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Advances in Output Feedback Control of Transient Energy Growth in a Linearized Channel Flow

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Transient energy growth (TEG) is a primary mechanism for bypass transition in many wall-bounded shear flows. Here, we investigate the efficacy of reducing TEG in a linearized channel flow with feedback controllers that use wall shear-stress sensors and wall-normal blowing/suction actuators. Owing to established performance limitations of observer-based controller designs within the context of TEG, we study static output feedback linear quadratic regulation (SOF-LQR) strategies for control. SOF-LQR is found to outperform optimal observer-based feedback designs, and to reduce TEG of spanwise disturbances relative to the uncontrolled flow. We further show that by introducing an appropriate set of additional observables, SOF-LQR controllers can reduce TEG associated with streamwise and oblique disturbances as well. In fact, we show that by selecting a small number of appropriate observables, SOF-LQR controllers can fully recover full-state LQR performance.

I. Introduction

Transient energy growth (TEG) of flow perturbations provides a linear non-modal mechanism for bypass transition [1]. As a result, TEG reduction has been a common goal of many feedback flow control strategies aiming to delay transition to turbulence. Full-state-feedback control strategies—e.g., linear quadratic regulation (LQR)—have been shown to successfully delay transition in numerical simulations [2–4]. Unfortunately, full-state information is rarely available in practice; thus, a controller must be designed to operate using only a limited set of sensor measurements (i.e., sensor-based output feedback control).

The standard approach for sensor-based output feedback design combines an optimal observer to estimate the flow state from available sensor measurements, and to then determine the control action based on the state estimate (i.e., observer-based feedback). Although observer-based feedback guarantees modal stability, it was recently established that such strategies have inherent performance limitations with regards to non-modal mechanisms for TEG [5]. Indeed, observer-based feedback is incapable of completely eliminating TEG whenever the uncontrolled system exhibits TEG.

Further, our previous investigations of channel flow control have shown that observer-based feedback by means of linear quadratic Gaussian (LQG) optimal control can degrade TEG relative to the uncontrolled flow, making LQG and other observer-based feedback strategies poor control candidates in the context of TEG reduction [6]. In the same study, we showed that a static output feedback LQR (SOF-LQR) approach could be leveraged to overcome these TEG performance limitations. SOF-LQR controllers tend to be more difficult to design than LQG controllers, which is part of the reason for the slow adoption of these controllers.

In the present study, we investigate the SOF-LQR approach further. As we will show here, for a channel flow actuated using wall-normal transpiration and sensed using wall shear-stress information, SOF-LQR controllers are able to improve worst-case TEG performance relative to the uncontrolled flow for purely spanwise disturbances. However, for oblique and streamwise disturbances, SOF-LQR performance is inconsistent. In some instances, SOF-LQR is found to result in larger TEG than the uncontrolled flow—though SOF-LQR always yields superior TEG performance than LQG control. The inconsistency in TEG reduction of SOF-LQR control does not come as a surprise: linear quadratic control approaches do not target TEG directly, but rather aim to minimize the balance of perturbation and actuation energies integrated over an infinite

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time horizon [7]. Thus, an optimal linear quadratic control may yield large TEG, while still providing an optimal (minimal integrated energy) solution. Nonetheless, linear quadratic techniques have demonstrated sufficient TEG reduction to yield success in transition delay in past numerical studies with the luxury of (full information) state feedback control. Motivated by this fact, in the present work, we seek to extend the performance capabilities of SOF-LQR control by introducing a small set of additional (and appropriately selected) observables. To do so, we propose a method for determining an appropriate set of observables that enables SOF-LQR control to recover full-state LQR performance. The method is then applied to and demonstrate on the linearized channel flow configuration. We further show that by relaxing the requirement for full-state feedback performance recovery, the method is able to yield an even smaller set of observables that enhances TEG performance relative to the uncontrolled flow.

In Section II, we present the linearized channel flow model used in this study. We also discuss TEG and the importance of worst-case analysis for comparing alternative control strategies. In Section III, we discuss linear quadratic optimal control strategies, summarize the associated synthesis procedures, and present relevant performance results. In Section IV, we propose and investigate a method for introducing an appropriate set of additional observables to improve SOF-LQR performance. Conclusions are presented in Section V.

