Potential flow of a cylindrical vortex sheet in a viscous fluid

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Batchelor and Gill 1962 constructed a simple analysis of a cylindrical jet of one fluid into the same fluid. Their analysis is based on potential flow of an inviscid fluid. The interface in their problem is defined by a discontinuity of velocity. The fluid on either side of the discontinuity is the same. Here we show how to generalize the analysis to irrotational flow of a viscous fluid. Our analysis is a paradigm for generalizing problems of potential flow with discontinuous velocity profiles to include the effects of viscosity.

Laplace equations are given by

$$\nabla^2\phi_0 = 0, \quad \nabla^2\phi_1 = 0, \tag{0.1}$$

and Bernoulli equations are

$$\frac{\partial\phi_0}{\partial t} + U\frac{\partial\phi_0}{\partial x} + \frac{p_0}{\rho} = f_0(t), \quad \frac{\partial\phi_1}{\partial t} + \frac{p_1}{\rho} = f_1(t). \tag{0.2}$$

The boundary conditions at the cylindrical surface $r = a + \eta \approx a$ are the kinematic conditions

$$\frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x} = \frac{\partial\phi_0}{\partial r}, \quad \frac{\partial\eta}{\partial t} = \frac{\partial\phi_1}{\partial r}, \tag{0.3}$$

and the normal stress balance

$$p_0 - 2\mu\frac{\partial^2\phi_0}{\partial r^2} = p_1 - 2\mu\frac{\partial^2\phi_1}{\partial r^2}. \tag{0.4}$$

The solution takes the form

$$\eta = A\mathrm{e}^{\mathrm{i}\alpha x + \mathrm{i}n\theta - \mathrm{i}\alpha ct}, \quad \phi_0 = CI_n(\alpha r)\mathrm{e}^{\mathrm{i}\alpha x + \mathrm{i}n\theta - \mathrm{i}\alpha ct}, \quad \phi_1 = DK_n(\alpha r)\mathrm{e}^{\mathrm{i}\alpha x + \mathrm{i}n\theta - \mathrm{i}\alpha ct}. \tag{0.5}$$

[**Computations**] The kinematic conditions give

$$\mathrm{i}\alpha\left(U_0 - c\right)A = \alpha CI_n'(\alpha a), \quad -\mathrm{i}\alpha cA = \alpha DK_n' \tag{0.6}$$

The normal stress balance gives

$$\frac{\partial\phi_0}{\partial t} + U\frac{\partial\phi_0}{\partial x} + 2\nu\frac{\partial^2\phi_0}{\partial r^2} = \frac{\partial\phi_1}{\partial t} + 2\nu\frac{\partial^2\phi_1}{\partial r^2}, \tag{0.7}$$

which is then written as

$$\mathrm{i}\alpha\left(U_0 - c\right)CI_n(\alpha a) + 2\nu C\left(\frac{\mathrm{d}^2 I_n(\alpha r)}{\mathrm{d}r^2}\right)_{r=a} = -\mathrm{i}\alpha cDK_n(\alpha a) + 2\nu D\left(\frac{\mathrm{d}^2 K_n(\alpha r)}{\mathrm{d}r^2}\right)_{r=a}, \tag{0.8}$$

$$\mathrm{i}\alpha\left(U_0 - c\right)CI_n(\alpha a) + 2\nu C\alpha^2 I_n''(\alpha a) = -\mathrm{i}\alpha cDK_n(\alpha a) + 2\nu D\alpha^2 K_n''(\alpha a) \tag{0.9}$$

where $\nu = \mu/\rho$.

The dispersion relation for $c = c_R + \mathrm{i}c_I$ is given by

$$\mathrm{i}\alpha\left(U_0 - c\right)\left[\mathrm{i}\alpha\left(U_0 - c\right)\frac{I_n(\alpha a)}{I_n'(\alpha a)} + 2\nu\alpha^2\frac{I_n''(\alpha a)}{I_n'(\alpha a)}\right] = -\mathrm{i}\alpha c\left[-\mathrm{i}\alpha c\frac{K_n(\alpha a)}{K_n'(\alpha a)} + 2\nu\alpha^2\frac{K_n''(\alpha a)}{K_n'(\alpha a)}\right], \tag{0.10}$$

$$\left(c - U_0\right)^2\frac{I_n(\alpha a)}{I_n'(\alpha a)} + 2\mathrm{i}\nu\alpha\left(c - U_0\right)\frac{I_n''(\alpha a)}{I_n'(\alpha a)} - c^2\frac{K_n(\alpha a)}{K_n'(\alpha a)} - 2\mathrm{i}\nu\alpha c\frac{K_n''(\alpha a)}{K_n'(\alpha a)} = 0 \tag{0.11}$$

