated; for a heavy particle, both of its two stable equilibrium positions are investigated. Table II shows the results for a neutrally buoyant particle and Table III shows those for a heavy particle.

### Table II

<table>
<thead>
<tr>
<th>(y_p/d)</th>
<th>(\hat{L}(\rho_p \pi d^2))</th>
<th>(\Omega_s/(2 \hat{y}_b))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(4.33 \times 10^{-2})</td>
</tr>
<tr>
<td>(\hat{L}(\rho_p \pi d^2))</td>
<td>(2.5 \times 10^{-4})</td>
<td>(-4.5 \times 10^{-4})</td>
</tr>
<tr>
<td>(\Omega_s/(2 \hat{y}_b))</td>
<td>(-1.4 \times 10^{-5})</td>
<td>(-7.9 \times 10^{-5})</td>
</tr>
</tbody>
</table>

**Downloaded 09 Sep 2009 to 128.101.98.21. Redistribution subject to AIP license or copyright; see http://pof.aip.org/pof/copyright.jsp**
TABLE III. The steady state values of the net lift force $L_n$ and $\Omega_s-\Omega_{se}$ in dimensionless form at fixed positions above ($y_e>y_p$) and below ($y_e<y_p$) the equilibrium positions of a heavy particle ($\rho_p/\rho_f=1.024$) in the flow with $n=0.9$ and $R=40$. Two stable equilibrium positions exist: $y_e/d=0.918$ with $\Omega_s/2y_e=1.16\times 10^{-2}$ and $y_e/d=2.266$ with $\Omega_s/(2y_e)=4.95\times 10^2$. For either one of the equilibrium positions, $\Omega_s-\Omega_{se}>0$ and $L_n>0$ when the particle is fixed below; and $\Omega_s-\Omega_{se}<0$ and $L_n<0$ when the particle is fixed above.

<table>
<thead>
<tr>
<th>$y_e/d$</th>
<th>$\Omega_s/(2y_e)$</th>
<th>$L_n/(\rho_f \pi d^3/4)$</th>
<th>$(\Omega_s-\Omega_{se})/(2y_e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.918</td>
<td>0.9</td>
<td>$7.16\times 10^{-2}$</td>
<td>$-6.4\times 10^{-3}$</td>
</tr>
<tr>
<td>2.26</td>
<td>1.0</td>
<td>$-6.4\times 10^{-3}$</td>
<td>$2.58\times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>2.26</td>
<td>$1.44\times 10^{-4}$</td>
<td>$-3.26\times 10^{-3}$</td>
</tr>
</tbody>
</table>

VI. LIFT CORRELATIONS

Motivated by the conclusion that $\Omega_s-\Omega_{se}$ has the same sign as $L_n$ across a stable equilibrium position, we seek the correlations between $L_n$ and $\Omega_s-\Omega_{se}$. Such correlations may be constructed by analogy with the classical lift formula $L=CU^2$ of aerodynamics. The proper analogs of $U$ and $F$ in the present context are $U_s$ and $\Omega_s-\Omega_{se}$ as first proped in Joseph and Ocando.\textsuperscript{10} We proceed as follows to obtain the correlations. First we compute $L_n$, $U_s$, and $\Omega_s$ as functions of $y$ by constrained simulations in a flow characterized by $(R, n)$. Then we correlate dimensionless parameters based on $L$ and $U_s$ ($\Omega_s-\Omega_{se}$) to power law formulas. These steps are repeated for different flows identified by $(R, n)$ pairs and lead to correlations for each flow. The coefficients in such correlations are functions of $R$ and $n$ which can be obtained by data fitting analyses. Finally, we obtain correlations between dimensionless $L$ and $U_s$ ($\Omega_s-\Omega_{se}$) with coefficients expressed as functions of $R$ and $n$.

Figure 6 shows the relative values of $L$, $U_s$, and $\Omega_s$ obtained from constraint simulations in the flow with $R=20$ and $n=0.9$.

