# A Bounded Real Lemma for Markovian Jump Linear Systems

Pete Seiler Raja Sengupta

**Abstract**

This paper presents a bounded real lemma for discrete-time Markovian jump linear systems (MJLS). We show that, given a class of stochastic inputs, the matrix inequality in the bounded real lemma is both a necessary and sufficient condition. For the case of one plant mode, this condition reduces to the standard necessary and sufficient condition for discrete-time systems. We envision this lemma being used to build necessary and sufficient LMI analysis and synthesis conditions for MJLS.

**Index Terms**

Markovian Jump Linear Systems, Bounded Real Lemma, Discrete-time, $H_\infty$ norm

## I. Introduction

THIS paper presents a bounded real lemma for discrete-time Markovian jump linear systems (MJLS). As we discuss below, these systems have previously received significant attention and many theoretical results are available. One motivation for the continued research on these systems is the recent interest in networked control. Specifically, several authors have modeled the packet delivery characteristics of a network by a discrete-time jump system [5], [15], [20], [21], [22]. Experimental testing suggests that a jump system is a reasonable model for the packet delivery characteristics of a wireless link [16].

Before proceeding, we briefly review some of the work on discrete-time jump systems that

P. Seiler is with the Department of Mechanical Engineering, University of California, Berkeley, Berkeley CA, 94720, USA, Phone: (510) 848-7374, email: pseiler@vehicle.me.berkeley.edu

R. Sengupta is with the Department of Civil Engineering, University of California, Berkeley, Berkeley CA, 94720, USA, email: raja@PATH.berkeley.edu

is most relevant to the results in this paper. We will refer to the definitions of controllability and observability for MJLS given by Ji and Chizeck [12]. We mention in passing that these ideas were used to solve the Jump Linear Quadratic Gaussian control problem [7], [6], [13]. We will also apply stability results by Ji, et.al. [14] and Costa and Fragoso [9]. We discuss these stability results in greater detail in Section III. Finally, we note that several authors have developed bounded real lemmas for MJLS [2], [11], [4]. These results show that satisfying a matrix inequality condition is sufficient for the MJLS to have $H_\infty$ norm less than some specified level. However, a proof of the necessity is lacking in the literature.

In this paper, we show that the matrix inequality in the bounded real lemma is both a necessary and sufficient condition for a given class of stochastic inputs into the plant. The proof uses ideas from [18] which gives a dynamic game interpretation to the continuous time $H_\infty$-control of jump linear systems. Reference [1] gives relevant information on generalized Riccati equations related to dynamic games.

The remainder of the paper has the following structure: In the next section, we give the notation that will be used throughout the paper. In Section III we review several useful results related to the stability of MJLS. Section IV contains the main result: the statement and proof of the bounded real lemma. The proof uses several auxiliary results which are contained in the appendix. Conclusions are then given in the final section.

## II. Markov Jump Linear Systems (MJLS)

Consider the following stochastic system, denoted $P$:

$$\begin{bmatrix} x(k+1) \\ e(k) \end{bmatrix} = \begin{bmatrix} A_{\theta(k)} & B_{\theta(k)} \\ C_{\theta(k)} & D_{\theta(k)} \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state, $d(k) \in \mathbb{R}^{n_d}$ is the disturbance vector and $e(k) \in \mathbb{R}^{n_e}$ is the error vector. The state matrices are functions of a discrete-time Markov chain taking values in a finite set $\mathcal{N} = \{1, \dots, N\}$. The Markov chain has transition probabilities $p_{ij} = \Pr(\theta(k+1) = j \mid \theta(k) = i)$ which are subject to the restrictions $p_{ij} \geq 0$ and $\sum_{j=1}^{N} p_{ij} = 1$ for any $i \in \mathcal{N}$. The plant initial conditions are given by specifying $\theta(0)$ and $x(0)$. When the plant is in mode $i \in \mathcal{N}$ (i.e. $\theta(k) = i$), we will use the following notation: $A_i := A_{\theta(k)}$, $B_i := B_{\theta(k)}$, $C_i := C_{\theta(k)}$, and $D_i := D_{\theta(k)}$. Plants of this form are called discrete-time Markovian jump linear systems.

We will work with sequences, $x := \{x(k)\}_{k=0}^{\infty}$, that depend on the sequence of Markov parameters, $\Theta = \{\theta(k)\}_{k=0}^{\infty}$. For notation, define $\Theta_k := \{\theta(1), \dots, \theta(k)\}$. We define $\ell_2$ as the space of square summable (stochastic) sequences:

$$\ell_2^n := \left\{ \{x(k)\}_{k=0}^{\infty} \;:\; \forall k \; x(k) \in \mathbb{R}^n \text{ is a random variable depending on } \Theta_k \text{ and } \|x\|_2 < \infty \right\} \tag{2}$$

where the $\ell_2$-norm is defined by:

$$\|x\|_2^2 := \sum_{k=0}^{\infty} \underset{\Theta_{k-1}}{E} \left[ \|x(k)\|^2 \right] \tag{3}$$

$\|\cdot\|$ is the standard Euclidean norm defined on $\mathbb{R}^n$. Note that $\Theta_k$ does not contain $\theta(0)$ because this is assumed to be given as part of the plant initial conditions.

