A model for the onset of vortex breakdown

By K. Mahesh

1. Motivation and objectives

A large body of information exists on the breakdown of incompressible streamwise vortices. Less is known about vortex breakdown at high speeds. An interesting example of supersonic vortex breakdown is the breakdown induced by the interaction of vortices with shock waves. The flow in supersonic engine inlets and over high-speed delta wings constitute technologically important examples of this phenomenon, which is termed 'shock-induced vortex breakdown'.

In this report, we propose a model to predict the onset of shock-induced vortex breakdown. The proposed model has no adjustable constants, and is compared to both experiment and computation. The model is then extended to consider two other problems: the breakdown of a free compressible vortex, and free incompressible vortex breakdown. The same breakdown criterion is used in all three problems to predict the onset of breakdown. Finally, a new breakdown map is proposed that allows the simultaneous comparison of data from flows ranging from incompressible breakdown to breakdown induced by a shock wave.

This work is described in detail by Mahesh (1996); only the prominent results are presented in this report.

2. Accomplishments

2.1 Shock-induced breakdown

Figure 1 shows a schematic of the interaction between a streamwise vortex and a normal shock wave. The axial flow is from left to right. The variables x and r are used to denote the axial and radial coordinate respectively. The axial and swirl components of velocity are denoted by U and v_θ respectively, and p, ρ, and T represent the pressure, density, and temperature. The subscripts '∞' and 'c' correspond to values in the free-stream and the centerline of the vortex, and the states upstream and downstream of the shock wave are respectively denoted by the subscripts '1' and '2' (e.g., $p_{\infty 2}$ denotes the free-stream pressure downstream of the shock wave). This report will occasionally refer to the variables, γ, Γ, and $M_{\infty 1}$. γ denotes the ratio of specific heats and is taken as 1.4. Γ is the swirl number of the vortex and is defined as $\Gamma = v_{\theta m}/U_\infty$ where $v_{\theta m}$ denotes the maximum swirl velocity. M_∞ is the free-stream Mach number, defined as $M_\infty = U_\infty/c_\infty$.

The model is obtained as follows. First, a simple criterion for breakdown of the upstream vortex is proposed. The properties of the upstream vortex are then substituted into the criterion. The resulting equation is then rearranged to obtain an expression for the critical swirl number above which the vortex would break down.

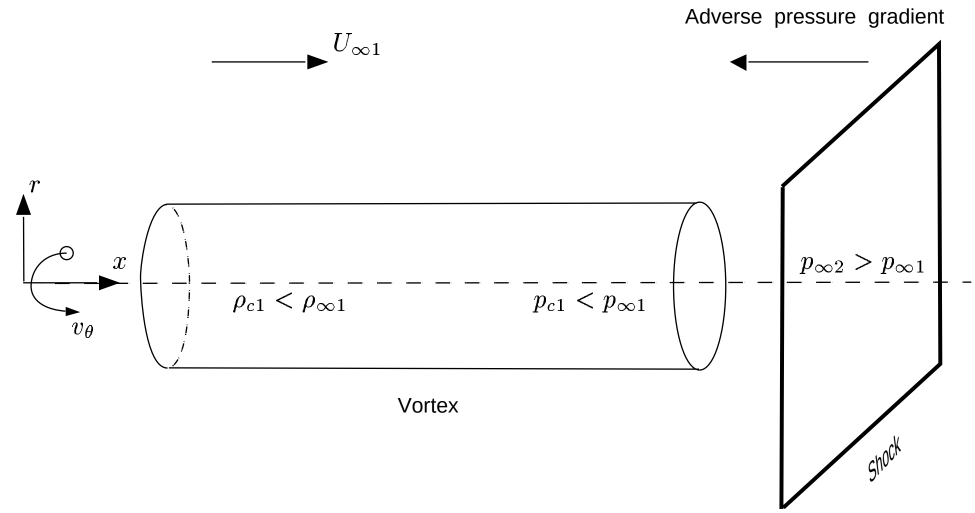

FIGURE 1. Schematic of the interaction between a streamwise vortex and a normal shock wave.

