Stress induced cavitation for the irrotational streaming motion of a viscous liquid past a sphere

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Abstract
Cavitation induced by stresses in the irrotational streaming flow of a viscous liquid past a stationary sphere is studied here using the theory of viscous potential flow. The maximum tension criterion for cavitation used here was proposed by Joseph 1995, 1998: “Liquids at atmospheric pressure which cannot withstand tension will cavitate when and where tensile stresses due to motion exceed one atmosphere. A cavity will open in the direction of the maximum tensile stress which is 45° from the plane of shearing in pure shear of a Newtonian fluid.” The analysis leads to a dimensionless expression for the maximum tensile stress as a function of position which depends on the cavitation and Reynolds numbers. The main conclusion is that at a fixed cavitation number the cavitation threshold decreases with the Reynolds number and the extent of the region of flow at risk to cavitation increases as the Reynolds number decreases. This prediction that more viscous liquids at a fixed cavitation number are at greater risk to cavitation seems not to be addressed, affirmed or denied, in the cavitation literature known to us.

1 Introduction
Cavitation of the streaming flow of a liquid past a stationary sphere is a good model of the classical problem of cavitation of blunt bodies. There is no literature other than the paper of Funada et al. (2004) on analysis of stress induced cavitation using potential flow. The analysis of this problem, given below, is completely transparent, though the effects of neglecting vorticity are not yet understood. We hope to evaluate the effects of vorticity generated in boundary layers in the flow over solid spheres by comparing the theoretical results based on potential flow with numerical simulation based on the Navier-Stokes equations. The effects of vorticity, however, on a spherical gas bubble are very confined for the wide range of moderately, large Reynolds numbers associated with the dissipation calculation of drag (cf. Figure 5.1 4.1, Batchelor 1967).

Irrotational flows of incompressible viscous fluids satisfy the Navier-Stokes equations and give rise to the usual Bernoulli equation because

$$\mu \nabla^2 u = \mu \nabla^2 \phi = 0$$  \hspace{1cm} (1.1)

no matter what the value of $\mu$. The stresses are given by

$$T = -pI + 2\mu \nabla \otimes \nabla \phi = \frac{1}{3} \text{Tr} (T) I + 2\mu \nabla \otimes \nabla \phi,$$  \hspace{1cm} (1.2)

where $p = -\frac{1}{3} \text{Tr} (T)$ is the average stress.

2 Analysis
A cartoon of the flow is given in figure 1. The flow is axisymmetrical and steady and the potential $\phi(r, \theta)$ satisfies $\nabla^2 \phi = 0$. 
Figure 1. The irrotational flow of a liquid past a sphere of radius $a$. The flow at infinity is $U$ and the potential there is $Ux$ where $x = r \cos \theta$ and $r, \theta$ are spherical polar coordinates. The pressure $p(r, \theta)$ is constant $p_\infty$ at infinity. The angle $\alpha$ puts the stresses into principal axes.

The potential for this flow is

$$\phi = U \left( r + \frac{1}{2} \frac{a^3}{r^3} \right) \cos \theta. \tag{2.1}$$

The velocity $u = e_r u_r + e_\theta u_\theta$ is given by

$$u_r = \frac{\partial \phi}{\partial r} = U \left( 1 - \frac{a^3}{r^3} \right) \cos \theta, \tag{2.2}$$
$$u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -U \left( 1 + \frac{1}{2} \frac{a^3}{r^3} \right) \sin \theta. \tag{2.3}$$

The pressure is given by

$$p = p_\infty + \rho \frac{U^2}{2} - \rho \left( u_r^2 + u_\theta^2 \right) = p_\infty + \rho \frac{U^2}{2} \left[ 1 - \left( 1 - \frac{a^3}{r^3} \right)^2 \cos^2 \theta - \left( 1 + \frac{1}{2} \frac{a^3}{r^3} \right)^2 \sin^2 \theta \right]. \tag{2.4}$$