II. Linearized Channel Flow and Transient Energy Growth

In the present study, we consider a channel flow configuration at $Re = 3000$ with wall-normal transpiration at the upper and lower walls serving as actuation. The channel walls are assumed to be infinite, resulting in periodic boundary conditions along the streamwise and spanwise directions. The flow control configuration is modeled using a Fourier-Chebyshev-Fourier spectral collocation method, described in detail in [8].

In order to study TEG, we consider the linearized dynamics of flow perturbations about a laminar equilibrium solution (i.e., a parabolic velocity profile). To do so, we linearize the Navier-Stokes equations about the laminar equilibrium solution, which yields a linear state-space description [8],

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where the state vector $x(t) \in \mathbb{R}^n$ represents perturbations about the laminar equilibrium; the input vector $u(t) \in \mathbb{R}^m$ represents the rate of change in wall-normal transpiration velocity; the output vector $y \in \mathbb{R}^p$ represents sensor measurements—taken to be shear-stress at the channel walls, unless specified otherwise—and $t \in \mathbb{R}$ denotes time.

For an initial flow perturbation $x(t_o) = x_o$, the system response is given in terms of the matrix exponential $x(t) = \mathrm{e}^{A(t-t_o)}x_o$. The associated perturbation energy is given as,

$$E(t) = x^T(t)Qx(t), \tag{2}$$

where $Q = Q^T > 0$. Further, the maximum TEG is defined as [5],

$$G = \max_{t \geq t_0} \max_{E(t_0) \neq 0} \frac{E(t)}{E(t_0)}, \tag{3}$$

which results from a so-called *worst-case* or *optimal disturbance* [9]. The perturbation energy will never exceed its initial value when $G = 1$. In contrast, certain perturbations will result in non-trivial TEG whenever $G > 1$. In the present study, we will always focus on the worst-case response of the system by considering the maxmimum TEG, G. For controlled systems, the worst-case response will be computed for the controlled system. For TEG analysis, this worst-case analysis is only fair; without performing an objective worst-case analysis, the performance limitations of certain control structures can remain hidden [5]—potentially giving a false impression that one controller outperforms another in terms of TEG.

In the present study, we consider a number of control strategies, with specific synthesis techniques described in Section III below. The goal of control is to improve worst-case TEG performance of the closed-loop system relative to the uncontrolled (open-loop) system (1). That is, the maximum TEG of the controlled flow (G_{CL}) should be less than the maximum TEG of the uncontrolled flow (G_{OL}). One means of doing so is to use full-state feedback control,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ u(t) &= Kx(t), \end{aligned} \tag{4}$$

where $K \in \mathbb{R}^{m \times n}$ is the control gain; however, full-state feedback control is usually restricted to numerical studies in which full-state information is directly available for feedback. An observer (state estimator) can be used to estimate the state from output measurements, which can then be fed to a full-state feedback controller. The resulting observer-based feedback system will be,

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(C\hat{x}(t) - y(t)) \\
y(t) &= Cx(t) \\
u(t) &= K\hat{x}(t),
\end{aligned} \tag{5}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the state estimate and $L \in \mathbb{R}^{n \times p}$ is the observer gain associated with the state estimator. The final control structure considered here is one associated with a static output feedback (SOF) controller, in which the inputs are determined directly from the outputs,

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) \\
u(t) &= Fy(t),
\end{aligned} \tag{6}$$

where $F \in \mathbb{R}^{m \times p}$ is the SOF control gain.

In the remainder, whenever the notion of worst-case response is discussed, we refer to the worst-case response of the associated dynamic system (i.e., (1) for uncontrolled; (4) for full-state feedback; (5) for observer-based feedback; and (6) for SOF control). Thus, in performing worst-case analysis for the controlled systems, we consider the optimal disturbance for the specific closed-loop system under consideration, not that of the uncontrolled flow. In the present study, we make use of the numerical algorithm presented in [10] to compute all relevant optimal disturbances.

III. Linear Quadratic Optimal Controllers

Feedback controllers have been shown to reduce TEG in various shear flows. In particular, the linear quadratic regulator (LQR) is a well-known design technique that has been demonstrated to reduce TEG and delay transition [2–4]. The "standard" LQR synthesis problem is based on solving,

$$\min_{u(t)} J = \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)dt \tag{7}$$

subject to the linear dynamic constraint $\dot{x}(t) = Ax(t) + Bu(t)$, where $R > 0$. The resulting LQR controller is a full-state feedback law of the form $u(t) = Kx(t)$, which can be determined from the solution of an algebraic Riccati equation [11]. LQR controllers exhibit robustness to various forms of uncertainty and can be tuned to reduce TEG, making them appealing candidates for control. Unfortunately, the full-state feedback nature of standard LQR controllers limits such approaches to numerical studies, in which full information is readily available for feedback.

