Irrotational viscous pressure and the dissipation method

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1 Analysis

Here we show that the analysis of the viscous correction of viscous potential flow (VCVPF) and the dissipation method (DM) are equivalent. Consider the equations of motion for an incompressible Newtonian fluid with gravity as a body force per unit mass

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla\Phi + \mu\nabla^2\mathbf{u} \tag{1.1}$$

where

$$\Phi = p + \rho g z. \tag{1.2}$$

The stress is given by

$$\mathbf{T} = -p\mathbf{1} + \boldsymbol{\tau} \tag{1.3}$$

where

$$\boldsymbol{\tau} = 2\mu\mathbf{D}[\mathbf{u}]$$

and

$$\nabla\cdot\boldsymbol{\tau} = \mu\nabla^2\mathbf{u}$$

The mechanical energy equation corresponding to (1.1) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho}{2}\left|\mathbf{u}\right|^2\ \mathrm{d}\Omega = \int_S \mathbf{n}\cdot\tilde{\mathbf{T}}\cdot\mathbf{u}\mathrm{d}S - 2\mu\int_\Omega \mathbf{D}:\mathbf{D}\ \mathrm{d}\Omega, \tag{1.4}$$

where S is the boundary of Ω, with outward normal $\mathbf{n}$. On solid boundaries no-slip is imposed; say $\mathbf{u} = 0$ there, and on the free surface

$$z = \eta(x, y, t)$$

and the shear stress

$$\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{e}_\mathrm{s} = 0, \tag{1.5}$$

where $\mathbf{e}_s$ is any vector tangent to the free surface S_f. On S_f we have

$$\int_S \mathbf{n}\cdot\tilde{\mathbf{T}}\cdot\mathbf{u}\mathrm{d}S = \int_{S_f} \mathbf{n}\cdot\tilde{\mathbf{T}}\cdot\mathbf{u}\mathrm{d}S$$

where

$$\tilde{\mathbf{T}} = \mathbf{T} - \rho g\eta\mathbf{1}, \tag{1.6}$$

$$\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n}=-\gamma\nabla_{\rm II}\cdot\mathbf{n}. \tag{1.7}$$

Hence, on $S_{\rm f}$

$$\mathbf{n}\cdot\tilde{\mathbf{T}}\cdot\mathbf{n}=-\rho g\eta-\gamma\nabla_{\rm II}\cdot\mathbf{n}. \tag{1.8}$$

and, since the shear stress vanishes on $S_{\rm f}$

$$\begin{aligned}\mathbf{n}\cdot\tilde{\mathbf{T}}\cdot\mathbf{u} &= \mathbf{n}\cdot\tilde{\mathbf{T}}\cdot(u_{\rm n}\mathbf{n}+u_{\rm s}\mathbf{e}_{\rm s})\\ &= -(\rho g\eta+\gamma\nabla_{\rm II}\cdot\mathbf{n})u_{\rm n}.\end{aligned} \tag{1.9}$$

Hence, (1.4) may be written as

$$\frac{\rm d}{{\rm d}t}\int_\Omega\frac{\rho}{2}\left|\mathbf{u}\right|^2\ {\rm d}\Omega=-\int_{S_{\rm f}}(\rho g\eta+\gamma\nabla_{\rm II}\cdot\mathbf{n})u_{\rm n}{\rm d}S_{\rm f}-2\mu\int_\Omega\mathbf{D}:\mathbf{D}\ {\rm d}\Omega, \tag{1.10}$$

Equation (1.10) holds for viscous fluids satisfying the Navier–Stokes equations (1.1) subject to the vanishing shear stress condition (1.5).

We turn now to potential flow $\mathbf{u}=\nabla\phi$, $\nabla^2\phi=0$. In this case, $\nabla^2\mathbf{u}=0$ but the dissipation does not vanish. How can this be? In Joseph (2006) we showed that the irrotational viscous stress is self-equilibrated and does give rise to a force density term $\nabla^2\nabla\phi=0$; however, the power of self-equilibrated irrotational viscous stresses

$$\int_{S_{\rm f}}\mathbf{u}\cdot 2\mu\nabla\otimes\nabla\phi{\rm d}S_{\rm f}$$

does not vanish, and it gives rise to an irrotational viscous dissipation

$$\begin{aligned}2\mu\int_\Omega\mathbf{D}[\mathbf{u}]:\mathbf{D}[\mathbf{u}]{\rm d}\Omega &= 2\mu\int_\Omega\frac{\partial^2\phi}{\partial x_{\rm i}\partial x_{\rm j}}\frac{\partial^2\phi}{\partial x_i\partial x_j}{\rm d}\Omega=2\mu\int_S n_{\rm j}\frac{\partial^2\phi}{\partial x_{\rm j}\partial x_{\rm i}}u_{\rm i}{\rm d}S\\ &= 2\mu\int_{S_{\rm f}}\mathbf{n}\cdot\mathbf{D}[\nabla\phi]\cdot\mathbf{u}{\rm d}S_{\rm f}=2\mu\int_{S_{\rm f}}\mathbf{n}\cdot\mathbf{D}\cdot(u_{\rm n}\mathbf{n}+u_{\rm s}\mathbf{e}_{\rm s})\,{\rm d}S_{\rm f}\\ &= \int_{S_{\rm f}}\left(2\mu\frac{\partial^2\phi}{\partial n^2}u_{\rm n}+\tau_{\rm s}u_{\rm s}\right){\rm d}S_{\rm f}\end{aligned} \tag{1.11}$$