Dimensionless parameters based on local quantities are used to express the correlations. The local dimensionless net force is

$$\lambda(y) = \frac{4\rho_f d[L(y)-\rho_p-\rho_f]g \pi d^3/4}{\pi \eta(y)^2} = \frac{4\rho_f d}{\pi \eta(y)^2} L_n(y).$$

Two local Reynolds numbers are based on $U_s$ and $\Omega_s-\Omega_{se}$, respectively,

$$R_U(y) = \frac{\rho_f U_s(y) d}{\eta(y)},$$

$$R_\Omega(y) = \frac{\rho_f [\Omega_s(y)-\Omega_{se}] d^3}{\eta(y)}.$$

The product of $R_U$ and $R_\Omega$ is defined as $F$,

$$F(y) = R_U \cdot R_\Omega = \frac{\rho_f^2 U_s(y) [\Omega_s(y)-\Omega_{se}] d^4}{\eta(y)^2}.$$

To compute $F(y)$ from (12), it is necessary to specify the equilibrium angular slip velocity $\Omega_s-\Omega_{se} = \Omega_s(y_e)$ where $y_e$ is the position at which the lift equals the buoyant weight. The $L$ vs $y/d$ curve (Fig. 2) shows that each and every value of $y/d$ on the stable branches is a possible equilibrium position ($y=y_e$) for some particle $\rho_p$. The range of possible $y_e$ may be covered by varying the density of the particle. Once $y_e$ is selected, $\Omega_{se}$ is given as $\Omega_s(y_e)$. The dependence of $\Omega_{se}$ and $L_n$ on $\rho_p$ makes the correlations between $\lambda(y)$ and $F(y)$ particle-density dependent. However, the steady state

FIG. 6. The relative values of $L$, $U_s$, and $\Omega_s$ in the flow with $R=20$ and $n=0.9$.

FIG. 7. The power law correlations between $\lambda(y)$ and $F(y)$ on the stable branch near the wall for the flows with $R=20$ and $n=0.7$, 0.8, 0.9, and 1.0 (Newtonian fluid).
values of \( L \) do not depend on particle density. If we derive the correlations between \( \lambda(y) \) and \( F(y) \) for one \( \rho_p \), the lift force is essentially obtained and can be applied to particles with different densities. We present the correlations with the single equilibrium position of a neutrally buoyant particle as the reference. There are two advantages of this choice: the complexity of multiequilibrium positions of a heavy particle is avoided; the correlations are in simple forms which are a power law for the stable branch near the wall and a linear relation for the stable branch near the centerline.

For a neutrally buoyant particle, a single equilibrium position exists at \( y = y^N \) (the superscript is for “neutral”) with \( L(y^N) = 0 \) and \( \Omega_s(y^N) = \Omega^N_r \). Thus the dimensionless parameters have the following form:

\[
\lambda(y) = \frac{4 \rho_p d L(y)}{\pi \eta(y)^2}
\]

and

\[
F(y) = \frac{\rho_f^2 U_s(y) [\Omega_s(y) - \Omega^N_r] d^3}{\eta(y)^2}.
\]

The correlations are in the following forms:

\[
\lambda(R, n, y/d) = a(R, n) F(R, n, y/d)^{m(R, n)}
\]

on the stable branch near the wall;

\[
\lambda(R, n, y/d) = k(R, n) F(R, n, y/d)
\]

on the stable branch near the centerline.

We obtain the correlations for flows with \( n = 0.7, 0.8, 0.9 \), and 1.0 (Newtonian fluid). In Fig. 7, the correlations on the stable branch near the wall are plotted for the flows with \( R = 20 \). The power law correlations along with the correlation coefficients \( \sigma^2 \) are shown in the figure. In Fig. 8, two examples of the linear correlation between \( \lambda(y) \) and \( F(y) \) on the stable branch near the centerline are plotted for the flows with \( R = 20, n = 0.7 \) and \( R = 80, n = 0.8 \). It can be seen that our correlations describe the data faithfully.