Hence the quadratic equation of c gives

$$A_2c^2 + 2A_1c + A_0 = 0 \quad \rightarrow \quad c = -\frac{A_1}{A_2} \pm \sqrt{\left(\frac{A_1}{A_2}\right)^2 - \frac{A_0}{A_2}} \tag{0.12}$$

with

$$A_2 = \frac{I_n(\alpha a)}{I_n'(\alpha a)} - \frac{K_n(\alpha a)}{K_n'(\alpha a)}, \quad A_1 = -U_0\frac{I_n(\alpha a)}{I_n'(\alpha a)} + \mathrm{i}\nu\alpha\frac{I_n''(\alpha a)}{I_n'(\alpha a)} - \mathrm{i}\nu\alpha\frac{K_n''(\alpha a)}{K_n'(\alpha a)}, \tag{0.13}$$

$$A_0 = U_0^2\frac{I_n(\alpha a)}{I_n'(\alpha a)} - 2\mathrm{i}\nu\alpha U_0\frac{I_n''(\alpha a)}{I_n'(\alpha a)} \tag{0.14}$$

For $n = 0$, α, α_a, b and b_a are defined as

$$\alpha = \frac{I_0(k/2)}{I_1(k/2)} = \frac{I_0(k/2)}{I_0'(k/2)}, \quad \alpha_a = \frac{K_0(k/2)}{K_1(k/2)} = -\frac{K_0(k/2)}{K_0'(k/2)}, \tag{0.15}$$

$$b = \alpha - \frac{2}{k} = \frac{I_0''(k/2)}{I_0'(k/2)}, \quad b_a = \alpha_a + \frac{2}{k} = -\frac{K_0''(k/2)}{K_0'(k/2)}. \tag{0.16}$$

It is noted for real k that $k\alpha \to 4$ and $\alpha_a \to 0$ as $k \to 0$, while $\alpha \to 1$ and $\alpha_a \to 1$ as $k \to \infty$; this will be shown in figure 0.1.

In the limit $\alpha a \to \infty$,

$$\frac{I_n(\alpha a)}{I_n'(\alpha a)} \to 1, \quad \frac{I_n''(\alpha a)}{I_n'(\alpha a)} \to 1, \quad -\frac{K_n(\alpha a)}{K_n'(\alpha a)} \to 1, \quad -\frac{K_n''(\alpha a)}{K_n'(\alpha a)} \to 1, \tag{0.17}$$

For $n \geq 1$, the ratios of Bessel functions will be checked later.

If $\nu = 0$, then

$$(c - U_0)^2 \frac{I_n(\alpha a)}{I_n'(\alpha a)} = c^2 \frac{K_n(\alpha a)}{K_n'(\alpha a)} \tag{0.18}$$

$\alpha a \to \infty$

$$(c - U_0)^2 \frac{I_n(\alpha a)}{I_n'(\alpha a)} = c^2 \frac{K_n(\alpha a)}{K_n'(\alpha a)} \;\to\; (c - U_0)^2 + c^2 = 0 \;\to\; c^2 - U_0 c + \frac{U_0^2}{2} = 0 \tag{0.19}$$

$$c = \frac{U_0}{2} \pm \sqrt{\frac{U_0^2}{4} - \frac{U_0^2}{2}} = \frac{U_0}{2}\left[1 \pm \mathrm{i}\right], \quad \to \quad \frac{c}{U_0} = \frac{1}{2}\left[1 \pm \mathrm{i}\right] \tag{0.20}$$

When $\nu \neq 0$ and $\alpha a \to \infty$, we have

$$(c - U_0)^2 + 2\mathrm{i}\nu\alpha\,(c - U_0) + c^2 + 2\mathrm{i}\nu\alpha c = 0 \tag{0.21}$$

$$c = \frac{U_0 - 2\mathrm{i}\nu\alpha}{2} \pm \sqrt{\left(\frac{U_0 - 2\mathrm{i}\nu\alpha}{2}\right)^2 - \frac{U_0^2 - 2\mathrm{i}\nu\alpha U_0}{2}} = \frac{U_0 - 2\mathrm{i}\nu\alpha}{2} \pm \mathrm{i}\sqrt{\frac{U_0^2}{4} + \nu^2\alpha^2} \tag{0.22}$$

$$c_R = \frac{U_0}{2}, \quad c_I = -\nu\alpha \pm \sqrt{\frac{U_0^2}{4} + \nu^2\alpha^2} = -\nu\alpha + \sqrt{\frac{U_0^2}{4} + \nu^2\alpha^2} \tag{0.23}$$