Finally we introduce some notation concerning collections of matrices, $\{M_i\}_{i \in \mathcal{N}}$. $\{M_i\} \geq 0$ means $M_i \geq 0 \; \forall i \in \mathcal{N}$. We will use similar notation whenever all matrices in the set satisfy a given condition. We will also define $\tilde{M}_i := \sum_{j=1}^{N} p_{ij} M_j$. This shorthand will be used frequently in the following sections.

## III. Stability of a MJLS

In this section, we review several useful results related to the stability of discrete-time jump linear systems. First we define several forms of stability for such systems [14].

*Definition 1:* For the system given by (1) with $d \equiv 0$, the equilibrium point at the origin is:

1. Mean-square stable if for every initial state $(x_0, \theta_0)$, $\lim_{k\to\infty} \underset{\Theta_{k-1}}{E}[\|x(k)\|^2 \mid x_0, \theta_0] = 0$.
2. Stochastically stable if for every initial state $(x_0, \theta_0)$, $\sum_{k=0}^{\infty} \underset{\Theta_{k-1}}{E}[\|x(k)\|^2 \mid x_0, \theta_0] < \infty$. In other words, $\|x\|_2 < \infty$ for every initial state.
3. Exponentially mean square stable if for every initial state $(x_0, \theta_0)$, there exists constants $0 < \alpha < 1$ and $\beta > 0$ such that $\forall k \geq 0$, $\underset{\Theta_{k-1}}{E}[\|x(k)\|^2 \mid x_0, \theta_0] < \beta\alpha^k \|x_0\|^2$.
4. Almost surely stable if for every initial state $(x_0, \theta_0)$, $\Pr[\lim_{k\to\infty} \|x(k)\| = 0] = 1$.

Ji, et.al. showed that the first three definitions of stability are actually equivalent for a MJLS [14]. They refer to the equivalent notions of mean-square, stochastic, and exponential mean square stability as second-moment stability (SMS). Moreover, SMS is sufficient but not necessary for almost sure stability. In the remainder of the paper, references to stability will be in the sense of second-moment stability. The major motivation for this choice is that straightforward necessary and sufficient conditions exist to check for SMS but not for almost-sure stability. Below we present a necessary and sufficient condition for SMS of the jump linear system.

*Theorem 1:* System (1) is SMS if and only if there exist matrices $\{G_i\} > 0$ that satisfy:

$$A_i^T \tilde{G}_i A_i - G_i < 0 \qquad\qquad i = 1, \ldots, N$$

where $\tilde{G}_i := \sum_{j=1}^{N} p_{ij} G_j$.

Proofs of Theorem 1 can be found in [13] and [9]. The theorem states that SMS is equivalent to finding $N$ positive definite matrices which satisfy $N$ coupled, discrete Lyapunov equations. It is interesting to note that stability of each mode is neither necessary nor sufficient for the system to be SMS. See [13] for several examples of this and other properties of MJLS.

## IV. Bounded Real Lemma

First we give the definition of the $H_\infty$ norm for discrete-time MJLS. We consider disturbances, $d \in \ell_2^{n_d}$, to the jump system given by Equation 1. The $H_\infty$ norm [8] of the system is defined below.

*Definition 2:* Assume the system, $P$, is an SMS system. Let $x(0) = 0$ and define the $H_\infty$ norm, denoted $\|P\|_\infty$, as:

$$\|P\|_\infty := \sup_{\theta(0) \in \mathcal{N}} \sup_{0 \neq d \in \ell_2^{n_d}} \frac{\|e\|_2}{\|d\|_2} \tag{4}$$

Below we state the bounded real lemma for a class of jump systems. To derive this condition, we need a definition of controllability for a MJLS.

*Definition 3:* The system, $P$, is weakly controllable if for every initial state/mode, $(x_0, \theta_0)$, and any final state/mode, $(x_f, \theta_f)$, there exists a finite time $T_c$ and an input $d_c(k)$ such that $Pr[x(T_c) = x_f \text{ and } \theta(T_c) = \theta_f] > 0$.

This definition of weak controllability is motivated by, but different from the definition given by Ji and Chizeck [12]. Suppose the Markov Chain is irreducible and the plant is controllable, in a deterministic sense, along a sequence of modes ending in $\theta_f$. If such a sequence occurs with some positive probability for every $\theta_f \in \mathcal{N}$, then $P$ is weakly controllable as defined above. The statement of the bounded real lemma assumes the system is weakly controllable. This assumption ensures that the disturbance can affect the system state. If the system is not weakly controllable, the matrix condition is still sufficient, but it may not be necessary.

*Theorem 2 (Bounded Real Lemma)* Assume the system, $P$, is weakly controllable. $P$ is

SMS and satisfies $\|P\|_\infty < \gamma$ if and only if there exist matrices $\{G_i\} > 0$ that satisfy:

$$R_i := \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}^T \begin{bmatrix} \tilde{G}_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} - \begin{bmatrix} G_i & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0 \qquad i = 1,\ldots,N \tag{5}$$

where $\tilde{G}_i := \sum_{j=1}^N p_{ij} G_j$.