The prefactor \( a \), the exponent \( m \) and the slope \( k \) in (13) and (14) are functions of \( R \) and \( n \). In Table IV, the coefficients \( a, k, \) and \( m \) are listed along with \( R, n, \) and the average Reynolds number \( \bar{R} \) which can be viewed roughly as a parameter for the combined effects of \( R \) and \( n \). Coefficients \( a, m, \) and \( k \) are also plotted against \( \bar{R} \) in Figs. 9–11.

Figures 9 and 10 reveal that the power law correlation (13) on the stable branch near the wall has two regimes. Flows of Newtonian fluids and weak shear thinning flows fall into regime 1 where the prefactor \( a \) increases with \( \bar{R} \) increasing and the exponent \( m \) is in the range of 0.4–0.5. Regime 2 has three flows \( (n = 0.7, R = 40), \ (n = 0.7, R = 80), \) and \( (n = 0.8, R = 80) \) and can be identified as a strong shear thinning regime where the prefactor \( a \) decreases with \( \bar{R} \) increasing and the exponent \( m \) is in the range of 0.77–0.9.

![Figure 8](image1.png)

FIG. 8. The linear correlation between \( \lambda(y) \) and \( F(y) \) on the stable branch near the centerline for the flows with \( (R = 20, n = 0.7) \) and \( (R = 80, n = 0.8) \).

![Figure 9](image2.png)

FIG. 9. The prefactor \( a \) vs the average Reynolds number \( \bar{R} \).

\[
\lambda(R, n, y/d) = k(R, n) F(R, n, y/d)
\]

on the stable branch near the centerline.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( R )</th>
<th>( \bar{R} )</th>
<th>( a )</th>
<th>( m )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>17.937</td>
<td>0.4003</td>
<td>53.171</td>
</tr>
<tr>
<td>0.9</td>
<td>20</td>
<td>24.28</td>
<td>21.589</td>
<td>0.4004</td>
<td>34.685</td>
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<td>0.8</td>
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<td>30.48</td>
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<td>0.423</td>
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<tr>
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<td>37.322</td>
<td>0.439</td>
<td>19.458</td>
</tr>
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<td>40</td>
<td>40</td>
<td>27.288</td>
<td>0.410</td>
<td>30.739</td>
</tr>
<tr>
<td>0.9</td>
<td>40</td>
<td>51.84</td>
<td>36.38</td>
<td>0.427</td>
<td>25.591</td>
</tr>
<tr>
<td>0.8</td>
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<td>69.97</td>
<td>40.808</td>
<td>0.481</td>
<td>22.166</td>
</tr>
<tr>
<td>0.7</td>
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<td>97.89</td>
<td>9.664</td>
<td>0.774</td>
<td>11.759</td>
</tr>
<tr>
<td>1</td>
<td>80</td>
<td>80</td>
<td>38.009</td>
<td>0.448</td>
<td>24.35</td>
</tr>
<tr>
<td>0.9</td>
<td>80</td>
<td>110.72</td>
<td>53.729</td>
<td>0.450</td>
<td>21.066</td>
</tr>
<tr>
<td>0.8</td>
<td>80</td>
<td>160.06</td>
<td>9.570</td>
<td>0.797</td>
<td>8.879</td>
</tr>
<tr>
<td>0.7</td>
<td>80</td>
<td>237.6</td>
<td>2.710</td>
<td>0.898</td>
<td>7.698</td>
</tr>
<tr>
<td>1</td>
<td>120</td>
<td>120</td>
<td>43.83</td>
<td>0.472</td>
<td>21.54</td>
</tr>
<tr>
<td>1</td>
<td>160</td>
<td>160</td>
<td>41.48</td>
<td>0.496</td>
<td>16.39</td>
</tr>
</tbody>
</table>