The non-zero components of the viscous stress

$$2\mu D [\nabla \phi] \tag{2.5}$$

are

$$2\mu D_{rr} = 2\mu \frac{\partial u_r}{\partial r} = 6\mu U \frac{a^3}{r^4} \cos \theta, \tag{2.6}$$
$$2\mu D_{\theta\theta} = 2\mu \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) = -3\mu U \frac{a^3}{r^4} \cos \theta, \tag{2.7}$$
$$2\mu D_{\varphi\varphi} = 2\mu \left( \frac{u_r}{r} + \frac{u_\theta}{r} \cot \theta \right) = -3\mu U \frac{a^3}{r^4} \cos \theta, \tag{2.8}$$
$$2\mu D_{r\theta} = \mu \left[ \frac{r}{\partial r} \left( \frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right] = 3\mu U \frac{a^3}{r^4} \sin \theta. \tag{2.9}$$
The matrix of components
\[ 2\mu \begin{bmatrix}
D_{rr} & D_{r\theta} & 0 & 0 \\
D_{r\theta} & D_{\theta\theta} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} = 3\mu U a^3 r^4 \begin{bmatrix}
2 \cos \theta & \sin \theta & 0 & 0 \\
\sin \theta & -\cos \theta & 0 & 0 \\
0 & 0 & -\cos \theta & 0 \\
0 & 0 & 0 & -\cos \theta
\end{bmatrix} \] (2.10)
can be rotated into diagonal form through an angle \( \alpha \) satisfying
\[ \tan 2\alpha = \frac{2}{3} \tan \theta. \] (2.11)
The diagonal form of \( 2\mu \nabla \otimes \nabla \phi \) is given by
\[ 3\mu U a^3 r^4 \begin{bmatrix}
\frac{1}{2} \cos \theta + \frac{\sin \theta}{\sin 2\alpha} & 0 & \frac{1}{2} \cos \theta - \frac{\sin \theta}{\sin 2\alpha} & 0 \\
0 & 0 & -\cos \theta & 0 \\
0 & 0 & -\cos \theta & 0 \\
0 & 0 & 0 & -\cos \theta
\end{bmatrix} \] (2.12)
at \( \theta = \pi/2 \), where the pressure is smallest, \( \alpha = \pi/4 \) and the diagonal form is
\[ 3\mu U a^3 r^4 \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix} \] (2.13)
giving rise to tension and compression.

We next consider the whole stress
\[ \mathbf{T} = -p \mathbf{1} + 2\mu \nabla \otimes \nabla \phi \] (2.14)
which may be written as
\[ \mathbf{T} + p_v \mathbf{1} = (-p + p_v) \mathbf{1} + 2\mu \nabla \otimes \nabla \phi \] (2.15)
where \( p_v \) is the vapor pressure
\[ \frac{\mathbf{T} + p_v \mathbf{1}}{\frac{1}{2} \rho U^2} = \left[ K + 1 - \left( 1 - \frac{a^3}{r^3} \right)^2 \cos^2 \theta - \left( 1 + \frac{a^3}{2r^3} \right)^2 \sin^2 \theta \right] \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
+ \frac{3}{Re} \left( \frac{a}{r} \right)^4 \begin{bmatrix}
\cos \theta + \frac{2 \sin \theta}{\sin 2\alpha} & 0 & 0 & 0 \\
0 & -\cos \theta - \frac{2 \sin \theta}{\sin 2\alpha} & 0 & 0 \\
0 & 0 & -2 \cos \theta & 0
\end{bmatrix} \] (2.16)
where
\[ K = \frac{p_\infty - p_v}{\frac{1}{2} \rho U^2} \] (2.17)
is the cavitation number and
\[ Re = \frac{U a \rho}{\mu} \] (2.18)
is the Reynolds number.
Suppose now that $T_{11}$ is the largest of the three principal values of stress. Then the locus of the cavitation threshold is given by

$$T_{11} + p_v = 0$$

(2.19)

giving rise to isolines $K = f(\frac{\alpha}{\pi}, Re)$ for the cavitation threshold.

The largest values of the viscous irrotational stress is at the boundary $r = a$ where the neglected vorticity is largest. It remains to establish that the criterion $T_{11} + p_v = 0$ given by (2.16) makes sense close to $r = a$.

Equation (2.16) gives the form of the diagonalized stress tensor at each point $(r, \theta)$ in the axially symmetric flow. $T_{11}, T_{22}$ and $T_{33} = T_{\phi\phi}$ are principal stresses in the principal axes coordinates with bases $e_1, e_2, e_\phi$. In the present case, the angle $\alpha$ changes with $\theta$, $\tan 2\alpha = \frac{2}{3} \tan \theta$. The solution of this equation (2.11) is displayed graphically in figure 2.