*Proof:*

($\Leftarrow$) Assume there exist $\{G_i\} > 0$ satisfying the matrix inequalities in Equation 5. These inequalities imply that the upper left blocks must also be negative definite:

$$A_i^T \tilde{G}_i A_i - G_i < C_i^T C_i \leq 0 \qquad i = 1,\ldots,N$$

By Theorem 1, we conclude that the system is SMS.

Next, define the function $V(x,i) := x^T G_i x$. Also, let $e_M$ denote the sequence $e$ truncated at time $M$:

$$e_M(k) = \begin{cases} e(k) & 0 \leq k \leq M \\ 0 & k \geq M \end{cases}$$

Given $x(0) = 0$, $V(x(0),\theta(0)) = 0$ for any initial mode $\theta(0) \in \mathcal{N}$ and hence:

$$\sum_{k=0}^{M} \underset{\Theta_{k+1}}{E} \left[ V(x(k+1),\theta(k+1)) - V(x(k),\theta(k)) \right] = \underset{\Theta_{M+1}}{E} \left[ \, V(x(M+1),\theta(M+1)) \, \right] \geq 0 \tag{6}$$

Inequality (a) below follows from Equation 6:

$$\begin{aligned}
\|e_M\|_2^2 - \gamma^2 \|d_M\|_2^2 &\overset{(a)}{\leq} \sum_{k=0}^{M} \underset{\Theta_{k+1}}{E} \left[ \|e(k)\|^2 - \gamma^2 \|d(k)\|^2 + V(x(k+1),\theta(k+1)) - V(x(k),\theta(k)) \right] \\
&\overset{(b)}{=} \sum_{k=0}^{M} \underset{\Theta_{k+1}}{E} \left[ \begin{bmatrix} x(k) \\ d(k) \end{bmatrix}^T \left( \begin{bmatrix} A_{\theta(k)} & B_{\theta(k)} \\ C_{\theta(k)} & D_{\theta(k)} \end{bmatrix}^T \begin{bmatrix} G_{\theta(k+1)} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\theta(k)} & B_{\theta(k)} \\ C_{\theta(k)} & D_{\theta(k)} \end{bmatrix} - \begin{bmatrix} G_{\theta(k)} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \right] \\
&\overset{(c)}{=} \sum_{k=0}^{M} \underset{\Theta_{k}}{E} \left[ \begin{bmatrix} x(k) \\ d(k) \end{bmatrix}^T R_{\theta(k)} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \right]
\end{aligned}$$

Equality (b) follows by using the system dynamics to replace $e(k)$ and $x(k+1)$ in terms of $x(k)$ and $d(k)$. By taking the expectation over $\theta(k+1)$ we obtain equality (c) where $R_{\theta(k)}$

is defined in Equation 5. By assumption $R_{\theta(k)} < 0 \quad \forall \theta(k) \in \mathcal{N}$. Since $d(k) \neq 0$ for some $k$, there exists (for sufficiently large $M$) an $\epsilon > 0$ such that $\|e_M\|_2^2 < \gamma^2 \|d_M\|_2^2 - \epsilon$. Taking limits as $M \to \infty$ gives $\|e\|_2^2 \leq \gamma^2 \|d\|_2^2 - \epsilon$ and hence for any $\theta(0) \in \mathcal{N}$ and any $d \in \ell_2^{n_d}$, $\|e\|_2 < \gamma \|d\|_2$. To conclude, if there exist $G_i > 0$ satisfying Equation 5, then the system is SMS and $\|P\|_\infty < \gamma$.

($\Rightarrow$) Next we prove that the necessity of the matrix inequalities. First we show that if $\|P\|_\infty < \gamma$ then there exist matrices, $\{G_i\} \geq 0$, that satisfy the following Generalized Riccati Equations:

$$G_i = A_i^T \tilde{G}_i A_i + C_i^T C_i + [B_i^T \tilde{G}_i A_i + D_i^T C_i]^T \; V_i^{-1} \; [B_i^T \tilde{G}_i A_i + D_i^T C_i] \qquad i = 1, \ldots, N \tag{7}$$

where $V_i := \gamma^2 I - B_i^T \tilde{G}_i B_i - D_i^T D_i$.

Consider the solution, $\{G_i(k)\}$, to the following Generalized Riccati Difference Equations (GRDE) with initial condition $\{G_i(0)\} = 0$:

$$G_i(k+1) = A_i^T \tilde{G}_i(k) A_i + C_i^T C_i + [B_i^T \tilde{G}_i(k) A_i + D_i^T C_i]^T \; V_i(k)^{-1} \; [B_i^T \tilde{G}_i(k) A_i + D_i^T C_i] \qquad i = 1, \ldots, N \tag{8}$$

where $\tilde{G}_i(k) := \sum_{j=1}^N p_{ij} G_j(k)$ and $V_i(k) := \gamma^2 I - B_i^T \tilde{G}_i(k) B_i - D_i^T D_i$. Note that the GRDE will not be defined for $k > k_o$ if $V_i(k_o)$ is singular for some $i \in \mathcal{N}$. We show below that $\|P\|_\infty < \gamma$ implies that this cannot occur and the solution exists for all $k$. There are two possible cases:

*Case 1:* There exists $\alpha > 0$ such that $\{V_i(k)\} > \alpha I \; \forall k$.