The breakdown criterion is based upon an approximation to the axial momentum equation at the centerline of the vortex. Consider the vortex impinging upon the shock wave. The vortex experiences an adverse streamwise pressure rise, which may be quantified by the pressure difference, $p_{\infty 2} - p_{c1}$. The fluid in the vortex has a certain inertia in the streamwise direction, which may be quantified by the streamwise momentum flux, $\rho_{c1} U_{c1}^2$. Breakdown is assumed to occur if the axial pressure rise exceeds the upstream streamwise momentum flux, thereby stagnating the flow; i.e., if

$$\begin{aligned} p_{\infty 2} - p_{c1} &\geq \rho_{c1} U_{c1}^2 \\ &\geq \rho_{c1} U_{\infty 1}^2 \left(1 + \frac{\Delta U}{U_{\infty 1}} \right)^2 \end{aligned} \tag{1}$$

where ΔU denotes the upstream excess in axial velocity at the centerline. If the axial velocity is uniform, then $\Delta U = 0$. Using $p_{\infty 1}, \rho_{\infty 1}$ and $T_{\infty 1}$ to non-dimensionalize the pressure, density, and temperature respectively, Eq. 1 may be rewritten for uniform axial velocity as

$$\widetilde{p}_{\infty 2} - \widetilde{p}_{c1} = \gamma \widetilde{\rho}_{c1} M_{\infty 1}^2, \tag{2}$$

where the 'tilde' is used to denote non-dimensional variables.

The Rankine vortex is used to approximate the swirl velocity in the upstream vortex. Two different idealizations of the thermodynamic field in the upstream vortex are considered: spatially uniform stagnation temperature and spatially uniform entropy. The radial momentum equation is then easily integrated to obtain the density and pressure at the centerline of the upstream vortex, i.e. $\widetilde{p}_{c1}$ and $\widetilde{\rho}_{c1}$, are obtained in terms of Γ and $M_{\infty 1}$. The Rankine-Hugoniot equations for a normal shock express $\widetilde{p}_{\infty 2}$ in terms of the upstream Mach number, $M_{\infty 1}$. Substituting for

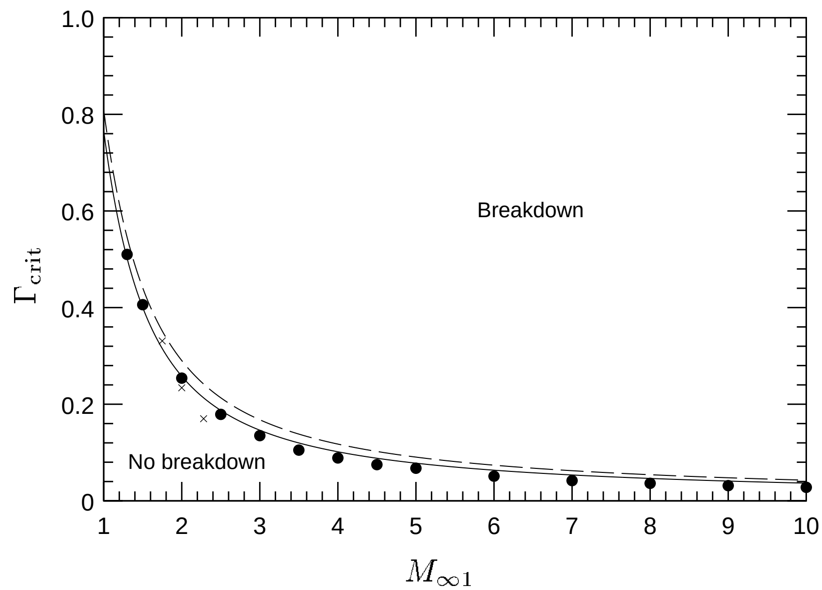

FIGURE 2. Comparison of predicted critical swirl number to experiment and computation of shock-induced vortex breakdown. ——— (Prediction: uniform stagnation temperature), ---- (Prediction: uniform entropy), • (Computation), × (Experiment).

$\widetilde{p}_{c1}$, $\widetilde{\rho}_{c1}$, and $\widetilde{p}_{\infty 2}$ into the above breakdown criterion will therefore yield an expression for the critical swirl number $\Gamma_{\rm crit}$ in terms of Mach number of the shock wave. This expression is given below.