TABLE IV. The prefactor \( a \), the exponent \( m \) and the slope \( k \) as functions of the shear index \( n \) and the Reynolds number \( \bar{R} \).
From the values of the exponent \( m \), we can tell that in regime 2 the dependence of the lift force on the product of slip velocities is stronger than that in regime 1. It is noted that the two flows \((n = 1.0, R = 160)\) and \((n = 0.8, R = 80)\) have very close values of \( \bar{R} \) but substantially different coefficients \( a \), \( m \), and \( k \) (see Table IV); this indicates that particle lift in strong shear thinning flows is different with that in flows of Newtonian fluids at high Reynolds number. Figure 11 exhibits one regime of the linear correlation (14) where the slope \( k \) decreases with \( \bar{R} \) increasing. Figures 9–11 also suggest that power law or linear functions of \( \bar{R} \) could be used to approximate the prefactor \( a \) and the exponent \( m \) in regime 1 and the slope \( k \). However, the error of such approximations would be considerable. The reason of such error is that \( a \), \( k \), and \( m \) depend on both \( n \) and \( R \); one single parameter \( \bar{R} \) cannot fully describe the dependence of the coefficients on the flow.

We cannot fully determine the coefficients \( a \), \( m \), and \( k \) as functions of \( \bar{R} \) and \( n \) because of insufficient data. If we focus on flows of Newtonian fluids \((n = 1)\), \( \bar{R} \) is the only active parameter and we expect to get satisfactory \( a(R) \), \( k(R) \), and \( m(R) \) approximations by data fitting analyses. The coefficients \( a \), \( k \), and \( m \) in flows of Newtonian fluids are listed as functions of \( \bar{R} \) in Table V.

Data fitting analyses yield
\[
\begin{align*}
  a &= 5.34\bar{R}^{0.428}, \quad \sigma^2 = 0.94, \quad \text{(15)} \\
  m &= 0.0007\bar{R} + 0.386, \quad \sigma^2 = 0.99, \quad \text{(16)} \\
  k &= 232.5\bar{R}^{-0.515}, \quad \sigma^2 = 0.96. \quad \text{(17)}
\end{align*}
\]
Inserting (15)–(17) into the correlations (13) and (14), we obtain correlations which apply to flows of Newtonian fluids with a Reynolds number in the range of 20–160:

<table>
<thead>
<tr>
<th>( R )</th>
<th>( a )</th>
<th>( m )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>17.94</td>
<td>0.400</td>
<td>53.17</td>
</tr>
<tr>
<td>40</td>
<td>27.29</td>
<td>0.410</td>
<td>30.74</td>
</tr>
<tr>
<td>80</td>
<td>38.01</td>
<td>0.448</td>
<td>24.35</td>
</tr>
<tr>
<td>120</td>
<td>43.83</td>
<td>0.472</td>
<td>21.54</td>
</tr>
<tr>
<td>160</td>
<td>41.48</td>
<td>0.496</td>
<td>16.39</td>
</tr>
</tbody>
</table>

Although correlations (20) and (21) are derived using the equilibrium of a neutrally buoyant particle as the reference, they can be applied to heavy particles. To demonstrate this, we first obtain \( U_s \) and \( \Omega_s \) for heavy particles at their equilibrium states from unconstrained simulations; these values are then inserted into (20) and (21) to calculate the lift forces which should match the values of the buoyant weight of the heavy particles. Two examples are shown in Table VI: a particle with \( \rho_p/\rho_f = 1.016 \) in a flow with \( \bar{R} = 40 \) and a particle with \( \rho_p/\rho_f = 1.045 \) in a flow with \( \bar{R} = 80 \). In both cases two stable equilibrium positions exist. The lift force for \( y_c \) close to the wall is computed using (20) and the lift force for \( y_c \) close to the centerline is computed using (21). It can be seen that the computed dimensionless lift forces are close to the values of the dimensionless buoyant weight \( (\rho_p/\rho_f - 1) \) of the particles. In this way we demonstrate that the correlations derived for neutrally buoyant particles can be applied to heavy particles.