*Case 2:* There does not exist $\alpha > 0$ such that $\{V_i(k)\} > \alpha I \; \forall k$.

Consider Case 2. There are several ways that such a $\alpha$ may fail to exist. Suppose there exists $T \geq 0$ such that $\{V_i(k)\} > 0$ for $0 \leq k \leq T-1$ and $V_{\theta_0}(T)$ has an eigenvalue $\lambda \leq 0$ for some

$\theta_0 \in \mathcal{N}$. By Lemmas 3 and 4 in the Appendix, this implies that $\|P\|_\infty \geq \gamma$. Alternatively, suppose that $\{V_i(k)\} > 0 \; \forall k$, but there does not exist $\alpha > 0$ such that $\{V_i(k)\} > \alpha I \; \forall k$. In other words, for some $\theta_0 \in \mathcal{N}$, one eigenvalue of $V_{\theta_0}(k)$ tends to zero as $k \to \infty$. The proof of Lemma 4 can also be used to show that $\|P\|_\infty \geq \gamma$ for this case.

By contraposition, $\|P\|_\infty < \gamma$ implies Case 1. Also by contraposition, $\|P\|_\infty < \gamma$ implies that $\{G_i(k)\}$ are uniformly bounded (Lemma 5). In summary, the GRDE is well-defined $\forall k$, it solutions $\{G_i(k)\}$ are uniformly bounded, and there exists $\alpha > 0$ such that $\{V_i(k)\} > \alpha I$ $\forall k$. We now prove $\exists G_i \geq 0$ that satisfy the Generalized Riccati Equation (Equation 7).

We show that the matrix sequences $\{G_i(k)\}$ are monotonically nondecreasing in $k$ and thus boundedness of these sequences implies convergence. Since $\{G_i(0)\} = 0$ implies $\{\tilde{G}_i(0)\} = 0$, it is clear from the GRDE that $\{V_i(0)\} > 0$ implies $\{G_i(1)\} \geq \{\tilde{G}_i(0)\} = 0$. Now make the induction assumption that $G_i(k_0) \geq G_i(k_0 - 1) \geq 0 \; \forall i \in \mathcal{N}$. Define another solution to the GRDE, $\{G_i^0(k)\}$, on $k \geq k_0$ with initial condition $\{G_i^0(k_0)\} = \{G_i(k_0 - 1)\}$. Define the difference $\Delta G_i(k) = G_i(k) - G_i^0(k)$ for $i \in \mathcal{N}$ and $k \geq k_0$. Note that, by the induction assumption, $\{\Delta G_i(k_0)\} \geq 0$ and hence $\{\Delta \tilde{G}_i(k_0)\} \geq 0$. Apply Lemma 1 to show $\{\Delta G_i(k_0 + 1)\} \geq 0$ which implies $G_i(k_0 + 1) \geq G_i(k_0) \; \forall i \in \mathcal{N}$. Thus $\{G_i(k)\}$ are monotonic matrix sequences and, as stated above, they are uniformly bounded. Consequently these sequences must have a limit, $\{G_i\} \geq 0$, and this limit matrix satisfies the Generalized Riccati Equation. Since $V_i(k) := \gamma^2 I - B_i^T \tilde{G}_i(k) B_i - D_i^T D_i$, it also has a well defined limit as $k \to \infty$ which we denote by $V_i$. Finally, $\{V_i(k)\} > \alpha I > 0 \; \forall k$ implies $\{V_i\} \geq \alpha I > 0$.

Define the perturbed plant, $P_\epsilon$:

$$\begin{bmatrix} x(k+1) \\ e(k) \end{bmatrix} = \begin{bmatrix} A_{\theta(k)} & B_{\theta(k)} \\ C^\epsilon_{\theta(k)} & D^\epsilon_{\theta(k)} \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \tag{9}$$

where the output equation matrices are given by:

$$C^{\epsilon}_{\theta(k)} := \begin{bmatrix} C_{\theta(k)} \\ \epsilon I_{n_x \times n_x} \end{bmatrix} \qquad D^{\epsilon}_{\theta(k)} := \begin{bmatrix} D_{\theta(k)} \\ 0_{n_x \times n_d} \end{bmatrix}$$

If a plant is SMS and has $\|P\|_\infty < \gamma$ then $\exists \epsilon > 0$ such that $P_\epsilon$ is also SMS and has $\|P_\epsilon\|_\infty < \gamma$. By the argument above, there exist matrices, $\{G^\epsilon_i\} \geq 0$, that satisfy the following Generalized Riccati Equations:

$$G^\epsilon_i = A_i^T \tilde{G}^\epsilon_i A_i + (C^\epsilon_i)^T (C^\epsilon_i) + [B_i^T \tilde{G}^\epsilon_i A_i + (D^\epsilon_i)^T C^\epsilon_i]^T \ (V^\epsilon_i)^{-1} \ [B_i^T \tilde{G}^\epsilon_i A_i + (D^\epsilon_i)^T C^\epsilon_i] \quad i = 1, \ldots, N \tag{10}$$

where $V^\epsilon_i := \gamma^2 I - B_i^T \tilde{G}^\epsilon_i B_i - (D^\epsilon_i)^T D^\epsilon_i > 0$. After multiplying out all the matrices we obtain:

$$G^\epsilon_i = A_i^T \tilde{G}^\epsilon_i A_i + C_i^T C_i + [B_i^T \tilde{G}^\epsilon_i A_i + D_i^T C_i]^T \ (V^\epsilon_i)^{-1} \ [B_i^T \tilde{G}^\epsilon_i A_i + D_i^T C_i] = \epsilon^2 I > 0 \quad i = 1, \ldots, N$$

where after multiplication, $V^\epsilon_i := \gamma^2 I - B_i^T \tilde{G}^\epsilon_i B_i - D_i^T D_i > 0$. It follows from these inequalities that $\{G^\epsilon_i\} > 0$. Apply Schur complements [1] to show that $\{G^\epsilon_i\}$ is a solution of Equation 5. ■

