Stability Region Estimates for SDRE Controlled Systems Using Sum of Squares Optimization

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Abstract

In this paper, we investigate the State-Dependent Riccati Equation method to control nonlinear systems. This method stabilizes the closed loop system around the origin. However, global asymptotic stability is not ensured. Moreover, stability analysis is complicated because the closed loop system is typically not known in a closed form. We present a theorem that turns stability region estimation into a functional search problem. Results on sum of squares polynomials are used to turn this search into a semidefinite programming problem. A simple example demonstrating this method is given.

1 Introduction

In this paper, we investigate the State-Dependent Riccati Equation (SDRE) method to control nonlinear systems [4]. The method is motivated by the standard Linear Quadratic Regulation problem. The name of the method comes from the construction of the control law. This construction uses the solution of an algebraic Riccati equation that depends on the state. The control law is, in general, a suboptimal solution to an infinite horizon, nonlinear regulation problem.

Numerous applications (see references in [5]) provide some testament to the ease of SDRE design. However, few stability results exist for this method. It is known that the control law makes the origin a locally asymptotically stable equilibrium point of the closed loop [4]. However, the SDRE controller provides no guarantees for global asymptotic stability. A global stability analysis for second order systems under SDRE control was done by Erdem and Alleyne [7]. For systems with state dimension greater than 2, the stability analysis is complicated by the difficulty of explicitly solving the state-dependent Riccati equation. Thus the closed-loop system is not known in a closed form. Consequently, local stability is occasionally demonstrated via simulation with many initial conditions [10]. More rigorous methods involve upper bounding the state trajectory to obtain an estimate of the region of attraction [12, 8]. These methods appear to give conservative results.

In this paper, we propose a method to estimate the region of attraction for SDRE controlled systems. The basic idea of the proposed method is given in a recent thesis by Parrilo [14, 15]. Consider $\dot{x} = f(x)$ and suppose we have a Lyapunov function, $V(x)$, that proves the origin is locally asymptotically stable. Thus we know that $\dot{V}(x) < 0$ for nonzero x in some neighborhood of the origin. If we can find a function $h(x)$ such that

$$(V(x) - \gamma)x^T x \geq h(x)\frac{\partial V}{\partial x}(x)f(x) \quad \forall x \in \mathbb{R}^n \qquad (1)$$

then $\dot{V}(\bar{x}) = 0$ implies $\bar{x} = 0$ or $V(\bar{x}) \geq \gamma$. This follows by evaluating Equation 1 at $\bar{x}$. $h(x)$ is a "multiplier" which proves $\dot{V}(x) \neq 0$ for $\forall x \in \hat{\mathcal{R}}_\gamma := \{x \in \mathbb{R}^n : V(x) < \gamma\}$. Figure 1 depicts the situation. Under some continuity assumptions, we conclude $\dot{V}(x) < 0 \; \forall x \in \hat{\mathcal{R}}_\gamma$ and $\hat{\mathcal{R}}_\gamma$ is an estimate of the region of attraction.

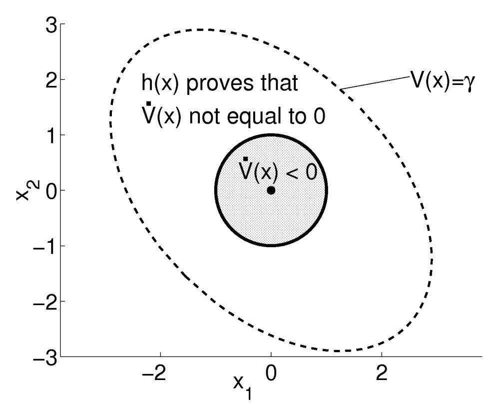

Figure 1: Stability Region Estimation

Notice that we have turned the stability region estimation problem into a search for a function, $h(x)$, which

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satisfies an inequality constraint. If we restrict f, h, and V to be polynomials, then there is a computationally tractable method to perform this search [14, 15]. This computational method uses ties between positive semidefinite matrices and sum of squares polynomials, i.e. polynomials that can be expressed as $\sum_{i=1}^{m} f_i^2(x)$ for some polynomials $\{f_i\}_{i=1}^{m}$. Given this tie, it not surprising that the search is converted into a semi-definite programming (SDP) problem. Moreover, a MATLAB toolbox is available to solve optimization problems involving sum of squares polynomials [16].