Correlations (20) and (21) apply to 2D motion of a particle in a Poiseuille flow. They may be compared to well-known lift expressions for a particle in a linear shear flow with shear rate \( \dot{\gamma} \). The comparisons are at best tentative because the linear shear neglects the effects of the shear gradient which is a constant in the Poiseuille flow and not small; also because the lift expressions in linear shear flows are for indefinitely small Reynolds number perturbing Stokes flow.
TABLE VI. Computation of the lift forces on heavy particles using the correlations (20) and (21). The computed dimensionless lift forces are close to the values of the dimensionless buoyant weight ($\rho_p/\rho_f - 1$) of the particles.

<table>
<thead>
<tr>
<th>$R$</th>
<th>$\Omega_s/(2 \gamma_s)$</th>
<th>$\rho_p/\rho_f - 1$</th>
<th>$y_s/d$</th>
<th>$\Omega_s/(2 \gamma_s)$</th>
<th>$U_s/(2 \gamma_s d)$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>$5.24 \times 10^{-3}$</td>
<td>0.016</td>
<td>1.093</td>
<td>3.94 $\times 10^{-2}$</td>
<td>7.17 $\times 10^{-3}$</td>
<td>0.018</td>
</tr>
<tr>
<td>80</td>
<td>$5.32 \times 10^{-3}$</td>
<td>0.045</td>
<td>2.377</td>
<td>2.96 $\times 10^{-2}$</td>
<td>1.35 $\times 10^{-2}$</td>
<td>0.014</td>
</tr>
</tbody>
</table>

We find that our expression for the lift on the stable branch near the centerline (21) is similar to the leading term in Saffman’s expression (23). If we make following changes in Eq. (21): $R = \rho_f V d/\eta_0 \rightarrow R = \rho_f \gamma d^2/\eta$, the power of $R (-0.515) \rightarrow (-0.5)$, and use $d = 2a$, Eq. (21) becomes

$$L = 365.2 \rho_f^{0.5} \eta^{-0.5} U_s \gamma^{-0.5} (\Omega_s - \Omega_s^{\infty}) a.$$  \hspace{1cm} (24)

Comparing (24) and the leading term in (23), we note that both expressions are linear in $U_s$; both have a similar dependence on $\rho_f$, $\eta$, and $a$ after noting that (24) is for the lift force per unit length. However, the dependence on $\gamma$ and $\Omega_s - \Omega_s^{\infty}$ is greatly different.

Another formula for the lift on a particle in an inviscid fluid in which uniform motion is perturbed by a weak shear was derived by Auton \cite{Auton} and a more recent satisfying derivation of the same result was given by Drew and Passman. \cite{Drew}

In a plane flow they find

$$L = \frac{\pi}{4} \alpha_3^2 \rho U_s \Omega_f = -\frac{\pi}{4} \alpha_3^2 \rho U_s \gamma,$$  \hspace{1cm} (25)

which is similar to our correlation (21) but differs from (21) in several ways: (25) has a constant prefactor for inviscid fluids whereas viscous effects enter into (21) through $R$; the lift force depends on $\Omega_f$—“spin” of the fluid in (25) but on the angular velocity discrepancy $\Omega_f - \Omega_f^{\infty}$ in (21); (25) is for 3D spheres and (21) is for 2D cylinders.

We compare the lift forces computed from the direct numerical simulation and from the lift expressions (21), (22), (23), and (25) in Fig. 12. Our correlation (21) and Bretherton’s expression (22) are for 2D cylinders and the dimensionless lift $\bar{L}$ is computed as $\bar{L} = L/(\rho_f g \pi a^2 d^4)$; the Saffman and Auton expressions (23) and (25) are for spheres and $\bar{L}$ is computed as $\bar{L} = L/(\rho_f g \pi a^3)$. The slip velocity $U_s$, which

