Irrotational motions of bubbles under the action of acceleration of added mass and viscous drag

J. Wang, T. Funada and D. D. Joseph

Abstract

We present elementary examples of how irrotational flows can be used to study the unsteady motions of bubbles in a viscous liquid suitable for the instruction of undergraduate students. A spherical gas bubble accelerates to steady motion in an irrotational flow of a viscous liquid induced by a balance of the acceleration of the added mass of the liquid with the viscous drag. The equation of rectilinear motion is linear and may be integrated giving rise to exponential decay with a decay constant $18\nu t/a^2$ where ν is the kinematic viscosity of the liquid and a is the bubble radius. The problem of decay to rest of a bubble moving initially when the forces maintaining motion are inactivated and the acceleration of a bubble initially at rest to terminal velocity are considered. We also discuss the motion of a massless cylindrical bubble under the combined action of Kutta-Joukowski lift, acceleration of added mass and viscous drag. The cylinder moves with a constant speed along a circular path if the liquid is inviscid, whereas it goes to rest in a spiral fashion when the viscous drag is added.

1 The rectilinear motion of a spherical bubble

We consider a body moving with the velocity U in an unbounded viscous potential flow. Let M be the mass of the body and M' be the added mass, then the total kinetic energy of the fluid and body is

$$T = \frac{1}{2}\left(M + M'\right)U^2. \tag{1.1}$$

Let D be the drag and F be the external force in the direction of motion, then the power of D and F should be equal to the rate of the total kinetic energy,

$$(F + D)U = \frac{\mathrm{d}T}{\mathrm{d}t} = (M + M')U\frac{\mathrm{d}U}{\mathrm{d}t}. \tag{1.2}$$

We next consider a spherical gas bubble, for which $M = 0$ and $M' = \dfrac{2}{3}\pi a^3\rho_f$. The drag can be obtained by direct integration using the irrotational viscous normal stress and a viscous pressure correction: $D = -12\pi\mu a U$ (see Joseph and Wang 2004). Suppose the external force just balances the drag, then the bubble moves with a constant velocity $U = U_0$. Imagine that the external force suddenly disappears, then (1.2) gives rise to

$$-12\pi\mu a U = \frac{2}{3}\pi a^3\rho_f\frac{\mathrm{d}U}{\mathrm{d}t}. \tag{1.3}$$

The solution is

$$U = U_0 \mathrm{e}^{-\frac{18\nu}{a^2}t}, \tag{1.4}$$

which shows that the velocity of the bubble approaches zero exponentially.

If gravity is considered, then $F = \dfrac{4}{3}\pi a^3\rho_f g$. Suppose the bubble is at rest at $t = 0$ and starts to move due to the buoyant force. Equation (1.2) can be written as

$$\frac{4}{3}\pi a^3\rho_f g - 12\pi\mu a U = \frac{2}{3}\pi a^3\rho_f\frac{\mathrm{d}U}{\mathrm{d}t}. \tag{1.5}$$

The solution is

$$U = \frac{a^2 g}{9\nu}\left(1 - \mathrm{e}^{-\frac{18\nu}{a^2}t}\right), \tag{1.6}$$

which indicates the bubble velocity approaches the steady state velocity

$$U = \frac{a^2 g}{9\nu} \tag{1.7}$$

exponentially.

Another way to obtain the equation of motion is to argue following Lamb (1932) and Levich (1949) that the work done by the external force F is equal to the rate of the total kinetic energy and the dissipation:

$$FU = (M + M')U\frac{\mathrm{d}U}{\mathrm{d}t} + \mathcal{D}. \tag{1.8}$$

Since $\mathcal{D} = -DU$, (1.8) is the same as (1.2).

