Optimal Pseudo-Steady-State Estimators for Systems with Markovian Intermittent Measurements

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Abstract

A state estimator design is described for discrete time systems having observably intermittent measurements. A stationary Markov process is used to model probabilistic measurement losses. The stationarity of the Markov process suggests an analagous stationary estimator design related to the Markov states. A precomputable time-varying state estimator is proposed as an alternative to Kalman's optimal time-varying estimation scheme applied to a discrete linear system with Markovian intermittent measurements. An iterative scheme to find optimal precomputed estimators is given. The results here naturally extend to Markovian jump linear systems.

1 Introduction

Discrete-time linear systems with a time-varying output matrix governed by a random process are considered. The optimal estimation problem described later in this paper has a well-known solution: the Time-Varying Kalman Estimator (TVKE). To implement the TVKE, real-time computation to update error covariance matrices must be carried out. For the systems under consideration the time-varying output matrix is not known a priori, so the TVKE error covariances cannot be pre-computed. The idea underlying the proposed design is that expected estimation errors far in the future have little dependence on measurement losses far in the past. Therefore, repeating patterns of measurement loss/reception should correspond to repeated error covariances maintained by a TVKE.

A pseudo-steady-state (PSS) for an estimator is defined in this paper by a collection of estimation error covariances paired with patterns of measurement loss/reception (referred to as measurement modes). The proposed estimator uses a fixed corrector gain for each defined measurement mode. For example, associate one mode with loss and one with reception – one corrector gain is when a measurement is received(R), and another when a measurement is lost(L). An extension is to use four modes. A different corrector gain is used after seeing each of the following possible loss/reception (L/R) sequences: LL; LR; RL; RR. Note that the gain to be used at the next instant depends on the current mode (RR may follow RR or LR, but not LL or RL).

After the effect of initial error covariance assumptions decays, the performance of these estimators and optimal time-varying Kalman estimators are expected to be comparable. However, the expected future error covariance (including expectation over possible measurement loss sequences) for TVKE may not be bounded. In such cases, PSS estimator designs cannot exist.

Very recently, Allam et. al. remarked that there has not been much attention paid to the problem of estimation subject to intermittent measurements [1]. They remark that sensors in manufacturing systems have been known to exhibit intermittent behavior. They use the terminology "marked point process" to denote that measurements occur at particular instances. Their approach is based upon using the entire history of measurements for estimation, whereas the proposed approach limits the amount of historical information used at the expense of optimality.

Intermittent communication processes are common in distributed control systems communicating over a network, as described by Seiler and Sengupta [7]. They analyze the effect of communication losses in a coordinated vehicle control problem. Markovian Jump Linear System descriptions are used in formulating control design problems as linear matrix inequalities. The Jump Linear Quadratic Gaussian problem was studied by Ji and Chizeck [5, 3]. A linear plant subject to measurement losses may be realized as a jump linear system with known plant modes. The estimation problem in this paper is closely related to work in [5, 3, 7].

The remainder of the paper is organized as follows: First, a class communication loss models are defined in terms of Markov chains. Generic time-varying estimation and TVKE are then briefly reviewed. A description of the proposed multi-modal time-varying estimator structure is followed by conditions for when its application will result in a pseudo-steady-state. An iterative method for finding the set of corrector gains which minimizes pseudo-steady-state error covariances is then given with existence/convergence conditions.

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2 Communication loss model

Consider information transmitted (perhaps across a wireless network) in the form of packets at equally spaced time intervals. Assume the transmission time is negligible, and that each successive packet can be immediately classified as received(R) or lost(L). Define $\Theta = \{1(\mathrm{R}), 2(\mathrm{L})\}$. If time is indexed by k, let the value of $\theta(k) \in \Theta$ indicate the status of the kth transmitted packet of information. A sequence $\theta(k)$ for $k = 0, 1, 2, \dots$ will be called a communication loss pattern. A communication loss process is a probability distribution on the space of all possible communication loss patterns.

