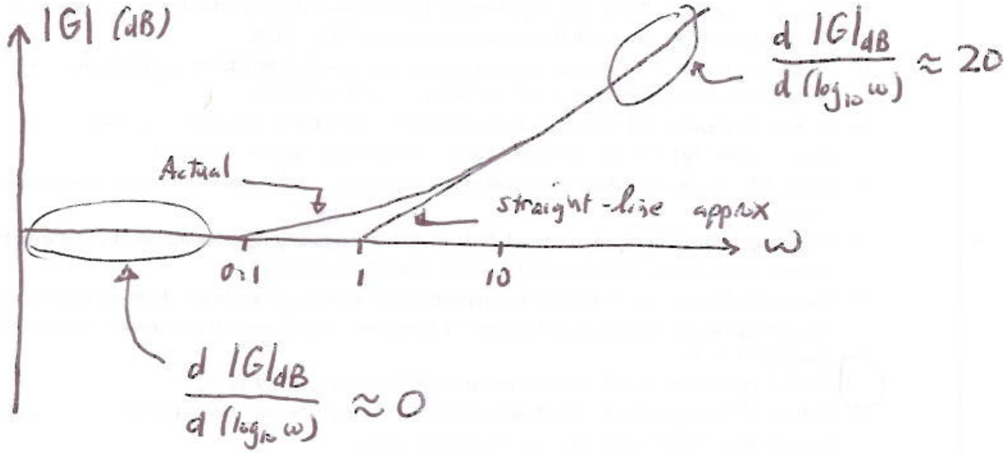
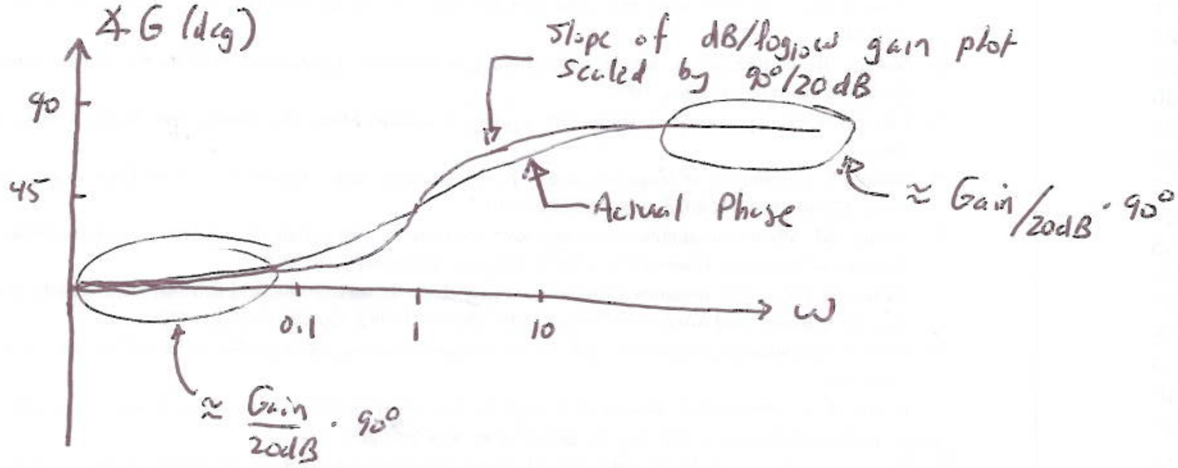
Bode Phase Formula

Bode derived a formula that connects the magnitude and phase of a transfer function with all poles and zeros in the open LHP. This formula will provide some insight into our design of "Lead" controllers.