We make several remarks concerning this $H_\infty$ condition. First, the condition given by Equation 5 reduces to the standard necessary and sufficient condition [17] for the case of one mode ($N = 1$). Second, the 'worst-case' disturbances constructed to prove necessity (Lemmas 2 - 5) depend on the plant state and mode, $(x(k), \theta(k))$. As noted in the introduction, this result has interesting game theory interpretations. Finally, we envision this lemma being used to derive necessary and sufficient LMI conditions for MJLS analysis and controller synthesis.

## V. Conclusions

This paper presented a bounded real lemma for discrete-time Markovian jump linear systems. We showed that, given a class of stochastic inputs, the matrix inequality in the

[1] See [3]: $R > 0$ and $Q - SR^{-1}S^T > 0$ if and only if $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0$

bounded real lemma is both a necessary and sufficient condition. A stochastic Lyapunov function was used to prove sufficiency while stochastic disturbances were constructed to prove the necessity of the lemma. Finally, we envision this lemma being used to derive necessary and sufficient LMI conditions for MJLS analysis and controller synthesis.

## References

[1] Tamer Başar. Generalized Riccati equation in dynamic games. In Sergio Bittanti, Alan J. Laub, and Jan C. Willems, editors, *The Riccati Equation*, chapter 11, pages 293–333. Springer Verlag, 1991.

[2] El-Kebir Boukas and Peng Shi. $H_\infty$ control for discrete-time linear systems with markovian jumping parameters. In *Proceedings of the 36th Conference on Decision and Control*, pages 4134–4139, December 1997.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, 1994.

[4] Yong-Yan Cao and James Lam. Stochastic stabilizability and $H_\infty$ control for discrete-time jump linear systems with time delay. *Journal of the Franklin Institute*, 336:1263–1281, 1999.

[5] H. Chan and Ü. Özgüner. Optimal control of systems over a communication network with queues via a jump system approach. In *Proceedings of the 4th IEEE Conference on Control Applications*, pages 1148–1153, September 1995.

[6] H.J. Chizeck and Y. Ji. Optimal quadratic control of jump linear systems with gaussian noise in discrete-time. In *Proceedings of the 27th Conference on Decision and Control*, pages 1989–1993, December 1988.

[7] H.J. Chizeck, A.S. Willsky, and D. Castanon. Discrete-time markovian-jump linear

quadratic optimal control. *International Journal of Control*, 43(1):213–231, 1986.

[8] O.L.V. Costa and Ricardo P. Marques. Mixed $H_2/H_\infty$-control of discrete-time markovian jump linear systems. *IEEE Transactions on Automatic Control*, 43(1):95–100, January 1998.

[9] Oswaldo L.V. Costa and Marcelo D. Fragoso. Stability results for discrete-time linear systems with markovian jumping parameters. *Journal of Mathematical Analysis and Applications*, 179:154–178, 1993.

[10] Carlos de Souza. On stabilizing properties of solutions of the Riccati difference equation. *IEEE Transactions on Automatic Control*, 34(12):1313–1316, December 1989.

[11] C.E. de Souza and M.D. Fragoso. $H_\infty$ filtering for discrete-time systems with markovian jumping parameters. In *IEEE Conference on Decision and Control*, pages 2181–2186, 1997.

[12] Y. Ji and H.J. Chizeck. Controllability, observability, and discrete-time markovian jump linear quadratic control. *International Journal of Control*, 48(2):481–498, 1988.

[13] Y. Ji and H.J. Chizeck. Jump linear quadratic gaussian control: Steady-state solution and testable conditions. *Control Theory and Advanced Technology*, 6(3):289–318, 1990.

[14] Y. Ji, H.J. Chizeck, X. Feng, and K.A. Loparo. Stability and control of discrete-time jump linear systems. *Control Theory and Advanced Technology*, 7(2):247–270, 1991.

[15] R. Krtolica, Ü. Özgüner, H. Chan, H. Göktas, J. Winkelman, and M. Liubakka. Stability of linear feedback systems with random communication delays. *International Journal of Control*, 59(4):925–953, 1994.

[16] G.T. Nguyen, R.H. Katz, B. Noble, and M. Satyanarayanan. A trace-based approach for modeling wireless channel behavior. In *Proceedings of 1996 Winter Simulation Conference*, pages 597–604, 1996.

[17] A. Packard and J. Doyle. The complex structured singular value. *Automatica*, 29(1):71–109, January 1993.