A Bernoulli process can be used to model communication losses. Each packet has an independent probability of being lost. A more sophisticated (and perhaps more realistic in the case of wireless channels) process should correlate $\theta(k)$ with $\theta(i)$ for $0 \leq i < k$. The two state markov chain shown in Figure 1(a) describes correlated $\theta(k)$. One state corresponds to communication loss, 2, and the other to reception, 1, as indicated in the figure. The probability that a received packet follows a lost packet (LR) is α, and that it follows a received packet (RR) is γ. A selection of γ large and α small corresponds to a scenario where missed communications are likely to occur in bursts of consecutive losses which themselves do not occur frequently. Given $\theta(k)$, the probability that $\theta(k+N) = 2$ (Loss) can be computed. Far in the future, this probability is independent of $\theta(k)$:

$$\lim_{N\to\infty} \Pr\{\ \theta(k+N) = 2 \mid \theta(k)\ \} = \frac{1-\gamma}{1-\gamma+\alpha}$$

To model more sophisticated loss processes, a Markov chain with more than two states may be used. Define the state of a stationary Markov chain, $n(k) \in \mathcal{N} = \{1, \dots, N_n\}$ and the sequence of states visited by the Markov chain as $\mathcal{N}_k = \{n(0), n(1), \dots, n(k)\}$. A probability distribution on the state-space, $\mathcal{N}$, is represented by a row of N_n probabilities summing to 1. The one-step transition matrix of the markov chain, $P = [p_{ij}]_{i,j\in\mathcal{N}}$, governs the evolution of probability distributions on $\mathcal{N}$, and has the following properties: **1)** $0 \leq p_{ij} \leq 1\ \ \forall\, i, j \in \mathcal{N}$; **2)** $p_{ij} = \Pr\{\ n(k) = j \mid n(k-1) = i\ \}\ \ \forall\, k > 0$; **3)** $\sum_{j\in\mathcal{N}} p_{ij} = 1\ \ \forall\, i \in \mathcal{N}$. When P satisfies several standard conditions (see [9], for instance), a steady-state probability distribution, v^{ss}, exists such that $v^{ss}P = v^{ss}$ and $\lim_{k\to\infty} \Pr\{\ n(k) = j\ \} = v^{ss}_j\ \ \forall j \in \mathcal{N}$ given any initial probability distribution for $n(0) \in \mathcal{N}$. A Markov chain will be identified with its one-step transition matrix, P.

To relate a Markov chain to communication losses, each state may be identified with loss or reception. The function $g(\cdot) : \mathcal{N} \to \Theta$ is used to represent this relationship. The sequence of states visited by the markov process, $n(k) \in \mathcal{N}$, defines the sequence of lost packets as $\theta(k) = g(n(k))$. The ordered pair $(g(\cdot), P)$ which defines communication loss probabilities in this way will be referred to as a Markovian communication loss process.

Several examples are shown in Figure 1(a)-(d). Transition matrices for (a) and (b) are shown in the figure in terms of the probabilities α and γ, and can be constructed with appropriate zero entries for (c) and (d). The mapping function value, $g(i)$, is inside the circle denoting state $i \in \mathcal{N}$. In chain (b), $n(k) = 1$ implies that $[\theta(k-1), \theta(k)] = [1, 1]$ and $n(k) = 2$ implies that $[\theta(k-1), \theta(k)] = [2, 1]$, differentiating between a reception following reception and reception following loss. In chain (c), receptions following one or two losses ($n(k) = 2$) are differentiated from receptions following more than two losses ($n(k) = 3$), with ($n(k) = 1$) again corresponding to a reception following a reception. In chain (d), $n(k) = 2$ and $n(k) = 3$ correspond to a reception following an odd or even number of consecutive losses, respectively. Thus, this definition of a communication loss process is quite flexible. Though Θ has only two elements, the number of modes which can be used in the markov chain, N_n, is not limited. Increased dimension of the Markov chain is exploited to increase the number of different estimator gains to use in our estimator design. Note that for fixed values of α and γ the Markov chains in Figure 1 (a) and (b) impose the same probability distribution on Θ_k when initial probability distributions on $\mathcal{N}$ are set to v^{ss}. Transition matrices for (c) and (d) can be easily found which preserve this probability distribution on Θ_k.

These communication loss processes were constructed so that $n(j)$, $1 \leq j \leq k$, can be determined uniquely from Θ_k. If this property holds, we refer to $(g(\cdot), P)$ as an observable pair. This condition allows designs dependent on $\mathcal{N}_k$ to be used in practice when Θ_k is available.

3 Time-varying Estimation

Assume a remote sensor is communicating measurement information over a wireless network. Communication losses will affect any filtering, estimation, or control which is based upon these measurements. Here, the transmitted information is assumed to be the output of a system for which a state estimator is sought. The communication losses (also referred to as measurement losses) are modeled with $(g(\cdot), P)$ as described in the previous section. In an otherwise standard linear estimation problem, a discrete time-invariant plant with intermittent measurements corrupted by white, gaussian, zero-mean measurement and process noises is modeled