When $c_I = 0$, the neutral state is given by

$$0 = -\nu\alpha + \sqrt{\frac{U_0^2}{4} + \nu^2\alpha^2} \;\to\; \frac{U_0^2}{4} = 0 \tag{0.24}$$

Thus KH instability may arise when $U_0^2 > 0$. The viscosity does not affect the neutral condition, as in the previous results for VPF. We then find that $c_I \to 0$ as $\alpha a \to \infty$.

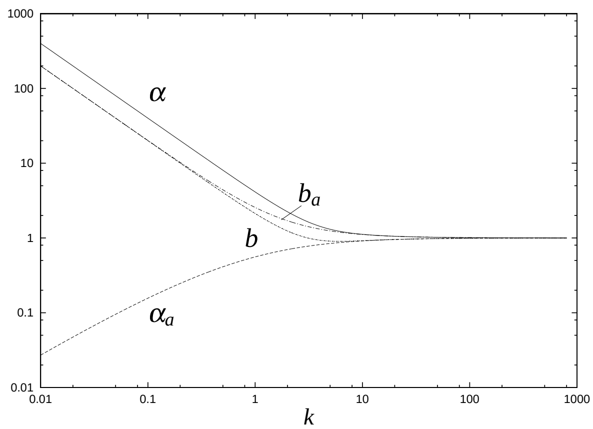

Figure 0.1: Functions α, b_a, b and α_a versus real k; these functions tend to one for $k > 10$.

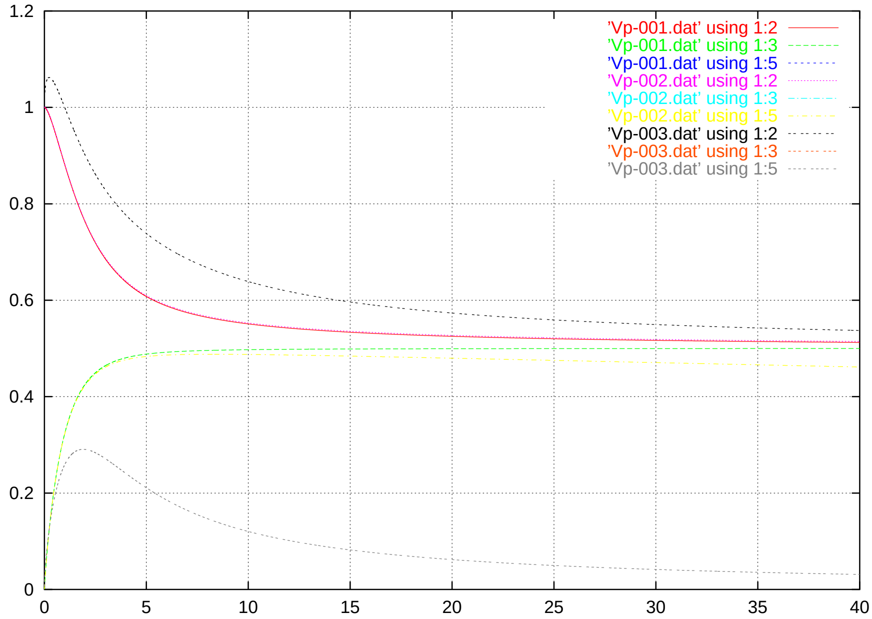

Fig.2a. c_R and c_I versus αa for $n = 0$. (1) c_R (red) c_I (green-top) for $\nu = 0$ (inviscid), (2) c_R (magenta) c_I (yellow) for $\nu = 0.001$, (3) c_R (black) c_I (gray) for $\nu = 0.1$. The positive value c_I is shown in the figure, since the solution is of the form $c = c_R \pm \mathrm{i}c_I$.