To overcome the full-information requirement of standard LQR controllers, the so-called *linear quadratic Gaussian* (LQG) controller is proposed as a (dynamic) output feedback alternative. The approach essentially combines a Kalman-Bucy filter—a linear quadratic optimal state estimator—with the standard LQR controller within an observer-based feedback structure. Despite "optimality" of the LQG solution, the approach has inherent performance limitations with regards to TEG suppression. Indeed, as proven in [5], LQG controllers will *always* exhibit $G > 1$ whenever the uncontrolled system exhibits $G > 1$. Further, our previous work suggests that LQG can dramatically degrade TEG performance within the context of channel flows [6].

To overcome the performance limitations of observer-based feedback structures, we previously proposed reconsidering the standard LQR problem above, but with a structural SOF constraint on the controller, $u(t) = Fy(t)$. The benefit of the SOF control structure is that the input is determined directly from the measured output, removing the need for an observer; indeed, SOF controllers constitute semi-proper compensators, and so satisfy the necessary conditions for overcoming the performance limitations of observer-based feedback with respect to TEG [5].

The design of SOF-LQR controllers is slightly more involved than the more commonly adopted LQR and LQG synthesis approaches; however, optimal controllers can be determined. The closed-loop dynamics under SOF-LQR control are of the form $\dot{x} = (A + BFC)x$. Thus, introducing the SOF constraint within the standard LQR objective function yields the modified objective function,

$$J = \int_0^\infty x^T(t)[Q + (FC)^T R(FC)]x(t)dt. \tag{8}$$

An iterative Anderson-Moore algorithm can be used to solve the resulting optimal control problem; here, we make use of the specific algorithm proposed in [6].

We investigate the worst-case response for each of these linear quadratic optimal control approaches on the linearized channel flow system described in Section II. Figure 1 shows each controller applied to the linearized channel flow at $Re = 3000$ for streamwise wavenumber $\alpha = 0$ and spanwise wavenumber $\beta = 1$. The response for the LQG controller is not actually the worst-case; rather, the initial condition on the estimator was initialized with zeros. Despite this fact, the LQG controller is found to significantly degrade TEG performance relative to the worst-case response of even the uncontrolled flow! The LQR controller performs better than all controllers, while the SOF-LQR controller manages to reduce TEG relative to the uncontrolled flow. In the remainder, we will no longer report the LQG controller's performance, since the associated TEG is consistently worse than that of both SOF-LQR control and the uncontrolled flow. The resulting worst-case TEG for other purely spanwise disturbances in the controlled (G_{CL}) and the uncontrolled (G_{OL}) systems are tabulated in Table 1, and the associated percent reductions in TEG relative to the uncontrolled flow (i.e., $(G_{\mathrm{OL}} - G_{\mathrm{CL}})/G_{\mathrm{OL}} \times 100\%$) are reported in Table 2.

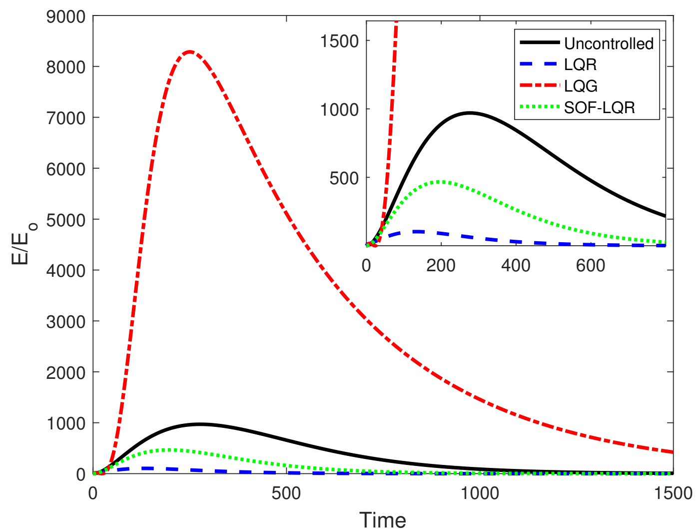


Figure 1: SOF-LQR outperforms LQG as an output feedback control approach for TEG reduction in the linearized channel flow. Here, $\alpha = 0$, $\beta = 1$ at $Re = 3000$. The LQR controller achieves the most TEG reduction, but requires direct access to full state information. The LQG response is not the worst-case; rather, the estimator is intialized to zero. All other responses are worst-case responses.