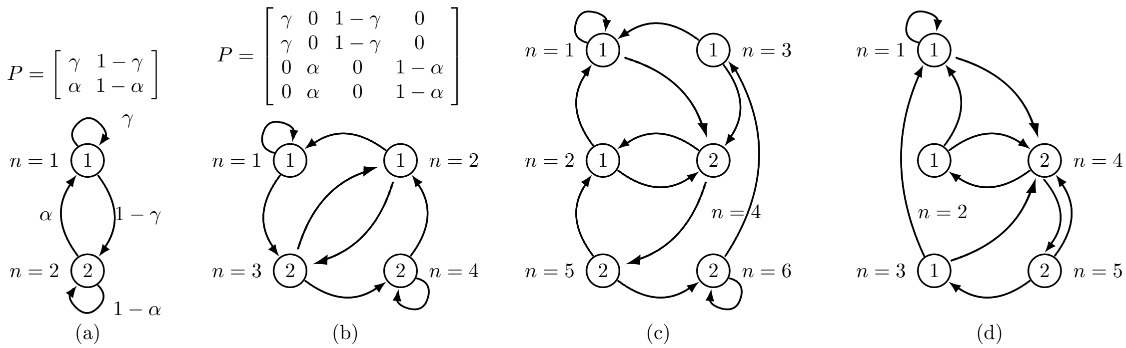

Figure 1: Markov Chains for communication loss processes

as:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bw(k) \\
y(k) &= C_{\theta(k)}x(k) + v(k)
\end{aligned}
\tag{1}
$$

Where $\theta(k) = g(n(k))$, $C_1 = C$ (noisy measurement received), and $C_2 = 0$ (only noise received). The initial condition, $x(0)$, is gaussian with mean, x_0, and variance, M_0. The probability distribution for $n(0)$ is assumed to be the steady state distribution, v^{ss}, of P. It is assumed that the estimator is aware of packet losses, $\theta(k)$. Finally, assume that the plant and sensor noise covariance matrices (W and V) are positive definite. Consider the following estimation problem:

Given measurements $\mathcal{Y}_k = \{y(0), \ldots, y(k)\}$ and loss process observations $\Theta_k = \{\theta(0), \ldots, \theta(k)\}$ for the plant (Eq 1), find the state estimate, $\hat{x}(k)$, which minimizes:

$$
J = \mathop{\mathrm{E}}_{\mathcal{Y}_k, \Theta_k} \left[\; \|x(k) - \hat{x}(k)\|^2 \; \right] \tag{2}
$$

While studying the Jump Linear Quadratic Gaussian problem, Ji and Chizeck note that the optimal state estimate (in terms of Eq (2)) for such systems is obtained with the time varying Kalman filter [3, 5], derived for linear, discrete, time-varying plants by Kalman in [6].

The loss process history, Θ_k, is known at time k. The notation used in the optimal Kalman filter design is:

Optimal Estimate: $\hat{x}(k|j) = E[x(k)|\mathcal{Y}_j, \Theta_j]$

Estimation Error: $\tilde{x}(k|j) = x(k) - \hat{x}(k|j)$

Error Covariances: $Z(k) = E[\tilde{x}(k|k)\tilde{x}(k|k)^T]$

$$M(k+1) = E[\tilde{x}(k+1|k)\tilde{x}(k+1|k)^T]$$

The state estimate is computed with the following predictor/corrector equations:

$$
\begin{aligned}
&\hat{x}(k|k) = \hat{x}(k|k-1) + F(k)\left[y(k) - C_{\theta(k)}\hat{x}(k|k-1)\right] \\
&\hat{x}(k|k-1) = A\hat{x}(k-1|k-1)
\end{aligned}
\tag{3}
$$

The Kalman filter uses a time-varying corrector gain, $F(k)$, computed recursively in real-time, according to the following relations:

$$
\begin{aligned}
&F(k) = M(k)C_{\theta(k)}^T\left[C_{\theta(k)}M(k)C_{\theta(k)}^T + V\right]^{-1} \\
&Z(k) = M(k) - F(k)\left[C_{\theta(k)}M(k)C_{\theta(k)}^T + V\right]F(k)^T \\
&M(k+1) = AZ(k)A^T + BWB^T
\end{aligned}
$$

The initial conditions for the predictor states and estimator error covariance are $\hat{x}(0|-1) = x_0$ and $M(0) = M_0$ respectively.

The Kalman filter is optimal, but can be computationally expensive. An alternative is to use a pre-computed filter gain. Recall that the prediction error covariance, $M(k)$, for the Kalman filter converges to a constant value if the plant is time invariant and detectable. This steady state error covariance can be used to precompute a corrector gain and significantly reduce real-time computation. Unfortunately, the plant with intermittent measurements (Eq 1) is time-varying due to its dependence on $\theta(k)$, and the values of $M(k)$ will typically not converge.