Consider: $G(s) = s+1$





Observation: The net phase change in G from $\omega=0$ to $\omega=\omega_0$ is approximately equal to the slope slope of the magnitude curve scaled by $90^\circ/20\,dB$.

$$\measuredangle G(j\omega_0) - \measuredangle G(j0) \approx \frac{90^\circ}{20dB}\cdot \left.\frac{d\,(|G|_{dB})}{d(\log_{10}\omega)}\right|_{\omega=\omega_0}$$

Exact Bode Gain / Phase Formula

Assume $G(s)$ has all of its poles and zeros in the open LHP. Then the net phase change in $G(j\omega)$ as ω goes from $\omega = 0$ to $\omega = \omega_0$ is given by:

$$\underbrace{\angle G(j\omega_0) - \angle G(j0)}_{\text{Net phase change in radians.}} = \int_{-\infty}^{\infty} \frac{d\,|G|_{dB}}{d\nu} \frac{1}{20\pi} \ln \coth \frac{|\nu|}{2} \, d\nu$$

where $\nu = \ln(\omega/\omega_0)$

Net phase change in radians.

This exact formula states that the net phase change is related to a weighted average of the slope on the dB-vs-$\log_{10}\omega$ gain plot of G.

For our purposes, two approximations to the exact Bode gain/phase formula will be useful:

A) If the slope of the gain plot is relatively constant over 1 decade of frequency (i.e. the slope is approx. constant for $\omega \in [\frac{\omega_0}{\sqrt{10}}, \sqrt{10}\,\omega_0]$) then the accumulated phase is:

$$\angle G(j\omega_0) - \angle G(0) \approx \frac{90^\circ}{20\,dB} \cdot \left. \frac{d(|G|_{dB})}{d(\log_{10}\omega)} \right|_{\omega=\omega_0}$$

$\Rightarrow$ Net phase change is $\approx 90^\circ$ for every $20\,dB/dec$ in slope

B) If the slope of the gain plot is $> \frac{20\,dB}{dec} \cdot M$ over 1 decade surrounding ω_0 then the accumulated phase approximately satisfies:

$$\angle G(j\omega_0) - \angle G(0) > 90^\circ \cdot M$$

Basic Loopshaping Theorem

Our basic loopshaping design procedure is based on 3 characteristics:

a) Make $|L|$ large at low frequencies so that $|S|$ is small, i.e. good tracking at low frequencies.

This should say "high" frequencies.

b) Make $|L|$ small at low frequencies so that $|T|$ is small, i.e. good noise rejection at high frequencies.

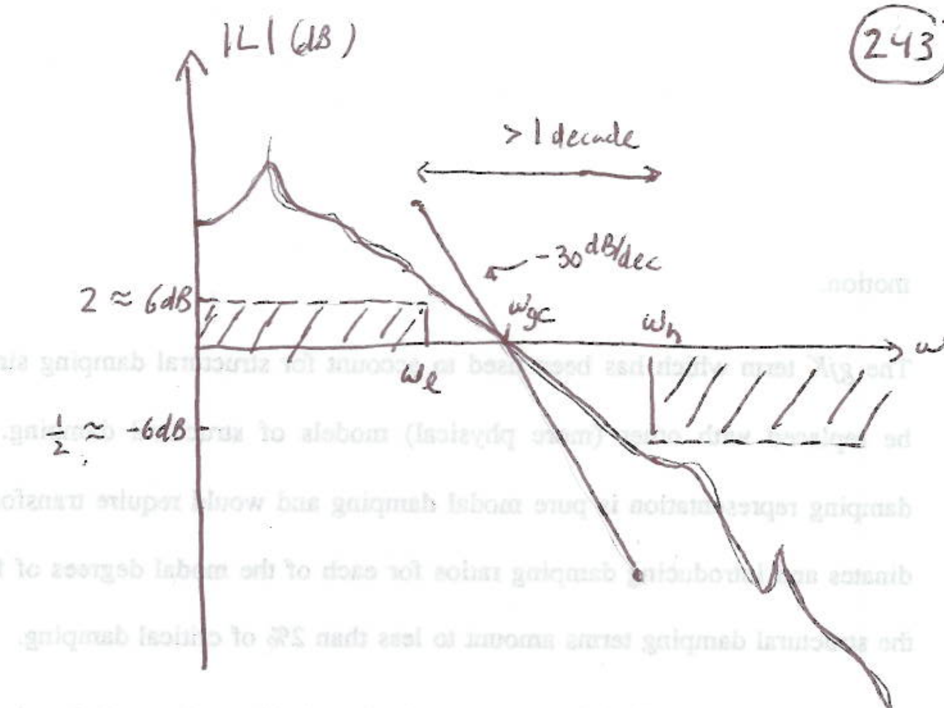
c) Make the slope of $|L|$ "shallow" (≥ -30 dB/dec) at the gain crossover (mid frequencies) so that the closed-loop is stable and has good robustness.

