Irrotational disturbances of the motion of a viscous fluid

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The approach of viscous potential flow has been applied to the analysis of stability of various flow configurations (see Joseph, Funada and Wang 2007). According to this theory, disturbances of the basic state are assumed to be irrotational but the viscosity of the fluid is not set to zero. Addressing the question, can we assume irrotational disturbances for any given basic flow, even if this is rotational? The answer is no. We reason this conclusion in the following paragraphs. In so doing, we obtain a necessary condition that may be verified such that the assumption of irrotational perturbations is plausible.

Consider an incompressible Newtonian fluid. Consider a basic state of motion for which the vorticity transport equation is

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{\Omega} = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{U} + \nu \Delta \boldsymbol{\Omega}, \tag{1}$$

where $\boldsymbol{U}$ is the basic state and $\boldsymbol{\Omega} = \nabla \times \boldsymbol{U}$ is the basic vorticity. Consider 3D disturbances $\boldsymbol{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$. Then, the vorticity transport equation for the disturbed flow may be writen as

$$\frac{\partial}{\partial t}(\boldsymbol{\Omega} + \boldsymbol{\omega}) + (\boldsymbol{U} + \boldsymbol{u}) \cdot \nabla(\boldsymbol{\Omega} + \boldsymbol{\omega}) = (\boldsymbol{\Omega} + \boldsymbol{\omega}) \cdot \nabla(\boldsymbol{U} + \boldsymbol{u}) + \nu \Delta \boldsymbol{\Omega} + \nu \Delta \boldsymbol{\omega}, \tag{2}$$

where $\omega = \nabla \times \boldsymbol{u}$. Substracting (1) from (2) yields:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{\omega} + \boldsymbol{u} \cdot \nabla \boldsymbol{\Omega} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{U} + \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} + \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nu \Delta \boldsymbol{\omega}. \tag{3}$$

If we set $\boldsymbol{\omega} = \mathbf{0}$ for irrotational disturbances, we are left with,

$$\boldsymbol{u} \cdot \nabla \boldsymbol{\Omega} = \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u}, \tag{4}$$

which can also be written as

$$\nabla \times \boldsymbol{u} \times \boldsymbol{\Omega} = \mathbf{0}. \tag{5}$$

Notice that $\boldsymbol{u} = \nabla \phi$ does not make (4) vanish, in general. Also notice from (4) that if $\boldsymbol{\Omega}$ is a non-zero constant vector, then $\nabla \boldsymbol{\Omega} = \mathbf{0}$ and we must satisfy,

$$\boldsymbol{\Omega} \cdot \nabla \boldsymbol{u} = 0,$$

which is always true for 2D flow. An example is the basic flow $\boldsymbol{U} = U(z)\, \boldsymbol{e}_1 = Kz\, \boldsymbol{e}_1$ for $K =$ constant. Moreover, conditions (4) or (5) hold for an irrotational basic motion, for which $\boldsymbol{\Omega} \equiv 0$. This is the case for Kelvin-Helmholtz instability, capillary instability and progressive/standing waves on a plane or sphere, among others.

The previous discussion leads us to formulate the following statement:

> Let $\boldsymbol{U}$ be the velocity of the basic state that satisfies the incompressible Navier-Stokes equations, with $\boldsymbol{\Omega} = \nabla \times \boldsymbol{U}$, and define
>
> $$f \equiv \boldsymbol{u} \cdot \nabla \boldsymbol{\Omega} - \boldsymbol{\Omega} \cdot \nabla \boldsymbol{u},$$
>
> where $\boldsymbol{u}$ is the velocity perturbation of the basic state. The constraint $f = 0$ is a necessary condition for the assumption of purely irrotational disturbances to hold for $\boldsymbol{u}$.

In other words, if one assumes the disturbances $\boldsymbol{u}$ to be purely irrotational and the basic state $\boldsymbol{U}$ is such that $f \neq 0$, then a contradiction arises in (3). This contradiction is resolved by considering rotational perturbations.