4 Modal Estimation

To avoid the computational cost of the Kalman Filter, an estimator is proposed that relies on a finite number of pre-computed gains. The number of precomputed gains and when they should be applied is determined by the measurement loss model, $(g(\cdot), P)$. The proposed estimator has the following form:

$$
\begin{aligned}
&\hat{x}(k|k) = \hat{x}(k|k-1) + F_{n(k)}[y(k) - C_{\theta(k)}\hat{x}(k|k-1)] \\
&\hat{x}(k|k-1) = A\hat{x}(k-1|k-1)
\end{aligned}
\tag{4}
$$

Note that the estimator uses the predictor/corrector form with a time-varying corrector gain. The corrector gain is chosen from a finite set, $\{F_1, \ldots, F_{N_n}\}$, based on the observed state of the markov chain, $n(k)$ (which corresponds to a measurement loss mode).

The flexibility of the measurement loss process allows us to design estimators of increasing complexity for a fixed probability distribution on Θ_k. If the communication loss process is taken as the markov chain in Figure 1(a), $N_n = 2$ and the estimator in Equation 4 uses two corrector gains, $\{F_1, F_2\}$. One corrector gain is used after a packet receipt (F_1) and another after a packet loss (F_2). For the loss process associated with Figure 1(b), $N_n = 4$ and the estimator uses one of four corrector gains, $\{F_1, \ldots, F_4\}$ based on the status of the previous two measurements.

The size of the prediction error covariance, $M(k)$, is used to quantify the performance of these state estimators. For the estimation scheme proposed above, the prediction error covariance evolves as follows:

$$\begin{aligned} M(k+1) =& A(I - F_{n(k)}C_{\theta(k)})M(k)(I - F_{n(k)}C_{\theta(k)})^T A^T \\ &+ AF_{n(k)}VF_{n(k)}^T A^T + BWB^T \end{aligned} \tag{5}$$

Note that $M(k)$ will depend upon $n(k)$. This recursion cannot be used to precompute $M(k)$ since $\mathcal{N}_k$ is not known in advance. A deterministic performance index is given by the *average* filter performance, $\mathrm{E}_{\mathcal{N}_{k-1}}[\ M(k)\ ]$, which can be pre-computed and used to measure the performance of an estimator based on a given set of corrector gains, $\{F_1, \ldots, F_{N_n}\}$. For the remainder of the paper $\{F_i\}$ will denote a set of corrector gains. Similar notation will be used for other sets. Notions of optimality for a set of gains, $\{F_i\}$, are implicitly relative to estimators with identical structure (meaning the same measurement modes identify different corrector gains).

Motivated by the desire to analyze the performance of each corrector gain in a given set, modal covariances are defined as $M_i(k) \doteq \mathrm{E}_{\mathcal{N}_{k-2}}[\ M(k) \mid n(k-1) = i\ ]$. This is a deterministic quantity. Assuming that the probability distribution of $n(k)$ is v^{ss} for all k, an average filter performance can be computed as $\mathrm{E}_{\mathcal{N}_{k-1}}[\ M(k)\ ] = \sum_{i=1}^{N_n} v_i^{ss} M_i(k)$.

These modal covariances satisfy a recursive relation. First, denote $\Pr\{\ n(k-1) = j \mid n(k) = i\ \}$ by p_{ij}^*. Using Bayes' Rule, $p_{ij}^* = v_j^{ss} p_{ji} / v_i^{ss}$. The affine operator $\mathcal{L}_{F_i}(\cdot)$ is defined for each $i \in \mathcal{N}$ as (see Eq 5):

$$\begin{aligned} \mathcal{L}_{F_i}(M) \doteq& A(I - F_iC_{g(i)})M(I - F_iC_{g(i)})^T A^T \\ &+ AF_iVF_i^T A^T + BWB^T \end{aligned} \tag{6}$$

where F_i is the corrector gain, $F(k)$, used when $n(k) = i$. The recursive relation is stated in the following lemma.

Lemma 1 (L-iteration) *Given a fixed set of corrector gains, $\{F_i\}$, the modal covariances satisfy the following recursion:*

$$M_i(k+1) = \mathcal{L}_{F_i}(M_{\mathrm{pre},i}(k)) \qquad \forall i \in \mathcal{N}$$

where $M_{\mathrm{pre},i}(k) \doteq \sum_{j=1}^{N_n} p_{ij}^* M_j(k)$ is the expected value of the prediction error covariance at time k, conditioned on $n(k) = i$.