where $\tau_{\rm s}$ is an irrotational shear stress

$$\begin{aligned}\tau_s &= 2\mu\mathbf{n}\cdot\mathbf{D}[\nabla\phi]\cdot\mathbf{e}_{\rm s},\\ u_{\rm s} &= \mathbf{u}\cdot\mathbf{e}_{\rm s}.\end{aligned} \tag{1.12}$$

Turning next to the inertial terms, we have

$$\left(\mathbf{u}\cdot\nabla\right)\mathbf{u}=\nabla\frac{|\mathbf{u}|^2}{2}$$

and

$$\begin{aligned}\frac{\rm d}{{\rm d}t}\int_\Omega\frac{\rho}{2}|\mathbf{u}|^2{\rm d}\Omega &= \int_\Omega\rho\left(\mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\frac{|\mathbf{u}|^2}{2}\right){\rm d}\Omega\\ &= \int_\Omega\rho\left[\frac{\partial\phi}{\partial x_{\rm i}}\frac{\partial}{\partial x_{\rm i}}\frac{\partial\phi}{\partial t}+\nabla\cdot\left(\mathbf{u}\frac{|\mathbf{u}|^2}{2}\right)\right]{\rm d}\Omega\\ &= \int_{S_{\rm f}}\rho u_{\rm n}\left(\frac{\partial\phi}{\partial t}+\frac{|\nabla\phi|^2}{2}\right){\rm d}S_{\rm f}\end{aligned} \tag{1.13}$$

Collecting the results (1.9),(1.11), (1.12) and (1.13) we need to evaluate the energy equation (1.4) for (1.1) when $\mathbf{u} = \nabla\phi$; we find that

$$\int_{S_\mathrm{f}} u_\mathrm{n} \left[\rho \left(\frac{\partial \phi}{\partial t} + \frac{|\nabla\phi|^2}{2} + g\eta \right) + 2\mu \frac{\partial^2 \phi}{\partial n^2} + \gamma \nabla_\mathrm{II} \cdot \mathbf{n} \right] \mathrm{d}S_\mathrm{f} = -\int_{S_\mathrm{f}} \tau_\mathrm{s} u_\mathrm{s} \mathrm{d}S_\mathrm{f}. \qquad (1.14)$$

This equation (1.14) is the energy equation for the irrotational flow of a viscous fluid.

The viscous pressure p_v for VCVPF can be defined by the equation

$$\int_{S_\mathrm{f}} (-p_\mathrm{v}) u_\mathrm{n} \mathrm{d}S_\mathrm{f} = \int_{S_\mathrm{f}} \tau_\mathrm{s} u_\mathrm{s} \mathrm{d}S_\mathrm{f}. \qquad (1.15)$$

This 'pressure' is important when it can be calculated; sometimes we can calculate p_v, especially in linear problems in which $\nabla^2 p_\mathrm{v} = 0$. More often, we do not know how to calculate p_v. VCVPF leads to a new set of PDE's in which p_v is a variable.

Suppose now that there is such a pressure correction and Bernoulli equation

$$p_\mathrm{i} + \rho \left(\frac{\partial \phi}{\partial t} + \frac{|\nabla\phi|^2}{2} + g\eta \right) = C$$

holds. Since

$$C \int_S \mathbf{u} \cdot \mathbf{n} \mathrm{d}S = C \int_\Omega \nabla \cdot \mathbf{u} \mathrm{d}\Omega = 0$$

we obtain

$$\int_{S_\mathrm{f}} u_\mathrm{n} \left[-p_\mathrm{i} - p_\mathrm{v} + 2\mu \frac{\partial^2 \phi}{\partial n^2} + \gamma \nabla_\mathrm{II} \cdot \mathbf{n} \right] \mathrm{d}S_\mathrm{f} = 0 \qquad (1.16)$$

The normal stress balance for VCVPF is

$$-p_\mathrm{i} - p_\mathrm{v} + 2\mu \frac{\partial^2 \phi}{\partial n^2} + \gamma \nabla_\mathrm{II} \cdot \mathbf{n} = 0 \qquad (1.17)$$

The dissipation method DM is equivalent to VCVPF; however, in DM we do not compute a pressure correction but the power of the irrotational shear stress on the right of (1.14), which satisfies (1.15) when p_v can be found and makes sense, is computed. In DM a new field p_v is not computed and is not needed. We use the word 'equivalent' rather than equal because one obtains the same results from (1.14) with or without (1.15).

In all the linear problems we have considered so far (e.g., capillary instability Wang *et al.* 2005a, Wang *et al.* 2005b; oscillations of drops and bubbles Padrino *et al.* 2007), we can actually calculate p_v, and VCVPF and DM give exactly the same result.

References

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