The motion of a single spherical gas bubble in a viscous liquid has been considered by some authors. Typically, these authors assemble terms arising in various situations, like Stokes flow (Hadamard-Rybczynski drag, Basset memory integral) and high Reynolds number flow (Levich drag, boundary layer drag, induced mass) and other terms into a single equation. Such general equations have been presented by Yang and Leal (1991) and by Park, Klausner and Mei (1995) and they have been discussed in the review paper of Magnaudet and Eams (2000, see their section 4). Yang and Leal's equation has Stokes drag and no Levich drag. Our equation is not embedded in their equation. Park *et al.* listed five terms for the force on a gas bubble; our equation may be obtained from theirs if the free stream velocity U is put to zero, the memory term is dropped, and the boundary layer contribution to the drag given by Moore (1963) is neglected. Park *et al.* did not write down the same equation as our equation (1.1) and did not obtain the exponential decay.

It is generally believed that the added mass contribution, derived for potential flow is independent of viscosity. Magnaudet and Eames say that "... results all indicate that the added mass coefficient is independent of the Reynolds, strength of acceleration and ... boundary conditions." This independence of added mass on viscosity follows from the assumption that the motion of viscous fluids can be irrotational. The results cited by Magnaudet and Eams seem to suggest that induced mass is also independent of vorticity.

Chen (1974) did a boundary layer analysis of the impulsive motion of a spherical gas bubble which shows that the Levich drag $48/R_e$ at short times evolves to the drag $\dfrac{48}{R_e}\left(1 - \dfrac{2.21}{\sqrt{R_e}}\right)$ obtained in a boundary layer analysis by Moore (1963). The Moore drag cannot be distinguished from the Levich drag when R_e is large. The boundary layer contribution is vortical and is neglected in our potential flow analysis.

2 The motion of a cylindrical bubble in circulation

We discuss the motion of a cylindrical body with a radius a moving in a viscous fluid. The analysis follows the appendix in Lundgren and Mansour (1991). Let

$$\boldsymbol{R}(t) = X(t)\boldsymbol{i} + Y(t)\boldsymbol{j} \tag{2.1}$$

be the instantaneous position of the center of the cylinder. The velocity is then

$$\dot{\boldsymbol{R}} = U\boldsymbol{i} + V\boldsymbol{j}. \tag{2.2}$$

The direction of the velocity $\boldsymbol{n}$ is

$$\boldsymbol{n} = \frac{U}{W}\boldsymbol{i} + \frac{V}{W}\boldsymbol{j} \text{ with } W = \sqrt{U^2 + V^2}. \tag{2.3}$$

The motion is approximated by a potential flow. The dissipation per unit length for a cylinder moving with a speed W and a circulation Γ evaluated using the potential flow is (Wang and Joseph 2006)

$$D = 8\pi\mu W^2 + \frac{\mu\Gamma^2}{\pi a^2}. \tag{2.4}$$

The dissipation should be equal to the power of the drag and torque on the cylinder. Ackeret (1952) did not consider the torque and used this dissipation to compute the drag on the cylinder:

$$\mathcal{D} = D/W = 8\pi\mu W + \frac{\mu\Gamma^2}{\pi a^2 W}. \tag{2.5}$$

Here we use this drag for the purpose of illustration of the viscous effect on the dynamics of the cylinder.

Lundgren and Mansour (1991) derived an equation (their Equation A7) governing the motion of the cylinder based on the assumption that the cylinder is a massless bubble and has no applied force. If the drag force (2.5) is added, the governing equation for the motion becomes

$$\rho\pi a^2 \ddot{\boldsymbol{R}} = \rho\Gamma\boldsymbol{k} \times \boldsymbol{R} - \mathcal{D}\boldsymbol{n}, \tag{2.6}$$

that is, the apparent mass times acceleration is balanced by the Kutta-Joukowski lift and the viscous drag. To make (2.6) dimensionless, we introduce following scales

$$[\text{length, velocity, time}] \sim \left[a,\ \frac{\Gamma}{a\pi},\ \frac{a^2\pi}{\Gamma}\right]. \tag{2.7}$$