Proof :

$$\begin{aligned} M_i(k+1) &= \mathop{\mathrm{E}}_{n(k-1)}\left[\mathop{\mathrm{E}}_{\mathcal{N}_{k-2}}[\ M(k+1) \mid n(k) = i\ ] \;\middle|\; n(k) = i \right] \\ &\overset{(a)}{=} \mathop{\mathrm{E}}_{n(k-1)}\left[\mathop{\mathrm{E}}_{\mathcal{N}_{k-2}}[\ \mathcal{L}_{F_i}(M(k))\ ] \;\middle|\; n(k) = i \right] \\ &\overset{(b)}{=} \sum_{j=1}^{N_n} p_{ij}^* \mathop{\mathrm{E}}_{\mathcal{N}_{k-2}}[\ \mathcal{L}_{F_i}(M(k)) \mid n(k-1) = j\ ] \\ &\overset{(c)}{=} \mathcal{L}_{F_i}(M_{\mathrm{pre},i}(k)) \end{aligned}$$

Equality (a) follows from the recursion for the error covariance (Eq 5) as well as the conditional knowledge that $n(k) = i$. Applying the outer conditional expectation and using the previously defined notation for p_{ij}^* yields equality (b). Finally, equality (c) follows because $\mathcal{L}_{F_i}(\cdot)$ is affine and $\sum_{j=1}^{N_n} p_{ij}^* = 1$. ■

4.1 Pseudo-steady-state

Using Lemma 1, pre-computed modal covariances may be used to judge the performance of the estimator based on a given $\{F_i\}$. These modal covariances will converge to steady state values in some cases. These indicate average performance of the estimator after a long period of time. This motivates the next definition.

Definition 1 *Given a fixed set of corrector gains, $\{F_i\}$, $\{M_i^{ss}\}$ is a* **pseudo-steady-state** *if for any set of positive semi-definite initial conditions, $\{M_i(0)\}$, $\lim_{k\to\infty} M_j(k)$ exists and is equal to M_j^{ss} $\forall j \in \mathcal{N}$.*

The choise of using $M_i()$ rather than $M_{\mathrm{pre},i}()$ to define the pseudo-steady-state was arbitrary; one can be found in terms of the other. A steady-state in the usual sense almost never exists; $M(k+1)$ in Eq 5 depends on $n(k)$. Moreover, the pseudo-steady-state need not exist for a given set of corrector gains. For example, the modal covariances may grow unbounded if the corrector gains are chosen poorly.

The L-iteration is governed by a collection of affine operators, $\{\mathcal{L}_{F_i}(\cdot)\}$. Thus the map from $\{M_i(k)\}$ to $\{M_i(k+1)\}$ is also affine. The linear portion of this operator can be extracted and its spectrum examined. If its spectral radius is strictly less than 1, then the operator is invertible and a pseudo-steady state exists for the given set of corrector gains. This is analysis, however, and gives no indication of how to synthesize a set of corrector gains such that a pseudo-steady-state exists, let alone a set of gains which minimizes the pseudo-steady-state covariances. The following iteration is proposed to synthesize $\{F_i\}$ indirectly:

Definition 2 *The* **R-iteration** *is:*

$$M_i(k+1) = \mathcal{R}_i(M_{\mathrm{pre},i}(k)) \qquad \forall i \in \mathcal{N}$$

where $M_{\text{pre},i}(k) \doteq \sum_{j=1}^{N_n} p_{ij}^* M_j(k)$. $\mathcal{R}_i(\cdot)$ *is defined for each* $i \in \mathcal{N}$ *as:*

$$\mathcal{R}_i(M) \doteq AMA^T + BWB^T - AMC_{g(i)}^T \left(C_{g(i)}MC_{g(i)}^T + V\right)^{-1} C_{g(i)}MA^T$$

A fixed point of this iteration, $\{M_i^{ss}\}$, is also a fixed point of the L-iteration when $\{F_i\}$ are taken as follows (complete the square):

$$F_i \doteq M_{\text{pre},i}^{ss} C_{g(i)}^T \left(C_{g(i)} M_{\text{pre},i}^{ss} C_{g(i)}^T + V\right)^{-1} \tag{7}$$

Monotonicity properties of the operators $\mathcal{L}_{F_i}(\cdot)$ and $\mathcal{R}_i(\cdot)$ as well as relative monotonicity of their respective iterations are given as Lemmas in the appendix. These Lemmas are used in the proof of the following theorem, which establishes a test for existence of $\{F_i\}$ such that a pseudo-steady-state exists.