Uniform stagnation temperature vortex

The critical swirl number is given by:

$$\Gamma_{\rm crit} = \frac{1}{M_{\infty 1}} \sqrt{\frac{2}{\gamma - 1} \left\{ 1 - \left(\frac{1}{1 + \gamma M_{\infty 1}^2} \left[1 + \frac{2\gamma}{\gamma + 1} (M_{\infty 1}^2 - 1) \right] \right)^{\frac{\gamma - 1}{2\gamma}} \right\}}. \tag{3}$$

Uniform entropy vortex

The critical swirl number is given by the following implicit equation:

$$1 + \frac{2\gamma}{\gamma + 1}(M_{\infty 1}^2 - 1) - [1 - (\gamma - 1)\Gamma_{\rm crit}^2 M_{\infty 1}^2]^{\frac{\gamma}{\gamma - 1}} = \gamma M_{\infty 1}^2 [1 - (\gamma - 1)\Gamma_{\rm crit}^2 M_{\infty 1}^2]^{\frac{1}{\gamma - 1}}. \tag{4}$$

Results for the critical swirl number are presented for the case where the axial velocity is uniform. Figure 2 shows the predicted values of the critical swirl number as a function of the Mach number of the shock. The predicted values are compared to the experimental values reported by Delery *et al.* (1984) and the computations by Erlebacher *et al.* (1996). Good agreement is seen. The critical swirl number is predicted to decrease with increasing Mach number as observed. According to the proposed criterion, this decrease in $\Gamma_{\rm crit}$ is due to a combination of two factors:

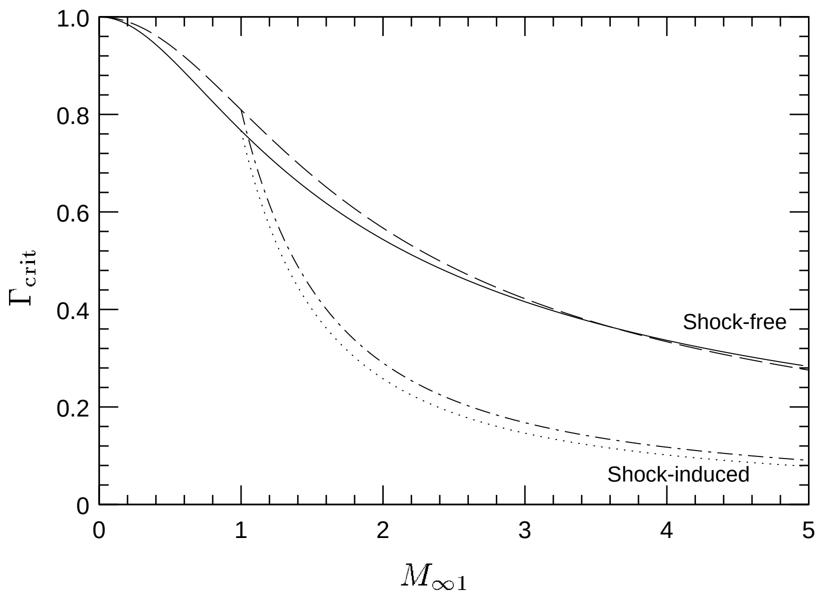

FIGURE 3. Predicted critical swirl number for shock-free vortex breakdown compared to the prediction for shock-induced breakdown. ——— (Shock-free: uniform stagnation temperature), ---- (Shock-free: uniform entropy), ······· (Shock-induced: uniform stagnation temperature), —-— (Shock-induced: uniform entropy).

increase in the adverse pressure rise (due to $\widetilde{p}_{\infty 2}$ increasing while $\widetilde{p}_{c1}$ decreases) and decrease in streamwise momentum flux (due to $\widetilde{\rho}_{c1}$ decreasing) with increasing Mach number.