[18] Z. Pan and T. Başar. $H_\infty$-control of markovian jump systems and solutions to associated piecewise-deterministic differential games. In G.J. Olsder, editor, *New Trends in Dynamic Games and Applications*, pages 61–94. Birkhäuser, 1995.

[19] P. Seiler. *Coordinated Control of Unmanned Aerial Vehicles.* PhD thesis, University of California, Berkeley, 2001.

[20] P. Seiler and R. Sengupta. Analysis of communication losses in vehicle control problems. In *Proceedings of the American Control Conference*, pages 1491–1496, June 2001.

[21] Lin Xiao, Arash Hassibi, and Jonathan How. Control with random communication delays via a discrete-time jump system approach. In *Proceedings of the American Control Conference*, pages 2199–2204, June 2000.

[22] W. Zhang, M.S. Branicky, and S.M. Phillips. Stability of networked control systems. *IEEE Control Systems Magazine*, 21(1):84–99, February 2001.

## Appendix

### A. Auxiliary Results

Prior to stating and proving the required lemmas, we need to introduce some notation. First define the cost function:

$$J(d, x_0, \theta_0) := \|e\|_2^2 - \gamma^2 \|d\|_2^2 = \sum_{k=0}^{\infty} \underset{\Theta_k}{E} \left[ e(k)^T e(k) - \gamma^2 d(k)^T d(k) \right] \tag{11}$$

where $d(k)$ and $e(k)$ are related by the jump system, $P$, defined in Equation 1 with the initial conditions $x(0) = x_0$ and $\theta(0) = \theta_0$. It follows from the definition of the $H_\infty$ norm that $\|P\|_\infty > \gamma$ if and only if there exists $d \in \ell_2$ and an initial mode $\theta_0$ such that $J(d, 0, \theta_0) > 0$.

In the deterministic LQ problem, a Riccati Difference Equation is used to generate optimal control inputs. Similarly, we use the GRDE (Equation 8) to generate optimal inputs (disturbances) that maximize $J(d,0)$ over a finite horizon. We now present several lemmas that will be used in the proof of the Bounded Real Lemma. The first is a technical lemma concerning GRDE. The second lemma constructs a 'bad' disturbance from the solution of the GRDE. The remaining lemmas use this 'bad' disturbance to show that $\|S\|_\infty \geq \gamma$ or $\|S\|_\infty > \gamma$ under various conditions on the GRDE. The results are used to prove the necessity of matrix inequality.

*Lemma 1:* Let $\{G_i^{\gamma_1}(k)\}$ and $\{G_i^{\gamma_2}(k)\}$ be solutions of the GRDE with initial conditions $\{G_i^{\gamma_1}(0)\}$ and $\{G_i^{\gamma_2}(0)\}$, respectively. The superscript denotes the value of $\gamma$ used in the iteration. Define $\Delta G_i(k) := G_i^{\gamma_2}(k) - G_i^{\gamma_1}(k)$. Then for $i = 1, \ldots, N$:

$$\Delta G_i(k+1) = \bar{A}_i(k)^T \Delta\tilde{G}_i(k)\bar{A}_i(k) + (\gamma_1 - \gamma_2)K_i(k)^T K_i(k) +$$
$$\left[(\gamma_1-\gamma_2)K_i(k) + B_i^T \Delta\tilde{G}_i(k)\bar{A}_i(k)\right]^T \; (V_i^{\gamma_2}(k))^{-1} \left[(\gamma_1-\gamma_2)K_i(k) + B_i^T \Delta\tilde{G}_i(k)\bar{A}_i(k)\right]$$

where $K_i(k) := (V_i^{\gamma_1}(k))^{-1}\left[B_i^T \Delta\tilde{G}_i^{\gamma_1}(k)A_i(k) + D_i^T C_i\right]$ and $\bar{A}_i(k) := A_i + B_i K_i(k)$.

*Proof:* The proof is a simple, albeit algebraically intensive, extension of a result by C. de Souza (Lemma 3.1 in [10]). A proof under the assumption that $p_{ij} = p_j \; \forall i,j \in \mathcal{N}$ is given in [19]. The proof of this lemma follows similarly. ■

*Lemma 2:* Let $\{G_i(k)\}$ be the solution to the GRDE with initial condition $\{G_i(0)\} = 0$. Fix $T$ and assume $\{V_i(k)\} > 0$ for $0 \leq k \leq T-1$. Define the following plant disturbance:

$$\bar{d}(k) = \begin{cases} V_{\theta(k)}(T-k-1)^{-1}\left(B_{\theta(k)}^T \tilde{G}_{\theta(k)}(T-k-1)A_{\theta(k)} + D_{\theta(k)}^T C_{\theta(k)}\right)x(k) & 0 \leq k \leq T-1 \\ 0 & else \end{cases} \tag{12}$$

Then, $J(\bar{d}, x_0, \theta_0) \geq x_0^T G_{\theta_0}(T) x_0$.