Controller Type	$\alpha = 0, \beta = 1$	$\alpha = 0, \beta = 2$	$\alpha = 0, \beta = 3$	$\alpha = 0, \beta = 4$	$\alpha = 0, \beta = 5$
Uncontrolled	970	1761	1430	956	630
LQR	104	288	359	327	264
SOF-LQR	466	971	971	717	502

Table 1: SOF-LQR reduces the worst-case TEG relative to the uncontrolled system for purely spanwise disturbances. The table lists the maximum TEG.

Controller Type	$\alpha = 0, \beta = 1$	$\alpha = 0, \beta = 2$	$\alpha = 0, \beta = 3$	$\alpha = 0, \beta = 4$	$\alpha = 0, \beta = 5$
LQR	89%	84%	75%	66%	58%
SOF-LQR	52%	45%	32%	25%	20%

Table 2: SOF-LQR reduces the worst-case TEG relative to the uncontrolled system for purely spanwise disturbances. The table lists the percent reduction in maximum TEG: i.e., $(G_{\mathrm{OL}} - G_{\mathrm{CL}})/G_{\mathrm{OL}} \times 100\%$.

IV. Introducing Additional Observables to Enhance Control Performance

Despite the success of SOF-LQR control at reducing TEG, the results reported in Section III were only associated with purely spanwise disturbances (i.e., $\alpha = 0$, $\beta \neq 0$). For oblique ($\alpha \neq 0$, $\beta \neq 0$) and purely streamwise ($\alpha \neq 0$, $\beta = 0$) disturbances, SOF-LQR control based on wall shear-stress measurements alone resulted in inconsistent TEG performance. In some instances, SOF-LQR was found to reduce TEG relative to the uncontrolled system; other times, SOF-LQR was found to degrade this performance—albeit, the TEG was still consistently less than that resulting using LQG control. In contrast, full-state feedback LQR consistently reduced TEG relative to the uncontrolled flow, for all wave numbers pairs considered.

Since we cannot realize full-state feedback in practice, and since observer-based feedback (e.g., LQG) dramatically degrades the worst-case performance and possesses inherent performance limitations [5], we instead propose to introduce additional observables to enhance the TEG performance of SOF-LQR control. In the limit that the number of linearly independent observables p is the same as the number of state variables n, full-state feedback performance can be recovered. Here, we will investigate the possibility of using fewer observables ($p < n$) for recovering full-state feedback performance with SOF control.