The ability of the model to predict the onset of shock-induced breakdown was further evaluated (Mahesh 1996) by comparing to the experimental data of Metwally *et al.* (1989). The 'strong interactions' observed experimentally were seen to lie in the region where the model predicts breakdown, while the 'weak interaction' regions were seen to lie in the predicted region of non-breakdown. Also, the influence of an excess/deficit in the centerline axial velocity, and obliquity of the shock wave on the critical swirl number was considered (Mahesh 1996). Jet-like profiles of the axial velocity were observed to delay breakdown, while a wake-like profile made the vortex more susceptible to breakdown. For fixed upstream Mach number, Γ_{crit} was predicted to increase as the shock became increasingly oblique.

2.2 Shock-free breakdown of a compressible vortex

The breakdown of a free axisymmetric vortex, i.e. breakdown in the absence of an externally imposed pressure gradient, is considered in this section. The arguments used are identical to those in breakdown induced by a shock. In the absence of the shock, the vortex discharges into the atmosphere. The difference between atmospheric pressure, ($p_{\infty 1}$), and the pressure at the vortex centerline, (p_{c1}), provides the adverse pressure rise that causes breakdown. Breakdown of the vortex is therefore assumed to occur when

$$1 - \widetilde{p}_{c1} = \gamma \widetilde{\rho}_{c1} M_{\infty 1}^2 \tag{5}$$

S_{crit}	
Quasi-cylindrical	1.41
Axisymmetric N-S	1.35
Bossel	1.12
Squire	1.4
Benjamin	1.4
Num. simulation	1.28
Spall *et al.*	1.37
Present	**1.4**

TABLE 1. Prediction of critical swirl number for incompressible vortex breakdown compared to other approaches. All data other than the present reproduced from review article by Delery (1993).

which is identical to the expression obtained when $\widetilde{p}_{\infty 2}$ is set equal to one in Eq. 2. The corresponding expressions for the critical swirl number are given below.
Uniform stagnation temperature vortex

$$\Gamma_{\text{crit}} = \frac{1}{M_{\infty 1}} \sqrt{\frac{2}{\gamma - 1}\left[1 - \left(\frac{1}{1 + \gamma M_{\infty 1}^2}\right)^{\frac{\gamma - 1}{2\gamma}}\right]}. \tag{6}$$

Uniform entropy vortex

$$1 - [1 - (\gamma - 1)\Gamma_{\text{crit}}^2 M_{\infty 1}^2]^{\frac{\gamma}{\gamma - 1}} = \gamma M_{\infty 1}^2 [1 - (\gamma - 1)\Gamma_{\text{crit}}^2 M_{\infty 1}^2]^{\frac{1}{\gamma - 1}}. \tag{7}$$

Figure 3 shows the predicted values of the critical swirl number as a function of the free-stream Mach number. Also shown (for supersonic flow) are the values obtained for breakdown induced by a shock wave at the same Mach number. Compressibility is seen to make the vortex more susceptible to breakdown. A similar trend was noted by Keller (1994). This trend may be explained by noting that increasing the free-stream Mach number decreases the centerline pressure and density, thereby increasing the adverse pressure rise while decreasing the axial momentum flux. The predicted values of Γ_{crit} in the absence of the shock are seen to be greater than those predicted for shock-induced breakdown. This trend can be explained by noting that the pressure rise across the shock wave produces a larger adverse pressure rise for the same upstream momentum flux.

2.3 Incompressible vortex breakdown

Figure 3 shows that as $M_{\infty 1}$ tends towards 0, Γ_{crit} tends towards 1. An incompressible vortex in the absence of externally imposed adverse pressure gradients is