![FIG. 12. A comparison of the lift forces computed from the direct numerical simulation and from the lift expressions (21), (22), (23), and (25). The lift forces on the stable branch near the centerline in a flow of Newtonian fluid with $R = 80$ are plotted.](Image)

![FIG. 13. The steady state values of the dimensionless angular slip velocity $\Omega_s/(2 \gamma_s)$ in flows of Newtonian fluids as a function of $y/d$.](Image)
is a functional of the solution, is prescribed in the Bretherton, Saffman, and Auton expressions and undetermined in their theories. To calculate the lift forces from these expressions, we use the values of \( U_s \) obtained from our DNS. The values of \( U_s, \Omega_s, \) and \( \Omega_N^s \) obtained from the DNS are used in the calculation of (21).

We draw the readers attention to the fact that the lift expressions (21), (22), (23), and (25) apply to different scenarios and are not strictly comparable. Our correlation (21) is for a freely rotating 2D cylinder without accelerations in a plane Poiseuille flow. Bretherton’s expression (22) and Saffman’s expression (23) are both for the lift on a particle in an unbounded linear shear flow with an indefinitely small Reynolds number; the difference is that the former applies to a nonrotating 2D cylinder while the latter applies to a rotating 3D sphere. Auton’s expression (25) applies to a fixed 3D sphere in an inviscid fluid in which uniform motion is perturbed by a weak shear. Expressions (22), (23), and (25) cannot predict the change of sign across the equilibrium position; whereas our correlation (21) reproduces the DNS results faithfully.

Our correlations provide explicit expressions for the lift force on a particle in terms of the slip velocity \( U_s \) and the angular slip velocity discrepancy \( \Omega_s - \Omega_{se} \). We emphasize that the relative angular motion is characterized by \( \Omega_s - \Omega_{se} \) rather than \( \Omega_s \) or \( \Omega_f \). By using the discrepancy, we are able to account for the Segré and Silberberg effect. Our correlations cover the whole channel except the unstable regions. We believe that our correlations capture the essence of the mechanism of the lift force.

Correlations (20) and (21) are derived for the steady state values of \( L, U_s, \) and \( \Omega_s \), i.e., they apply to particles with zero acceleration. For a migrating particle, correlations (20) and (21) are not valid, although they might give good approximations when the acceleration of the particle is small. The application of such correlations is to determine parameters of a particle at equilibrium, e.g., the equilibrium position, translational velocity and angular velocity. For this end, correlations which relate \( U_s \) and \( \Omega_s \) to prescribed parameters are needed. We will show derivation of such correlations is feasible in the next section.

VII. CORRELATIONS FOR SLIP VELOCITY AND ANGULAR SLIP VELOCITY

To make correlations (20) and (21) completely explicit, we need correlations which relate \( U_s \) and \( \Omega_s \) to \( R \) and \( y/d \) in steady flows of Newtonian fluids. We illustrate the procedure for \( \Omega_s \). In Fig. 13, the steady state values of \( \Omega_s/(2 \gamma_w) \) obtained in constrained simulations are plotted against \( y/d \) for five values of \( R \). If these data are plotted on a log–log plot of \( \Omega_s/(2 \gamma_w) \) versus \( R \), we obtain straight lines one for each value of \( y/d \) from the wall to the centerline (five of which are shown in Fig. 14), leading to power law correlations:

\[
\frac{\Omega_s(y/d,R)}{2 \gamma_w} = b(y/d)R^{(y/d)}\Omega_s(y/d,R)
\]

\[
= b(y/d)R^{(y/d)} \frac{\eta_0}{\rho j d^2}.
\]

The prefactor \( b \) and exponent \( r \) in these power law correlations, which are functions of \( y/d \), are plotted in Fig. 15. With more data points, these functions could be fitted to splines, making (26) completely explicit.