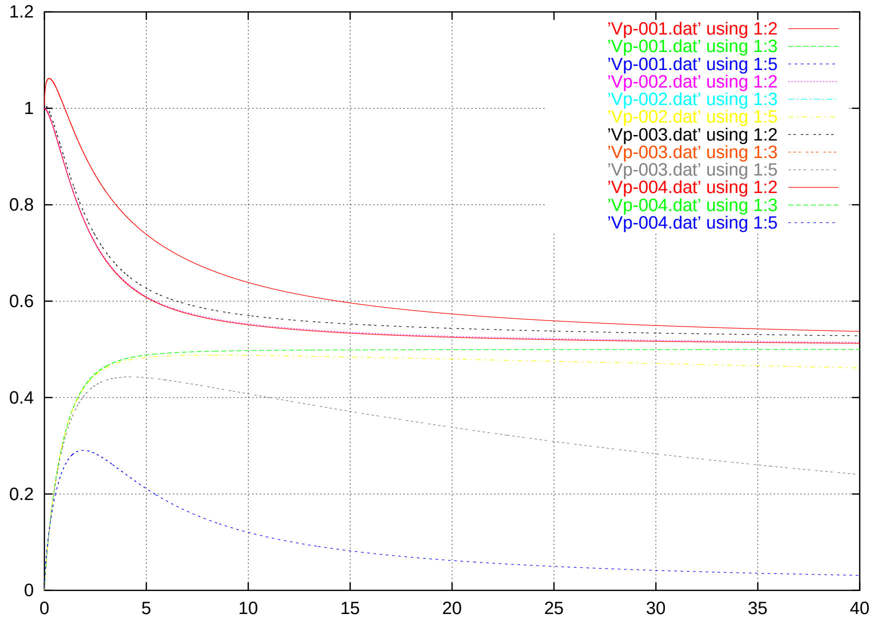

Fig.2b. c_R and c_I versus αa for $n = 0$. (1) c_R (red) c_I (green-top) for $\nu = 0$ (inviscid), (2) c_R (magenta) c_I (yellow) for $\nu = 0.001$, (3) c_R (black) c_I (gray) for $\nu = 0.01$, (4) c_R (red-top) c_I (blue-bottom) for $\nu = 0.1$. The data (3) is added to Fig.2a.

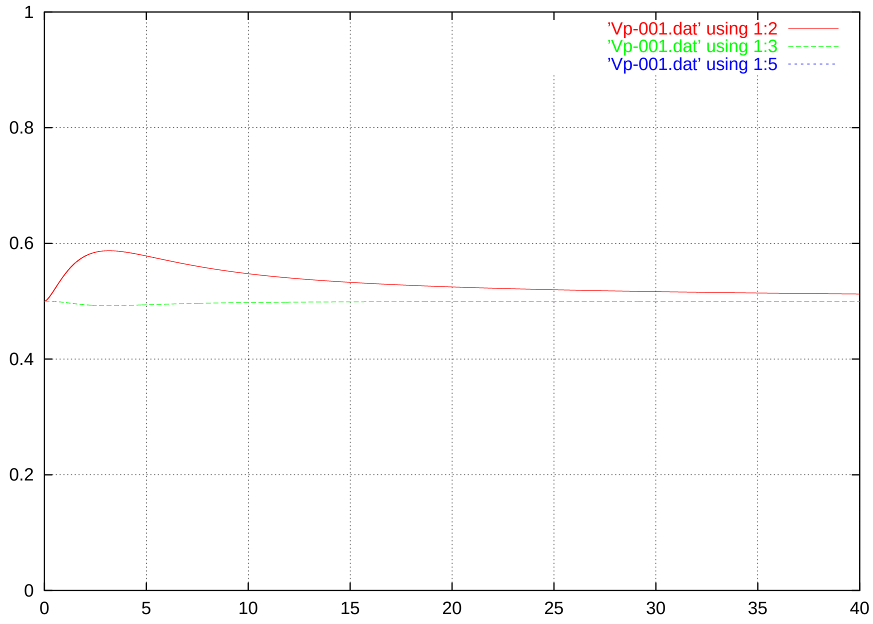

Fig.3a. c_R and c_I versus αa for $n = 1$. (1) c_R (red) c_I (green) for $\nu = 0$ (inviscid).