*Proof:* The lemma is proved by the following string of equalities / inequalities:

$$
\begin{aligned}
J(\bar{d},x_0,\theta_0) &\overset{(a)}{\geq} \sum_{k=0}^{T-1} \mathop{E}_{\Theta_k}\left[e(k)^T e(k) - \gamma^2 \bar{d}(k)^T \bar{d}(k)\right] \\
&\overset{(b)}{=} x_0^T G_{\theta_0}(T) x_0 + \sum_{k=0}^{T-1} \mathop{E}_{\Theta_{k+1}}\Big[x(k+1)^T G_{\theta(k+1)}(T-k-1)x(k+1) - x(k)^T G_{\theta(k)}(T-k)x(k) \\
&\qquad\qquad + e(k)^T e(k) - \gamma^2 \bar{d}(k)^T \bar{d}(k)\Big] \\
&\overset{(c)}{=} x_0^T G_{\theta_0}(T) x_0 + \sum_{k=0}^{T-1} \mathop{E}_{\Theta_k}\Big[x(k+1)^T \tilde{G}_{\theta(k)}(T-k-1)x(k+1) - x(k)^T G_{\theta(k)}(T-k)x(k) \\
&\qquad\qquad + e(k)^T e(k) - \gamma^2 \bar{d}(k)^T \bar{d}(k)\Big] \\
&\overset{(d)}{=} x_0^T G_{\theta_0}(T) x_0
\end{aligned}
$$

Inequality (a) follows because $\bar{d}(k) = 0$ for $k > T-1$. Since $\{G_i(0)\} = 0$, the extra terms appearing after equality (b) are a net zero quantity. Equality (c) is obtained after taking the expectation over $\theta(k+1)$. Next, substitute for $e(k)$, $x(k+1)$ using the system dynamics and for $G_{\theta(k)}(T-k)$ using the GRDE. Then complete the square with the resulting terms. Equality (d) follows by noting that the choice of $\bar{d}(k)$ in Equation 12 makes the summation on line 3 equal to zero. ■

*Lemma 3:* Let $\{G_i(k)\}$ be the solution to the GRDE with initial condition $\{G_i(0)\} = 0$. Assume there exists $T \geq 0$ such that $\{V_i(k)\} > 0$ for $0 \leq k \leq T-1$ and $V_{\theta_0}(T)$ has a negative eigenvalue for some $\theta_0 \in \mathcal{N}$. Then $\|P\|_\infty > \gamma$.

*Proof:* By assumption, there exists $r, \lambda$ such that $\lambda < 0$ and $V_{\theta_0}(T) r = \lambda r$. Define the disturbance:

$$
\bar{d}(k) = \begin{cases} r & k = 0 \\ V_{\theta(k)}(T-k)^{-1}\left(B_{\theta(k)}^T \tilde{G}_{\theta(k)}(T-k) A_{\theta(k)} + D_{\theta(k)}^T C_{\theta(k)}\right) x(k) & 1 \leq k \leq T \\ 0 & else \end{cases}
$$

Applying this disturbance to the system with $x(0) = 0$ and $\theta(0) = \theta_0$:

$$\begin{aligned}
J(\bar{d}, 0, \theta_0) &\overset{(a)}{=} r^T(D_{\theta_0}^T D_{\theta_0} - \gamma^2 I)r + \sum_{k=1}^{\infty} \underset{\Theta_k}{E}\left[e(k)^T e(k) - \gamma^2 \bar{d}(k)^T \bar{d}(k)\right] \\
&\overset{(b)}{=} r^T(D_{\theta_0}^T D_{\theta_0} - \gamma^2 I)r + \underset{\theta(1)}{E}\left[J(\bar{d}, B_{\theta_0} r, \theta(1))\right] \\
&\overset{(c)}{\geq} r^T(D_{\theta_0}^T D_{\theta_0} - \gamma^2 I)r + \underset{\theta(1)}{E}\left[r^T B_{\theta_0}^T G_{\theta(1)}(T) B_{\theta_0} r\right] \\
&\overset{(d)}{=} -r^T V_{\theta_0}(T) r = -\lambda \|r\|^2 > 0
\end{aligned}$$

Equality (a) follows from the choice of $d(0)$. Equality (b) follows from a slight abuse of notation concerning $J(\cdot, \cdot, \cdot)$. By $J(\bar{d}, B_{\theta_0} r, \theta(1))$,we mean the cost function with the system starting at $B_{\theta_0} r$ and applying $\bar{d}(k)$ for $k \geq 1$. Inequality (c) then follows from Lemma 2. Equality (d) follows from the definition of $V_{\theta_0}(T)$ after taking the expectation over $\theta(1)$. It follows that $J(\bar{d}, 0, \theta_0) > 0$ and we conclude that $\|P\|_\infty > \gamma$. ■

*Lemma 4:* Let $\{G_i(k)\}$ be the solution to the GRDE with initial condition $\{G_i(0)\} = 0$. Assume there exists $T \geq 0$ such that $\{V_i(k)\} > 0$ for $0 \leq k \leq T-1$ and $V_{\theta_0}(T)$ has an eigenvalue at zero for some $\theta_0 \in \mathcal{N}$. Then $\|P\|_\infty \geq \gamma$.