Theorem 1 (Existence) *Assume (A,B) controllable. The R-iteration converges with initial conditions* $M_i(0) = 0 \ \forall\ i \in \mathcal{N}$ **iff** *there exists a set of gains* $\{F_i\}$ *such that a pseudo-steady-state exists.*

Proof (⇐): Assume that the R-iteration fails to converge when started from zero initial conditions. Lemma 2 in the appendix states that this iteration is monotonically nondecreasing, $M_i(k+1) \geq M_i(k) \geq 0 \quad \forall i \in \mathcal{N}$. The R-iteration therefore diverges, and there exists at least one $i^* \in \mathcal{N}$ such that $M_{i^*}(k)$ grows unbounded as $k \to \infty$.

Using Lemma 3 in the appendix, the iterates of the L-iteration for any $\{F_i\}$ are bounded below by the iterates of the R-iteration started with zero initial conditions. Therefore, one of the L-iterates diverges, prohibiting the existence of a pseudo-steady-state for any set of corrector gains, $\{F_i\}$. ■

Proof (⇒): Assume the R-iteration converges to a limit when started from zero initial conditions. Denote the limit matrices by $M_j^{ss} \doteq \lim_{k\to\infty} M_j(k) \ \forall j \in \mathcal{N}$. $\{M_j^{ss}\}$ must be a fixed point of the R-iteration and by Lemma 2, all of the matrices be positive semidefinite. Choose $\{F_j\}$ according to Eq 7 using $\{M_j^{ss}\}$.

To complete the proof, it is sufficient to show that a pseudo-steady-state exists for this set of corrector gains. It is easily shown (completing the square) that $\{M_i^{ss}\}$ is a fixed point for the L-iteration using $\{F_i\}$ and that each M_i^{ss} is positive definite (Lemma 4 and the controllability assumption).

Let $\{\tilde{M}_i(k)\}$ denote the L-iterates for the corrector gains in Equation 7. Assume initial conditions for the L-iteration satisfying $0 \leq \tilde{M}_i(0) \leq M_i^{ss} \ \forall i \in \mathcal{N}$. The L-iterates can be bounded above and below $\forall i \in \mathcal{N}$ as follows: $M_i(k) \leq \tilde{M}_i(k) \leq M_i^{ss}$. The first inequality follows from Lemma 3. The second inequality follows through application of Lemma 5 using $\{\tilde{M}_i(k)\}$ and the L-iteration started at the fixed point $\{M_i^{ss}\}$. By assumption, $\lim_{k\to\infty} M_i(k) = M_i^{ss}$ and hence $\lim_{k\to\infty} \tilde{M}_i(k) = M_i^{ss}$.

Since the limit matrices are positive definite, the above sandwich argument shows that when the initial conditions deviate from the limit matrices in any negative semidefinite direction, the L-iteration converges to the limit matrices. The L-iteration is affine with the set of symmetric matrices as an invariant set. A negative semidefinite basis exists for the set of symmetric matrices, therefore the spectral radius of the linear portion of the L-iteration restricted to symmetric matrices is strictly less than 1, and the L-iteration will converge to the given limit matrices for any initial symmetric iterates, satisfying the pseudo-steady-state conditions. ■

The gains constructed in the proof of the above theorem are "optimal" in a certain sense described below. Note that the R-iteration is a deterministic method for finding these optimal corrector gains, and will fail only if no set of gains exists. Finally, it follows from the proof that if a pseudo-steady-state fails to exist, the *average* filter performance, $\mathrm{E}_{\mathcal{N}_{k-1}}[\ M(k)\ ]$, grows unbounded for any set of filter gains. In other words, the lack of a pseudo-steady-state implies that the estimator is unstable, on average, for any set of filter gains.

Definition 3 *The set* $\{F_i\}$ *is a set of* **optimal corrector gains** *if the pseudo-steady state,* $\{M_i^{ss}\}$, *exists and for any other set of gains,* $\{\tilde{F}_i\}$, $\liminf_{k\to\infty} \tilde{M}_j(k) \geq M_j^{ss} \ \forall j \in \mathcal{N}$.

The following theorem is easily proven using Lemma 3. It states that the gains used in the proof of the previous theorem are in fact optimal in the sense defined above.

Theorem 2 (Optimality) *Assume that the R-iteration with initial conditions* $M_i(0) = 0 \ \forall\ i \in \mathcal{N}$ *converges to* $\{M_i^{ss}\}$. *Let* $M_{\text{pre},i}(k) \doteq \sum_{j=1}^{N_n} p_{ij}^* M_j(k)$. *The following set of corrector gains is optimal:*

$$F_i \doteq M_{\text{pre},i}^{ss} C_{g(i)}^T \left(C_{g(i)} M_{\text{pre},i}^{ss} C_{g(i)}^T + V\right)^{-1} \quad \forall i \in \mathcal{N}$$

This R-iteration based design is used for illustrative examples in the following section.