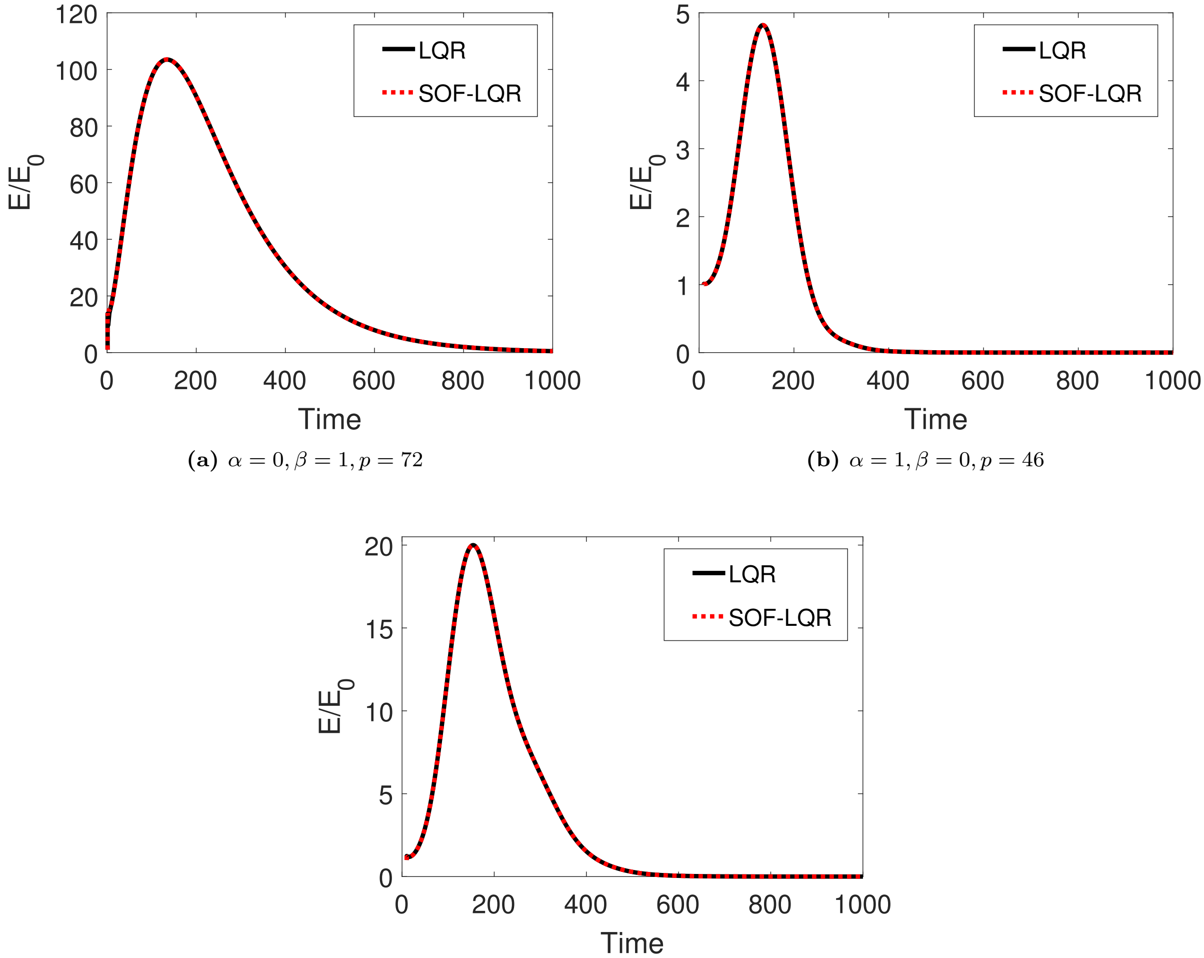
In our approach, we begin by defining the output to be the full state,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Ix, \end{aligned} \tag{9}$$

where I is the $n \times n$ identity matrix. In this case, the full-state LQR controller and the SOF-LQR controller will be the same. By comparing the norms of each column in the resulting full-state feedback gain matrix K, we can determine the relative importance of various outputs (here, simply states) for control. In the present study, we make use of the 2-norm in making these comparisons. Columns of K with a small norm contribute less to the closed-loop response than columns of K with a large norm. Consider, for example, that a column of all zeros and the associated row in I could be removed completely without altering the closed-loop system response. Therefore, we identify the q columns in K with the least norm relative to the other columns. Denoting the column-indices of these q columns as $\{i_1, \ldots, i_q\}$, we proceed to remove columns $\{i_1, \ldots, i_q\}$ from K and the associated rows $\{i_1, \ldots, i_q\}$ from I. This elimination procedure yields a reduced gain matrix for SOF-LQR control $F = K_p \in \mathbb{R}^{m \times p}$, where $p = n - q$. The associated set of observables can be determined by setting $C = I_p$ in the output equation, where I_p is the identity matrix with rows $\{i_1, \ldots, i_q\}$ removed. Note that no effort was made to recompute the optimal SOF-LQR gain with the new set of observables in this study, though doing so may improve performance further. Further, the procedure can be generalized to the case where I is replaced with an arbitrary output matrix C, corresponding to other physically relevant outputs for the proposed down-selection approach. For simplicity, we focus on the specific case described above, in which the resulting set of observables will be a subset of the state variables. Doing so also allows physical and modeling insights to be extracted from the analysis.