![Graph](image)

Fig. 2 $\alpha/\pi$ versus $\theta/\pi$, derived from $\tan 2\alpha = \frac{2}{3} \tan \theta$. A linear approximation of this graph is $\alpha = 0.5\theta$. A cavitation bubble will open asymmetrically with the axis of maximum tension rotated through angle $\alpha$ at each point $r, \theta$ as in figure 2.

We note that

$$\text{Tr} \mathbf{T} = -3p = T_{11} + T_{22} + T_{33}$$

(2.20)

because the trace of the viscous part of the stress tensor vanishes. In the classical theory of cavitation, the viscous part of the stress tensor is not considered and

$$\mathbf{T} + p_v \mathbf{1} = (-p + p_v) \mathbf{1}.$$  

(2.21)

The classical theory assumes that the cavitation threshold is given by the average stress, called the pressure. The fluid cannot average its stresses; it sees only principal stresses and when the actual state of stress is considered there is at least one stress which is more compressive and another which is more tensile than the average stress. The most conservative criterion is the one which requires that the most compressive stress is larger than the vapor pressure; if $T_{22}$ is the most compressive and $T_{11}$ is the most tensile stress, then if

$$T_{22} + p_v > 0$$

(2.22)
for cavitation, it will surely be true that
\[ -p + p_v > 0 \text{ and } T_{11} + p_v > 0. \] (2.23)

The maximum tension theory, which perhaps embodies the statement that liquids which are not specially prepared will cavitate when they pass into tension, can be expressed by the condition that supposing \( T_{11} \) to be the maximum of the three principal stresses,
\[ T_{11} + p_v > 0. \] (2.24)

The maximum tension criterion (2.24) has recently been studied in a numerical simulation of bubble growth in Newtonian and viscoelastic filaments undergoing stretching by Foteinopoulou et al. 2004. They base their analysis on the Navier-Stokes equations for Newtonian fluids and the Phan-Thien/Tanner model for viscoelastic fluids. They compute the principal stresses and evaluate the cavitation threshold for the maximum tension criterion (2.24), the classical pressure theory (2.21) and the minimum principal theory (2.22). They find that the capillary number at inception is smallest for (2.24).

Suppose now that \( K = K_c \) at the marginal state separating cavitation from no cavitation. This marginal state is defined by an equality in one of the three criteria (2.22), (2.23) or (2.24). For the maximum tension theory \( K = K_c \) when \( T_{11} + p_v = 0 \). From (2.16), we see that
\[ T_{11} + p_v < 0 \text{ when } K > K_c \] (2.25)
and
\[ T_{11} + p_v > 0 \text{ when } K < K_c. \] (2.26)

It is apparent from (2.16) that the largest stresses are at the boundary of the sphere where \( r = a \). In figure 3, we have plotted \( K_c \) versus \( \theta/\pi \) for the three principal stresses when \( r = a \) at \( Re = 10^{-2}, 10, 100, 10^3 \).

Certainly the liquid will cavitate when \( K = (p_\infty - p_v) / (\frac{\rho}{2} U^2) < 0 \); only \( K > 0 \) is of interest. Using now the maximum tension criterion we see that cavitation occurs for \( 0 < K < K_c \) and the fluid is most at risk to cavitation for \( \theta \) at which \( K_c(\theta) \) is greatest. This most dangerous \( \theta \) is at \( \theta = 0 \) when \( Re \) is small and at \( \theta = \pi/2 \) when \( Re \) is large. It follows that the place most at risk to cavitation runs from the rear stagnation point at \( \theta = 0 \) when \( Re \) is small to \( \theta = \pi/2 \) when \( Re \) is large.
Fig. 3 $K_c$ versus $\theta/\pi$ from (2.16) with $r = a$. For all values of $Re$ we have $T_{11} \geq T_{33} \geq T_{22}$. The strong effects of viscosity have disappeared when $Re > 10^3$; then the classical theory holds. (a) $Re = 10^{-2}$, (b) $Re = 10$, (c) $Re = 100$ and (d) $Re = 1000$. 
Fig. 4 The cavitation threshold curves on which $T_{11} + p_v = 0$ for different $K$ at a given $Re$. Cavitation occurs inside the curve on which $T_{11} + p_v = 0$; (a) $Re = 10^{-2}$ with $K = 600, 800, 1000$, (b) $Re = 10$ with $K = 0, 0.5, 1.0$ and $1.5$, and (c) $Re = 1000$ with $K = 0, 0.5, 1.0$.