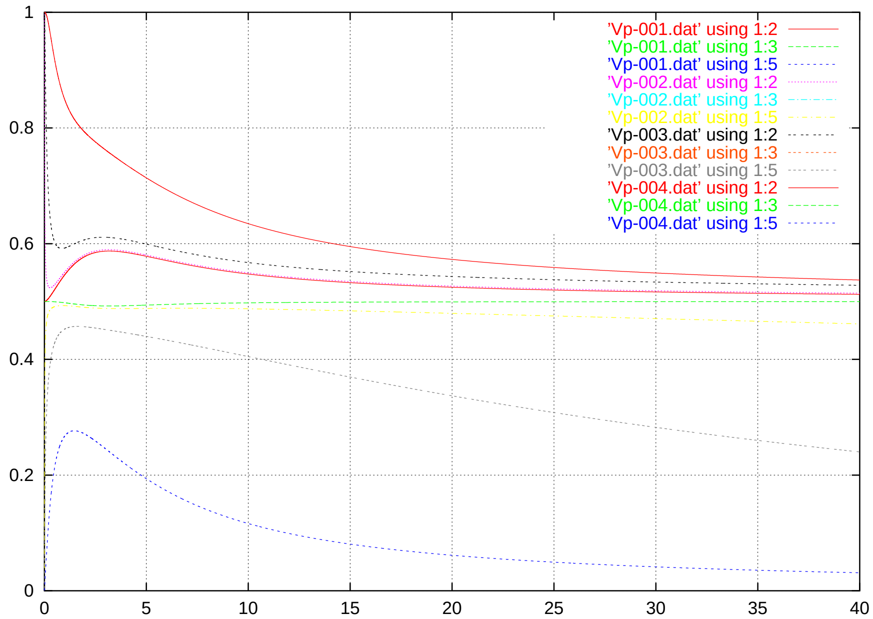

Fig.3b. c_R and c_I versus αa for $n = 1$. (1) c_R (red) c_I (green) for $\nu = 0$ (inviscid), (2) c_R (magenta) c_I (yellow) for $\nu = 0.001$, (3) c_R (black) c_I (gray) for $\nu = 0.01$, (4) c_R (red-top) c_I (blue-bottom) for $\nu = 0.1$.

References

[1] Batchelor, G. K. and Gill, A. E. 1962, Analysis of the stability of axsymmetric jets, *J. Fluid Mech.* **14**, 529-551.

Formulae of the modified Bessel functions

$$K_{n-1}(z)+K_{n+1}(z)=-2K_n'(z),\ \ K_{n-1}(z)-K_{n+1}(z)=-\frac{2n}{z}K_n(z) \tag{0.25}$$

$$\rightarrow\ K_{n-1}(z)=-K_n'(z)-\frac{n}{z}K_n(z)\ \rightarrow\ K_0(z)=-K_1'(z)-\frac{1}{z}K_1(z) \tag{0.26}$$

$$\rightarrow\ K_1'(z)=-K_0(z)-\frac{1}{z}K_1(z)\ \rightarrow\ -\frac{K_1'(z)}{K_1(z)}=\frac{K_0(z)}{K_1(z)}+\frac{1}{z} \tag{0.27}$$

$$I_{n-1}(z)+I_{n+1}(z)=2I_n'(z),\ \ I_{n-1}(z)-I_{n+1}(z)=\frac{2n}{z}I_n(z) \tag{0.28}$$

$$\rightarrow\ I_{n-1}(z)=I_n'(z)+\frac{n}{z}I_n(z)\ \rightarrow\ I_0(z)=I_1'(z)+\frac{1}{z}I_1(z) \tag{0.29}$$

$$\rightarrow\ I_1'(z)=I_0(z)-\frac{1}{z}I_1(z)\ \rightarrow\ \frac{I_1'(z)}{I_1(z)}=\frac{I_0(z)}{I_1(z)}-\frac{1}{z} \tag{0.30}$$

$$\begin{aligned} K_1''(z)&=-K_0'(z)-\frac{1}{z}K_1'(z)+\frac{1}{z^2}K_1(z)=\left(1+\frac{1}{z^2}\right)K_1(z)-\frac{1}{z}\left(-K_0(z)-\frac{1}{z}K_1(z)\right)\\ &=\left(1+\frac{2}{z^2}\right)K_1(z)+\frac{1}{z}K_0(z) \end{aligned} \tag{0.31}$$

$$\begin{aligned} I_1''(z)&=I_0'(z)-\frac{1}{z}I_1'(z)+\frac{1}{z^2}I_1(z)=\left(1+\frac{1}{z^2}\right)I_1(z)-\frac{1}{z}\left(I_0(z)-\frac{1}{z}I_1(z)\right)\\ &=\left(1+\frac{2}{z^2}\right)I_1(z)-\frac{1}{z}I_0(z) \end{aligned} \tag{0.32}$$