In this analysis, we assume that the circulation Γ is a constant and does not depend on time. The dimensionless equations are written in the scalar form as follows

$$\dot{\tilde{U}} = -\tilde{V} - \frac{1}{R_e}\left(8\tilde{W} + \frac{1}{\tilde{W}}\right)\frac{\tilde{U}}{\tilde{W}}, \tag{2.8}$$

$$\dot{\tilde{V}} = \tilde{U} - \frac{1}{R_e}\left(8\tilde{W} + \frac{1}{\tilde{W}}\right)\frac{\tilde{V}}{\tilde{W}}, \tag{2.9}$$

where "$\sim$" indicates dimensionless parameters and the Reynolds number is defined as

$$R_e = \frac{\rho\Gamma}{\mu\pi}. \tag{2.10}$$

We set the initial conditions for (2.8) and (2.9) arbitrarily to be

$$\tilde{U}(t=0) = 10, \text{ and } \tilde{V}(t=0) = 0. \tag{2.11}$$

The set of equations (2.8), (2.9) and (2.11) can be solved analytically. First we assume

$$\tilde{U} = \tilde{W}\cos\theta, \quad \tilde{V} = \tilde{W}\sin\theta, \tag{2.12}$$

thus we have a set of equations:

$$\left.\begin{aligned}
&\frac{\mathrm{d}\tilde{U}}{\mathrm{d}t}=\frac{\mathrm{d}\tilde{W}}{\mathrm{d}t}\cos\theta-\tilde{W}\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\tilde{V}-\frac{1}{R_e}\left(8\tilde{W}+\frac{1}{\tilde{W}}\right)\frac{\tilde{U}}{\tilde{W}}\\
&=-\tilde{W}\sin\theta-\frac{1}{R_e}\left(8\tilde{W}+\frac{1}{\tilde{W}}\right)\cos\theta,\\
&\frac{\mathrm{d}\tilde{V}}{\mathrm{d}t}=\frac{\mathrm{d}\tilde{W}}{\mathrm{d}t}\sin\theta+\tilde{W}\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}=\tilde{U}-\frac{1}{R_e}\left(8\tilde{W}+\frac{1}{\tilde{W}}\right)\frac{\tilde{V}}{\tilde{W}}\\
&=\tilde{W}\cos\theta-\frac{1}{R_e}\left(8\tilde{W}+\frac{1}{\tilde{W}}\right)\sin\theta.
\end{aligned}\right\}\tag{2.13}$$

Multiply the first of (2.13) by $\cos\theta$ and the second by $\sin\theta$, then the sum gives

$$\frac{\mathrm{d}\tilde{W}}{\mathrm{d}t}=-\frac{1}{R_e}\left(8\tilde{W}+\frac{1}{\tilde{W}}\right)\ \rightarrow\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\tilde{W}^2\right)=-\frac{1}{R_e}\left(8\tilde{W}^2+1\right)\tag{2.14}$$

$$\tilde{W}^2+\frac{1}{8}=C_1\exp\left(-16t/R_e\right).\tag{2.15}$$

Multiply the first by $\sin\theta$ and the second by $\cos\theta$, then the subtraction gives

$$\tilde{W}\frac{\mathrm{d}\theta}{\mathrm{d}t}=\tilde{W}\ \rightarrow\ \frac{\mathrm{d}\theta}{\mathrm{d}t}=1\ \rightarrow\ \theta=t+C_2.\tag{2.16}$$

The integration constants C_1 and C_2 are to be determined by initial conditions; $C_1=10^2+\frac{1}{8}$ and $C_2=0$. When $R_e\rightarrow\infty$, $\tilde{W}$=constant and $\theta=t+C_2$. $\tilde{U}$ and $\tilde{V}$ are integrated to obtain the position of the cylinder. The calculation results are shown in figure 2.1.

When $R_e\rightarrow\infty$, the results tend to be the same as the inviscid solution given by Lundgren and Mansour (1991); the cylinder moves with a constant speed along a circular path. When the Reynolds number is finite, the cylinder moves in a spiral fashion. The speed decreased continuously because of the viscous effect and the cylinder eventually stops. The dimensionless stopping time is about 8.36 for R_e =20 and 41.79 for R_e =100. The paths from the start to the end of the motion are shown for R_e =20 and 100 in figure 2.1.