3 Stokes flow

The most stringent possible comparison of stress induced cavitation in the streaming irrotational motion of a viscous fluid over a sphere, is with the same problem when the stream is creeping in the Stokes flow limit. In the irrotational case we are thinking of a flow outside the boundary layer where $r > a$ and, as usual in boundary layer theory, we do the evaluation at $r = a$. This kind of thinking does not work in Stokes flow where there is no boundary layer and the flow for $r > a$ is very little different than at $r = a$. 
For Stokes flow, we have

\[ u_r = U \left[ 1 - \frac{3a^3}{2r^3} \right] \cos \theta, \]  
\[ u_\theta = -U \left[ 1 - \frac{3a^3}{4r^3} \right] \sin \theta, \]  
\[ p = p_\infty - \frac{3\mu}{2a} U \cos \theta. \]  

The normal components of the stress deviator \( 2\mu D[u] \) vanish and

\[ 2\mu \begin{bmatrix} D_{rr} & D_{r\theta} & 0 \\ D_{r\theta} & D_{\theta\theta} & 0 \\ 0 & 0 & D_{\phi\phi} \end{bmatrix} = \frac{3\mu}{2a} U \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sin \theta. \]  

The principal axes form of this tensor is achieved for \( \alpha = -45^\circ \)

\[ = \frac{3\mu}{2a} U \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sin \theta. \]  

The cavitation criteria may then be formed from

\[ \frac{T_{11} + p_v}{2U^2} = -K \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{3}{Re} \begin{bmatrix} \cos \theta + \sin \theta & 0 & 0 \\ 0 & \cos \theta - \sin \theta & 0 \\ 0 & 0 & \cos \theta \end{bmatrix}. \]  

The maximum tension is achieved at the equator \( \theta = 90^\circ \) due to shear or at the rear stagnation point \( \theta = 0 \) due to pressure.

4 Discussion

The paper is motivated by the desire to understand the effect of viscosity on stress induced cavitation of liquids. The analysis is based on the irrotational flow of viscous fluids around a sphere which allows for a particularly transparent analysis in which all the predictions are given by simple explicit formulas. The main points about stress induced cavitation, in particular the role of the maximum tensile stress in principal axes coordinates, and the comparison of this criterion with the classical one depending only on pressure and the most conservative one which requires that all the principal stresses be in tension, are very clearly expressed in the simple analysis based on viscous potential flow.

Irrotational motions of viscous liquids account for viscous stresses but not for vorticity created by the no slip condition at the boundary of solids which is neglected. The analysis is not restricted to small viscosity but is restricted to small vorticity.

In steady flows over a sphere the effects of vorticity are greatest in the wake regions behind the separated boundary layers. This is just the region that the irrotational theory predicts that the liquid is at greatest risk to cavitation. The irrotational theory cannot be used in the wake region. However stress induced cavitation is also predicted on the front face of the sphere where the boundary layer is
thin and the exterior flow is irrotational or very closely irrotational. A good discussion of this can be found in section 4.10.4 of White 1991. There, it is shown that the skin friction based on the surface velocity of the irrotational flow is close to the actual friction near the front stagnation point and not so hugely different up to the point of separation. Measured maximum surface velocity is about $1.3 \times U_\infty$ with separation at about $\theta = 107^\circ$. The irrotational flow theory given here is off the mark not because it is rotational but due to the displacement of the irrotational flow by the separated wake.

There are cases in which irrotational theory for very viscous liquids should be close to exact; the early time flow for start from rest for flow around spheres or small amplitude, high frequency oscillations of a sphere in a viscous liquid are examples. The rise of a spherical gas bubble is another case in which the theory of irrotational motions of a viscous liquid could be applied. In this problem, the vorticity at the gas liquid surface is small (see Batchelor, p.367). Here, we could study how the liquid is eroded by vaporization to stress induced cavitation.

The theory of stress induced cavitation does not require one to assume irrotational flow; we worked the same theory for Stokes flow. The predictions here are for how a very thick liquid could erode in creeping flow around a sphere. We find that the cavities would develop at the equator $45^\circ$ from the direction of shearing or at the rear stagnation point in a spherically symmetric way. The same theory can be worked for exact numerical simulation, even for turbulent flow.

The main prediction of this work is that highly viscous fluids are at greater risk to cavitation at a fixed cavitation number. This prediction appears to be new since the question seems surprisingly, not to have been addressed in the cavitation literature.

References