5 Examples

The proposed estimator design is applied to a discrete double integrator system as illustrative example. The system is considered without control; the estimation error dynamics and analysis will remain unchanged if control were to be applied. Different designs corresponding to the Markov chains in Figure 1 are shown in comparison to the TVKE in simulation. MATLAB was used to calculate the estimator gains, generate random processes, and simulate the various estimation schemes.

The plant data as referred to in Eq 1 is

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

The following covariances, describing the process noise, measurement noise, and initial estimation error are used to define the estimation problem as in Section 3:

$$W = 0.1, \qquad V = 1.0, \qquad M_0 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

Designs are carried out for the different measurement loss models shown in Figure 1 (models (a), (b), and (c)). The same probability distribution on measurement loss patterns as in model (a) is assumed for models (b) and (c) in terms of probabilities $\alpha = 0.5$ and $\gamma = 0.7$. The analytical properties of the resulting multi-modal estimator designs for these three cases are followed by simulation comparisons for a given Θ_k.

Design (a) has the following $(g(\cdot), P)$ pair:

$$P_a = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}, \qquad \begin{bmatrix} g_a(1) \\ g_a(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The design results in the following multi-modal gains:

	$i=1$	$i=2$
F_i	$\begin{bmatrix} 0.745 \\ 0.202 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\mathrm{Tr}\, Z_i$	0.94	4.31
v_i^{ss}	0.625	0.375

The trace of Z_i is $\mathrm{E}\left[\ \tilde{x}(k|k)^T \tilde{x}(k|k)\ \middle|\ n(k) = i\ \right]$ which differs from the trace of M_i^{ss}. The steady-state probability distribution on the Markov chain states is $\{v_i^{ss}\}$. When a measurement is missed, the corrector gain is zero and the expected estimation error is larger. The Kalman filter cost (Equation 2) for this estimator, J_a can be computed in terms of the above data as follows:

$$J_a = \sum_{i=1}^{2} v_i^{ss} \cdot \mathrm{Tr}\, Z_i = 2.20$$

For design (b),

$$P_b = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.7 & 0 & 0.3 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0.5 \end{bmatrix}, \qquad \begin{bmatrix} g_b(1) \\ g_b(2) \\ g_b(3) \\ g_b(4) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

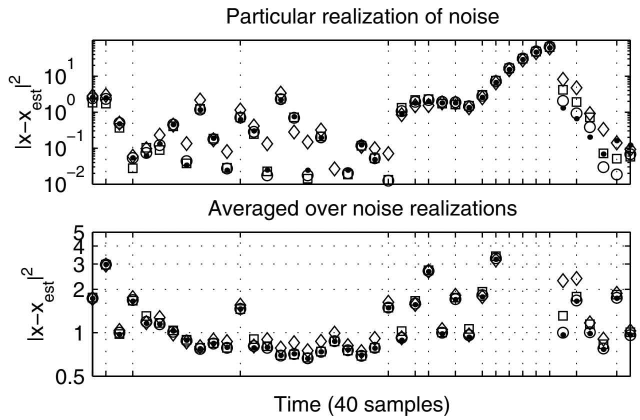

Figure 2: Simulation results, same measurement pattern; measurements lost at indicated samples. Legend: •-TVKE, ◇-(a), □-(b), ○-(c)

describe the measurement loss process. The resulting multi-modal design is described by

	$i=1$	$i=2$	$i=3$	$i=4$
F_i	$\begin{bmatrix} 0.576 \\ 0.208 \end{bmatrix}$	$\begin{bmatrix} 0.862 \\ 0.202 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\mathrm{Tr}\, Z_i$	0.759	1.05	1.64	6.72
v_i^{ss}	0.437	0.188	0.188	0.188

For this estimator, the Kalman filter cost is $J_b = 2.10$, an improvement relative to design (a), as expected. For design (c),

$$P_c = \begin{bmatrix} 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0.7 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 \end{bmatrix}, \qquad \begin{bmatrix} g_c(1) \\ g_c(2) \\ g_c(3) \\ g_c(4) \\ g_c(5) \\ g_c(6) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

The estimator (c) has the following properties:

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
F_i	$\begin{bmatrix} 0.574 \\ 0.208 \end{bmatrix}$	$\begin{bmatrix} 0.775 \\ 0.231 \end{bmatrix}$	$\begin{bmatrix} 0.935 \\ 0.176 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\mathrm{Tr}\, Z_i$	0.749	0.948	1.14	1.62	3.08	10.2
v_i^{ss}	0.437	0.141	0.0469	0.188	0.094	0.094

Relative to designs (a) and (b), an improvement in the Kalman estimation cost is observed: $J_c = 2.06$.