Acknowledgement

This work was supported in part by the NSF under grants from Chemical Transport Systems.

References

[1] Ackeret, J. 1952 Über exakte Lösungen der Stokes-Navier-Gleichungen inkompressibler Flüssigkeiten bei veränderten Grenzbedingungen. *Z. Angew. Math. Phys.* **3**, 259 - 270.

[2] Chen, J.L.S. 1974 Growth of the boundary layer on a spherical gas bubble, *Trans. ASME. J. Appl. Mech.* **41**, 873-878.

[3] Joseph, D. D. and Wang, J. 2004 The dissipation approximation and viscous potential flow, *J. Fluid Mech.* **505**, 365-377.

[4] Lamb, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press. (Reprinted by Dover,1945)

[5] Levich, V.G. 1949 The motion of bubbles at high Reynolds numbers, *Zh. Eksperim. & Teor. Fiz.* **19**, 18.

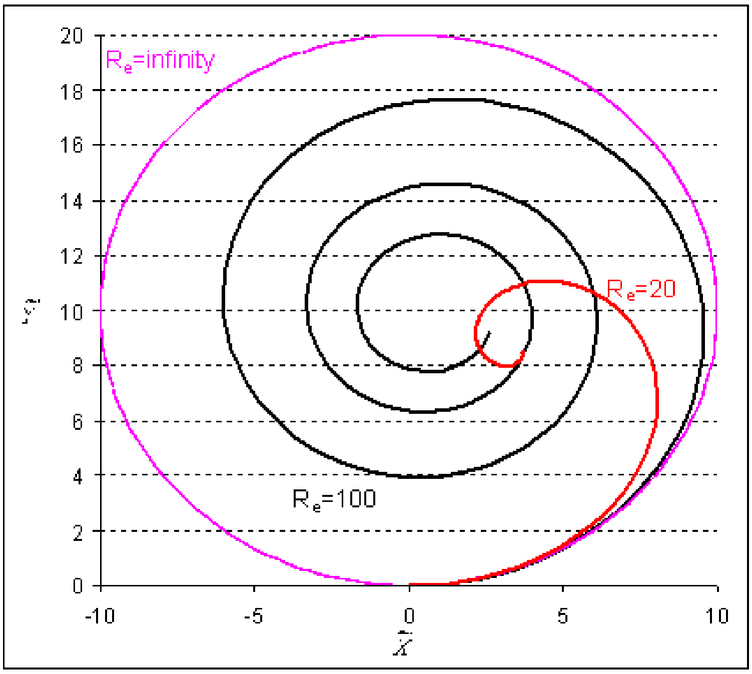

Figure 2.1: The path of the cylinder at different Reynolds numbers. —: $R_e \to \infty$; —: R_e=100; —: R_e=20.

[6] Lundgren, T.S. and Mansour, N.N. 1991 Vortex ring bubbles, *J. Fluid Mech.* **224**, 177 - 196.

[7] Magnaudet, J. and Eames, I. 2000 The motion of high-Reynolds-number bubbles in inhomogeneous flows, *Annu. Rev. Fluid Mech.* **32**, 659-708.

[8] Moore, D.W. 1963 The boundary layer on a spherical gas bubble. *J. Fluid Mech.* **16**, 161-176.

[9] Park, W.C., Klausner, J.F. and Mei, R. 1995 Unsteady forces on spherical bubbles, *Exp. Fluids* **19**, 167-172.

[10] Wang, J. and Joseph, D.D. 2006 Pressure corrections for the effects of viscosity on the irrotational flow outside Prandtl's boundary layer. *J. Fluid. Mech.* **557**, 145 - 165.

[11] Yang, S. and Leal, G.L. 1991 A note on memory-integral contributions to the force on an accelerating spherical drop at low Reynolds number. *Phys. Fluids* **A 3**, 1822-1823.