*Proof:* Using the notation of Lemma 1, let $\{G_i^\gamma(k)\}$ denote a solution of the GRDE with initial conditions $\{G_i^\gamma(0)\} = 0$. By assumption, $\{V_i^\gamma(k)\} > 0$ for $0 \leq k \leq T-1$ and $V_{\theta_0}^\gamma(T)$ has an eigenvalue at zero for some $\theta_0 \in \mathcal{N}$. Given $\epsilon > 0$, let $\{G_i^{\gamma-\epsilon}(k)\}$ denote a second solution of the GRDE with initial conditions $\{G_i^{\gamma-\epsilon}(0)\} = 0$. Thus $\{\Delta G_i(0)\} = 0$ which implies $\{\Delta \tilde{G}_i(0)\} = 0$. If $\epsilon > 0$ is sufficiently small, then $\{V_i^{\gamma-\epsilon}(k)\} > 0$ for $1 \leq k \leq T$. We can then use Lemma 1 and induction to show that $\{\Delta G_i(0)\} \geq 0$ for $0 \leq k \leq T$. It follows that $V_{\theta_0}^{\gamma-\epsilon}(T) < V_{\theta_0}^\gamma(T)$ and hence $V_{\theta_0}^{\gamma-\epsilon}(T)$ has a negative eigenvalue. By Lemma 3, $\|P\|_\infty > \gamma - \epsilon$. Since this holds $\forall \epsilon > 0$ which are sufficiently small, we conclude that $\|P\|_\infty \geq \gamma$. ■

*Lemma 5:* Assume the plant is weakly controllable. Let $\{G_i(k)\}$ be the solution to the

GRDE with initial condition $\{G_i(0)\} = 0$ and assume that $\forall k \geq 0$, $\{V_i(k)\} > 0$. If, for some $\theta_f \in \mathcal{N}$, the sequence $\lambda_{\max}(G_{\theta_f}(k))$ is unbounded, then $\|P\|_\infty > \gamma$.

*Proof:* By assumption, there exists a $\theta_f \in \mathcal{N}$ and a sequence $\{T_j\}_{j=0}^{\infty}$ such that $\lambda_{\max}(G_{\theta_f}(T_j)) \to \infty$ as $j \to \infty$. For each $j$, let $r_j$ be the eigenvector associated with $\lambda_{\max}(G_{\theta_f}(T_j))$ normalized to $\|r_j\| = 1$. Then there exists an $r^* \in \mathbb{R}^{n_x}$ and a subsequence $j_l$ such that $\lim_{j_l \to \infty} r_{j_l} = r^*$. Furthermore, $\lim_{j_l \to \infty} (r^*)^T G_{\theta_f}(T_{j_l}) r^* = \infty$. To ease some of the notation below, we now refer to this subsequence as $G_{\theta_f}(T_l)$ with $(r^*)^T G_{\theta_f}(T_l) r^* \to \infty$.

Now apply the assumption of weak controllability: Given $x(0) = 0$ and any initial mode, $\theta(0)$, there exists a time, $T_c$, and an input, $d_c(k)$, such that $p_c := \Pr[x(T_c) = r^* \text{ and } \theta(T_c) = \theta_f] > 0$. Define a second input on $T_c \leq k \leq T_c + T_l - 1$:

$$d_l(k) = V_{\theta(k)}(T_l - T_c - k - 1)^{-1} \left( B_{\theta(k)}^T \tilde{G}_{\theta(k)}(T_l - T_c - k - 1) A_{\theta(k)} + D_{\theta(k)}^T C_{\theta(k)} \right) x(k)$$

We now construct a disturbance which can make the cost function arbitrarily large:

$$\bar{d}_l(k) = \begin{cases} d_c(k) & 0 \leq k \leq T_c - 1 \\ d_l(k) & \text{if } (x(T_c), \theta(T_c)) = (r^*, \theta_f) \text{ and } T_c \leq k \leq T_c + T_l - 1 \\ 0 & else \end{cases}$$

The first portion of the disturbance attempts to move the system from $(0, \theta_0)$ to $(r^*, \theta_f)$. Note that $d_l(k)$ is only applied if $d_c(k)$ successfully moves the system to the desired state and mode. By construction of $d_c(k)$, this occurs with some positive probability. When applied, $d_l(k)$ is able to make the cost function arbitrarily large. Mathematically, this argument is:

$$\begin{aligned} J(\bar{d}_l, 0, \theta_0) &= \underset{\Theta_{T_c - 1}}{E} \left[ \sum_{k=0}^{T_c - 1} e(k)^T e(k) - \gamma^2 \bar{d}_l(k)^T \bar{d}_l(k) \right] + \underset{\Theta}{E} \left[ \sum_{k=T_c}^{\infty} e(k)^T e(k) - \gamma^2 \bar{d}_l(k)^T \bar{d}_l(k) \right] \\ &\overset{(a)}{\geq} \underset{\Theta_{T_c - 1}}{E} \left[ \sum_{k=0}^{T_c - 1} e(k)^T e(k) - \gamma^2 \bar{d}_l(k)^T \bar{d}_l(k) \right] + p_c \cdot (r^*)^T G_{\theta_f}(T_l) r^* \end{aligned}$$

Inequality (a) follows by the construction of $\bar{d}_l(k)$ and by Lemma 2. The first term on the second line is a fixed cost for all $l$. By construction, the second term can be made arbitrarily large as $l \to \infty$ and thus $\exists l$ such that $J(\bar{d}_l, 0, \theta_0) > 0$. Hence $\|P\|_\infty > \gamma$. ■