A similar procedure for \( U_s \) leads to

\[
\frac{U_s(y/d,R)}{2 \gamma_w d} = c(y/d)R^{(y/d)}U_s(y/d,R)
\]

\[
= c(y/d)R^{(y/d)} \frac{\eta_0}{\rho d^2}.
\]

As for \( b \) and \( r \) in (26), \( c \) and \( q \) could be fit to splines if more data points were available. Unlike correlation (26) which can be found at values of \( y/d \) from the wall to the centerline, correlation (27) can only be found at values of \( y/d \) on stable branches of steady solutions. It does not correlate well with the data for the unstable branches; in fact for some values of \( R \), \( U_s \) is slightly negative at some values of \( y/d \) on the unstable branch near the wall, which is incompatible with a power law in the form (27).

In addition to (26) and (27), we also need a correlation between \( \Omega_N^s \), the angular slip velocity of a neutrally buoyant
TABLE VII. The dimensionless angular slip velocity of a neutrally buoyant particle at equilibrium is essentially a constant in flows of Newtonian fluids with $R = 20–160$.

<table>
<thead>
<tr>
<th>$R$</th>
<th>$\Omega_{se}/(2 \gamma_w)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>$5.06 \times 10^{-3}$</td>
</tr>
<tr>
<td>40</td>
<td>$5.24 \times 10^{-3}$</td>
</tr>
<tr>
<td>80</td>
<td>$5.32 \times 10^{-3}$</td>
</tr>
<tr>
<td>120</td>
<td>$5.24 \times 10^{-3}$</td>
</tr>
<tr>
<td>160</td>
<td>$5.21 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

The distribution of these regions is affected by shear thinning. Our results show that when shear thinning effects become stronger, the unstable region at the centerline shrinks, indicating that the equilibrium position of a particle could be closer to the centerline.

The conclusion that the angular velocity discrepancy $\Omega_s - \Omega_{se}$ changes sign across an equilibrium position established by Joseph and Ocampo in Newtonian fluids is confirmed in shear thinning fluids. Across a stable equilibrium position, $\Omega_s - \Omega_{se}$ has the same sign as the net lift force $L_n$; across an unstable equilibrium position, $\Omega_s - \Omega_{se}$ has the opposite sign as the net lift force $L_n$.

Correlations for the lift force on a particle in terms of the slip velocity $U_s$ and the angular slip velocity discrepancy $\Omega_s - \Omega_{se}$ are derived. The correlations are a power law near the wall and a linear relation (which can be taken as a power law with the power of one) near the centerline. The correlations apply to both neutrally buoyant and heavy particles and cover the whole channel except the unstable regions. Two regimes, one with no or weak shear thinning effects and the other with strong shear thinning effects, are identified for the power law correlation (13) whereas only one regime is found for the linear correlation (14). It is noted that particle lift in strong shear thinning flows is different with that in flows of Newtonian fluids at high Reynolds number.

We are able to obtain correlations between $L$ and $U_s(\Omega_s - \Omega_{se})$ with coefficients expressed as functions of $R$; these correlations cover the flows of Newtonian fluids with the Reynolds number in the range of 20–160. The correlation is compared to well-known analytical expressions for lift force in shear flows and similarities between them are revealed. The major difference between them is that the angular slip velocity discrepancy $\Omega_s - \Omega_{se}$ is used in our correlations instead of the shear rate or $\Omega_s$. We also demonstrate that correlations which relate $U_s$ and $\Omega_s$ to prescribed parameters can be constructed and will make the correlations for $L$ completely explicit. Thus the lift force in steady flows can be calculated using correlations at any value of $y/d$ on stable branches from the prescribed parameters; the equilibrium position of a particle with a certain density can then be determined by the balance between the lift force and its buoyant weight.
ACKNOWLEDGMENTS

This work was partially supported by the National Science Foundation KDI/New Computational Challenge Grant (NSF/CTS-98-73236); by the US Army, Mathematics; by the DOE, Department of Basic Energy Sciences; by a grant from the Schlumberger foundation; from STIM-LAB Inc.; and by the Minnesota Supercomputer Institute. The authors thank T. Hesla for his help in preparing the paper.