We perform this design procedure on the linearized channel flow at $Re = 3000$ and report results in Figure 2. Interestingly, the number of states (observables here) needed for SOF-LQR to recover full-state feedback LQR performance is significantly less than the state dimension. For the purely spanwise disturbance $(\alpha, \beta) = (0, 1)$, the number of observables is reduced from $n = 198$ in the full-state feedback case to $p = 72$ in the SOF-LQR case, a reduction of 63.6%. For the purely streamwise disturbance $(\alpha, \beta) = (1, 0)$, the total number of observables is reduced from $n = 198$ to $p = 46$, corresponds to a reduction of 76.7% in the number of observables. For the oblique disturbance $(\alpha, \beta) = (1, 1)$ considered, the total number of observables is reduced from $n = 396$ to $p = 152$, a reduction of 61.6%.

Note that the selection of observables in the procedure above was determined so as to completely recover the full-state feedback LQR performance. If we relax this requirement, an even smaller number of observables can be found such that SOF-LQR will reduce TEG relative to the uncontrolled flow. For the oblique



(a) $\alpha = 0, \beta = 1, p = 72$

(b) $\alpha = 1, \beta = 0, p = 46$

(c) $\alpha = 1, \beta = 1, p = 152$

Figure 2: SOF-LQR can recover full-state LQR performance by feeding back an appropriate set of p outputs.

disturbance $(\alpha, \beta) = (1, 1)$ the number of observables needed for SOF-LQR to achieve LQR performance was found to be $p = 152$. If we further reduce the number of observables to $p = 38$ using the procedure outlined above, we find that SOF-LQR continues to reduce TEG relative to the uncontrolled flow (see Figure 3). We found that further attempts to reduce the number of observables would no longer result in a stabilizing controller for the linear system.

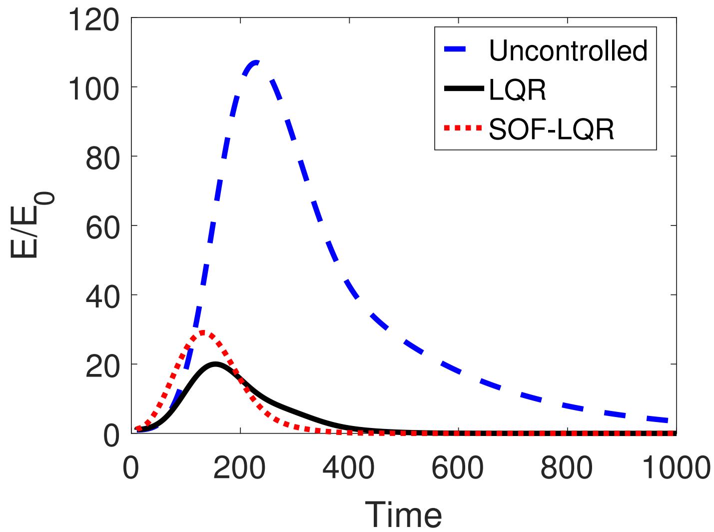


Figure 3: SOF-LQR with $p = 38$ observables reduces TEG relative to the uncontrolled flow with $(\alpha, \beta) = (1, 1)$. Recall, full-state LQR recovery was achieved with $p = 152$ for this case (see Figure 2c).

V. Conclusion and Discussion

In this study, we investigated the performance of alternative linear quadratic optimal control strategies for TEG reduction in a linearized channel flow. For the purposes of sensor-based output feedback using shear-stress measurements at the channel walls, worst-case analysis revealed SOF-LQR to be a more suitable approach than LQG control for TEG reduction. Despite the superior performance of SOF-LQR over LQG control for TEG reduction, the SOF-LQR controllers did not always reduce TEG relative to the uncontrolled flow. This short-coming was attributed to the fact that linear quadratic methods are designed to optimize the integrated energy over an infinite time horizon, and will not necessarily yield TEG minimizing controllers [7]. We further hypothesized that wall shear-stress information alone was insufficient to reduce TEG in all cases. To overcome this potential limitation, we proposed a method for introducing an appropriate set of additional observables to improve SOF-LQR performance. Using this approach, we found that SOF-LQR controllers based on an appropriate set of observables can recover full-state feedback LQR performance. The ability to do so suggests that these approaches may also provide insights to guide future investigations on control-oriented model reduction. Further, relaxing the requirement for full-state feedback performance recovery, we found that the number of observables could be reduced even further, while still leading to enhanced SOF-LQR control performance. Future work will focus on extending the proposed approach for observable selection. In particular, we plan to investigate the same problem with an additional constraint that the set of observables be "physical observables" (e.g., shear-stress, pressure, velocity) that can be measured using existing sensor technologies.

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