Some results of simulating these estimators are shown in Figure 2. A single measurement loss pattern was generated. The results in the top figure correspond to a single instance of process and measurement noise, drawn from the appropriate distribution. The results in the bottom figure are averaged over 500 different process and measurement noise instances using the same measurement loss pattern. The TVKE can be seen to perform better on average than the other schemes for this particular sequence of measurement losses. The estimation errors resulting from the long sequence of consecutive measurement losses are not shown in the lower graph (outside axis limits) but are shown on the upper plot.

For the particular instance of process noise shown in the upper plot, a clear advantage of using TVKE over one of the multi-modal estimators is apparent only looking at the errors after a long string of measurement losses. The TVKE recovers better. In fact, the errors are very nearly ordered as (a)>(b)>(c)>TVKE during this recovery stage. Otherwise, there is not a significant apparent advantage to TVKE in terms of the averaged errors shown in the lower plot.

A correspondence between the averaged simulated errors and the precomputed expected estimation errors, $\operatorname{Tr} Z_i$, can be seen. For all designs, $n(k) = 1$ corresponds to the case of several received measurements in a row. For designs (b) and (c), $\operatorname{Tr} Z_1 \approx 0.75$ and this is roughly the best performance we can expect even if all measurements are received. The averaged estimation errors in the figure are lower bounded by this value.

6 Conclusions

In this paper, a state estimator design for discrete time systems having intermittent measurements was described. A stationary Markov process was used to model probabilistic measurement losses. It is assumed that the measurement losses are observable, i.e. the estimator has access to both the measurement and the state of the Markov process. A design for a precomputable time-varying estimator that uses one filter gain for each state of the Markov process was proposed. An iterative scheme was shown to compute optimal filter gains for this estimator structure, but the proposed estimator is sub-optimal when compared to the TVKE. In summary, estimator performance is sacrificed to alleviate the real-time computational burden of TVKE. The results shown here naturally extend to generic Discrete Time Markovian Jump Linear Systems.

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A Auxiliary Results

Lemma 2 (Monotonicity of R-Iteration) *The R-iteration starting with the initial conditions $M_i(0) = 0 \ \forall\ i \in \mathcal{N}$ satisfies $M_i(k+1) \geq M_i(k) \geq 0\ \forall i \in \mathcal{N}$.*

Proof : The proof is by induction using the following relation which follows from Lemma 3.1 in [4]: $M_1 \geq M_2 \geq 0$ implies $\mathcal{R}_i(M_1) \geq \mathcal{R}_i(M_2) \geq 0\ \forall i \in \mathcal{N}$. ∎

Lemma 3 (Minimum Property) *Given any set of corrector gains, $\{F_i\}$, let $\{\tilde{M}_i(k)\}$ denote the solution of the L-iteration starting at $\tilde{M}_i(0) \geq 0\ \forall i \in \mathcal{N}$. Let $\{M_i(k)\}$ denote the solution of the R-iteration starting at $M_i(0) = 0\ \forall i \in \mathcal{N}$. Then $M_i(k) \leq \tilde{M}_i(k)\ \forall i \in \mathcal{N}$ and $\forall k$.*

Proof : This is a generalization of a result by Caines and Mayne [2] for the standard Riccati equation, the proof again uses an inductive arguement. ∎

Lemma 4 *Assume (A, B) controllable. Let $\{M_i\}$ be a set of positive semi-definite matrices satisfying:*

$$M_i = \mathcal{L}_{F_i}\left(M_{pre,i}\right) \qquad \forall i \in \mathcal{N} \tag{8}$$

where $M_{pre,i} \doteq \sum_{j=1}^{N_n} p^*_{ij} M_j$. Then $M_i > 0\ \forall i \in \mathcal{N}$.

Proof : This is a non-trivial proof. The critical part of the proof is a generalization of Theorem 4.1 in [8]. The covariances are bounded below by controllability grammians of a synthetic time varying system. It is shown that controllability of the synthetic system can be identified with controllability of the pair (A, B), proving the lemma. ∎

Lemma 5 *Given any set of corrector gains, $\{F_i\}$, let $\{\tilde{M}_i(k)\}$ and $\{M_i(k)\}$ denote solutions of the L-iteration. Assume their initial conditions satisfy: $0 \leq \tilde{M}_i(0) \leq M_i(0)\ \forall i \in \mathcal{N}$. Then $\tilde{M}_i(k) \leq M_i(k)\ \forall i \in \mathcal{N}$ and $\forall k$.*

Proof : The proof is by induction using the definition of the L-iteration. ∎