It was fairly straightforward to make the connection between the loop gain $|L|$ and the requirements on $|S|$ and $|T|$ (characteristics a and b). For example, see lecture 31.

~~We do~~

We used the Bode gain/phase formula to show that if the slope of $|L| > -30$ dB/dec for approximately 1 decade of frequency surrounding the gain crossover then the system will have approximately $\pm 45°$ of phase margin. For example see p 212-215. This is part of characteristic c but we still don't have any assurances that the closed-loop will actually be stable. We'll use the Nyquist stability theorem to show that our loop-shaping design will achieve a stable closed-loop.

Theorem



Theorem Assume the loop transfer function $L(s)$ satisfies:

- $L(s)$ has no poles or zeros in the closed RHP
- $L(0) > 0$
- L has one gain crossover frequency, ω_{cg}.
- The slope of $|L| \geq -30$ dB/dec for at least 1 decade of frequencies around ω_{cg}: $\omega_l \leq \omega_{cg} \leq \omega_h$
- Outside this frequency interval around ω_{cg}:
  - $|L| \geq 2$ at lower frequencies ($\omega \leq \omega_l$)
  - $|L| \leq 1/2$ at higher frequencies ($\omega \geq \omega_h$)

If all these conditions are satisfied then the closed-loop is stable, approximately achieves good classical margins (± 6dB, $\pm 45°$) and has $|S(j\omega)| \leq 2.5$ for all ω

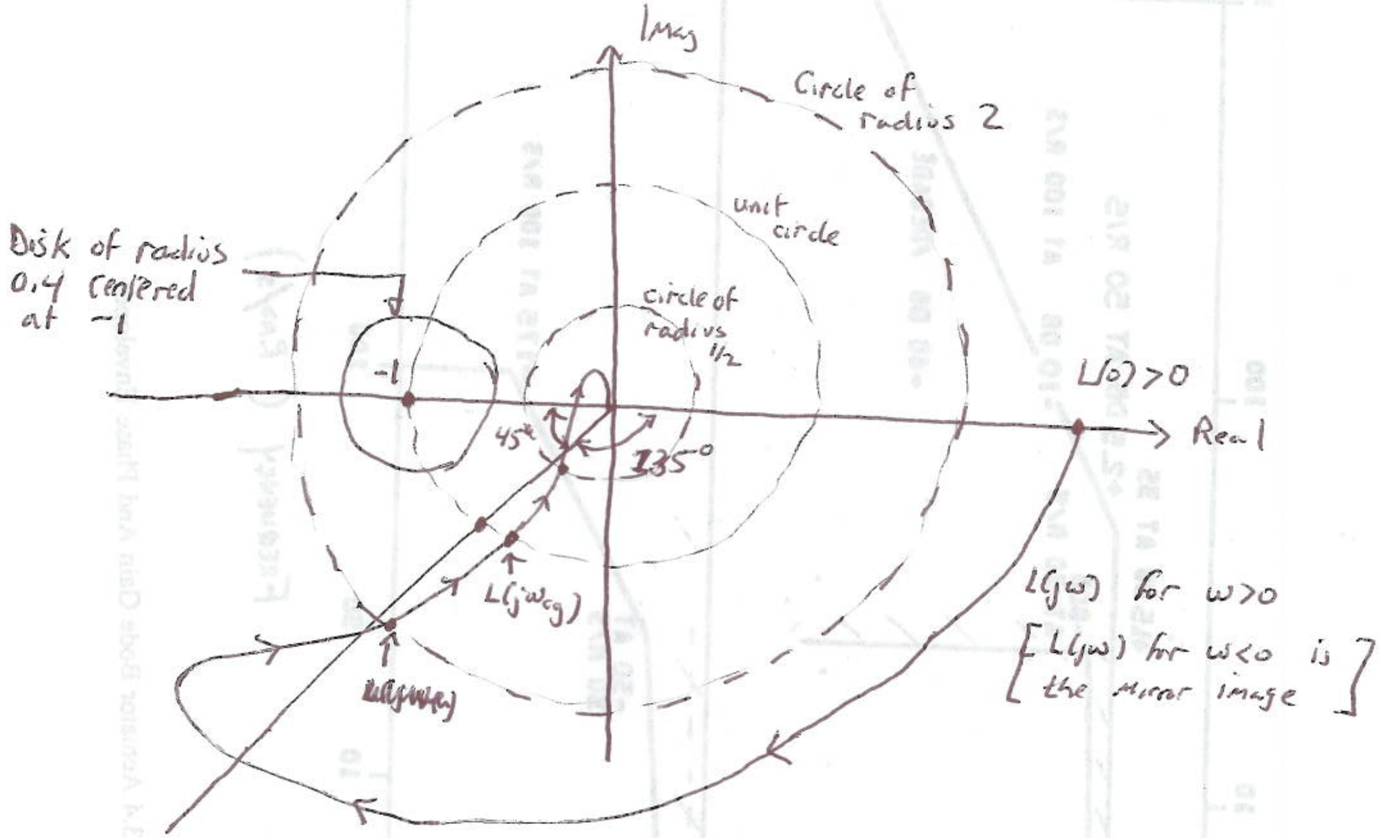
Before we sketch a proof of this result, there are a few important remarks:

a) The theorem can be extended/generalized to systems with poles and/or zeros in the closed RHP. An important case is loop transfer functions with integrators (poles at $s=0$)

b) The condition $L(0) > 0$ is easy to satisfy. If $G(0) > 0$ then choose $K(0) > 0$. If $G(0) < 0$ then choose $K(0) < 0$.

Proof (Due to A. Packard)

The proof uses the Bode gain/phase relation and the Nyquist stability theorem. We'll sketch the constraints on the Nyquist plot of L based on the given assumptions.



L has one gain crossover frequency where $|L(j\omega_{cg})| = 1$. Since the slope of $|L| > -30\,\mathrm{dB/dec}$ for at least one decade around ω_{cg}, the accumulated phase must approximately satisfy $\measuredangle L(j\omega_{cg}) > -135^\circ$ (by the Bode gain/phase formula). For lower frequencies ($\omega \le \omega_\ell$) the Nyquist plot lies outside the disk of radius 2 and for higher frequencies ($\omega \ge \omega_h$) the Nyquist plot lies inside a disk of radius 1/2. Thus it is not possible for the Nyquist plot of L to encircle -1. By the Nyquist theorem:

$$\begin{pmatrix}\#\text{ of closed-loop}\\ \text{RHP poles}\end{pmatrix} = \begin{pmatrix}\#\text{ of open loop}\\ \text{RHP poles}\end{pmatrix} - \begin{pmatrix}\#\text{ of CCW encirclements}\\ \text{of } -1\end{pmatrix} = 0$$

Thus the closed-loop will be stable. Moreover, the system will approx have $\pm 45^\circ$ of phase margin. In addition any phase crossover frequencies, $\measuredangle L(j\omega) = -180^\circ$), can only occur when $|L|$ is ≥ 2 or $\le 1/2$. Thus the system will approx have $\pm 6\,\mathrm{dB}$ of gain margin. Finally, $L(j\omega)$ will not enter the disk of radius 0.4 centered at -1, There fore $|S| \le 2.5$ at all ω. (because $|1+L| \ge 0.4$ at all ...)