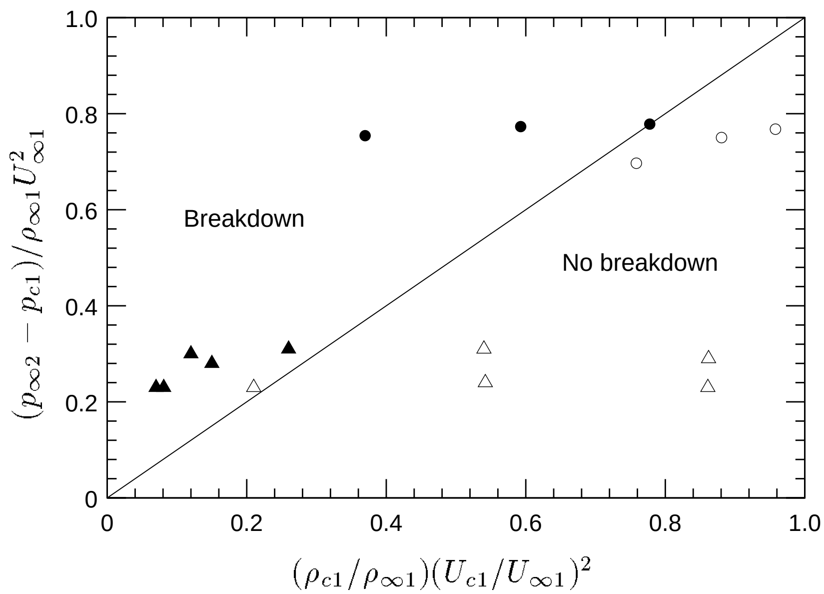

FIGURE 4. Evaluation of the proposed breakdown map in predicting the onset of vortex breakdown. • (Experiment: breakdown), ∘ (Experiment: no breakdown), ▴ (Computations: breakdown), ▵ (Computations: no breakdown).

therefore predicted to undergo breakdown at a critical swirl number of one. In a recent review article, Delery (1993) documents (section 3.4.5 of his paper) critical swirl numbers for incompressible vortex breakdown as predicted by different theories. He considers a Burger's vortex and defines a swirl parameter S, which is related to the swirl number Γ by,

$$S = \frac{\Gamma}{1 - \mathrm{e}^{-1.256}} = 1.398\ \Gamma. \tag{8}$$

Thus $\Gamma_{\text{crit}} = 1$ corresponds to $S_{\text{crit}} = 1.398 \sim 1.4$. Table 1 reproduces from Delery's paper the critical swirl numbers obtained by different approaches. Our simple criterion is seen to agree well with the other data.

2.4 A 'universal' breakdown map

The preceding sections presented results for the onset of vortex breakdown by plotting the critical swirl number as a function of Mach number. The curve $\Gamma_{\text{crit}} = \Gamma_{\text{crit}}(M_{\infty 1})$ defined the boundary between the regimes of breakdown and non-breakdown. However, the critical swirl number is not universal (as also observed by Delery, 1993). Different curves were obtained for Γ_{crit} for the different problems. In this section, a breakdown map that allows a common breakdown boundary to be defined for all of the above mentioned problems is proposed. It is suggested that a plot of $p_{\infty 2} - p_{c1}$ against $\rho_{c1} U_{c1}^2$ could be used to map the onset of vortex breakdown. The proposed map could even be used for incompressible vortex breakdown, and would be expected to adequately represent the onset of breakdown induced by pressure gradients acting over distances that are small as compared to a characteristic lengthscale of the vortex. The curve $p_{\infty 2} - p_{c1} = \rho_{c1} U_{c1}^2$ (the 45° line) would act as the boundary between the breakdown and non-breakdown regimes. The proposed map is illustrated in Fig. 4. Note that the pressure rise and momentum flux are non-dimensionalized by $\rho_{\infty 1} U_{\infty 1}^2$ to allow incompressible data to be plotted. Experimental data on normal shock/vortex interaction from Metwally *et al.* (1989) is also shown. The breakdown and non-breakdown cases are seen to be appropriately delineated. Also plotted are data on oblique shock/vortex interaction from the recent computations by Nedungadi and Lewis (1996). Numerical solutions of the constant stagnation temperature Burger's vortex were used to obtain the centerline pressure and density since the vortices considered had a velocity deficit. With the exception of one point (run 8 in their paper), the proposed map appropriately delineates the 'strong' and 'weak' interactions observed in the computations.

3. Conclusions

A simple inviscid model was proposed to predict the onset of breakdown in an axisymmetric vortex. Three problems were considered: shock-induced breakdown, free compressible breakdown, and free incompressible breakdown. A formula with no adjustable constants was derived for the critical swirl number in all three problems. Comparison to experimental and computational data showed good agreement. Finally, a new breakdown map that allows a common breakdown boundary to be defined for a wide range of flows was proposed.

Acknowledgments

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