In the remainder of this paper, we seek to generalize this idea with application to SDRE controlled systems. Similar work can be found in [13]. In the next section, we briefly describe the SDRE method to control nonlinear systems. In Section 3, we generalize the idea described above for stability region estimation of SDRE controlled systems. Theorem 1 turns the stability region estimation problem into a search for several functions that satisfy inequality constraints. In Section 4, we introduce the required results on sum of squares optimization. Then we use this machinery to obtain an SDP problem that can be used to estimate the region of attraction. We present an example demonstrating the proposed method in Section 5. Conclusions are given in Section 6.

2 The SDRE Method

In this section, we give a brief description of the SDRE method to control nonlinear systems. This method can be viewed as a suboptimal solution to an infinite horizon nonlinear regulation problem. We consider input-affine systems:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ x(0) &= x_0 \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f(0) = 0$. The regulation problem is to find the control, $u(t)$ defined on $[0, \infty)$, that solves the following minimization:

$$V(x_0) = \min_u \int_0^\infty x^T Q(x) x + u^T R(x) u \; dt \tag{3}$$

where $Q(x) \geq 0$ and $R(x) > 0$ $\forall x$. We assume that the functions f, g, Q, and R are continuously differentiable. The Hamilton-Jacobi equation (HJE) can be used to solve this regulation problem [1]. Specifically, a solution of the HJE yields the optimal performance index, $V(x_0)$. The optimal control can then be constructed from the solution of the HJE. Unfortunately, the HJE is a partial differential equation that is typically difficult to solve.

The SDRE method is motivated by the standard Linear Quadratic Regulator problem. It yields a control law that is, in general, suboptimal. First, the nonlinear system is represented in state-dependent coefficient form:

$$\dot{x} = A(x)x + B(x)u \tag{4}$$

where $f(x) = A(x)x$, $B(x) = g(x)$, and $A(x)$ is continuously differentiable. We note that f continuously differentiable and $f(0) = 0$ ensures the existence of a representation in state-dependent coefficient form. We assume that $(A(x), B(x))$ is stabilizable $\forall x$ and $(A(x), Q^{1/2}(x))$ is detectable $\forall x$.

Imitating the standard Linear Quadratic Regulator, the following state feedback law is used:

$$u(x) = -R^{-1}(x) B^T(x) P x \tag{5}$$

P is the positive semidefinite solution of the State-Dependent Riccati Equation (SDRE):

$$A^T(x)P + PA(x) - PB(x)R^{-1}(x)B^T(x)P + Q(x) = 0 \tag{6}$$

Let $\mathcal{S}^{n \times n}$ denote the set of symmetric, $n \times n$ matrices. The SDRE is compactly written as $H(x, P) = 0$ where $H(x, P) : \mathbb{R}^n \times \mathcal{S}^{n \times n} \to \mathcal{S}^{n \times n}$ is defined as:

$$\begin{aligned} H(x, P) :=& A^T(x)P + PA(x) - PB(x)R^{-1}(x)B^T(x)P \\ &+ Q(x) \end{aligned}$$

The SDRE has a unique, positive semidefinite solution $\forall x$ (Corollary 13.8 in [17]). In fact, there exists a continuously differentiable function, $P(x)$, such that for all $x \in \mathbb{R}^n$, $H(x, P(x)) = 0$ and $P(x) \geq 0$ [6]. Moreover, $P(x)$ has the following property: $A(x) - B(x)R^{-1}(x)B^T(x)P(x)$ is Hurwitz $\forall x$.

Finally, we note that the state-dependent coefficient representation is not unique. In fact, there are an infinite number of ways to represent the system in this form [4] and a parameterization of all such representations is given in [9]. Huang and Lu [9] proved that if the gradient of $V(x_0)$ has a particular form, then there exists a representation such that the SDRE method recovers the optimal control. However, choosing $A(x)$ to recover the optimal controller is difficult and particular choices of $A(x)$ may lead to a cost that is far from the optimal. On the other hand, there are heuristics to obtain a good representation [5]. In the following section, we assume that $A(x)$ has been chosen and the goal is to analyze the stability properties of the closed loop system.

3 Estimation of Stability Regions

Using the SDRE control law (Equation 5), leads to the following closed loop system:

$$\dot{x} = \left[A(x) - B(x)R^{-1}(x)B^T(x)P(x)\right]x := A_{cl}(x)x \tag{7}$$

As stated above, $A_{cl}(x)$ is Hurwitz $\forall x$ so the SDRE control law (Equation 5) is pointwise stabilizing. In particular, $A_{cl}(0)$ is Hurwitz and the origin of the closed loop is locally asymptotically stable [4]. A brief outline of the proof is given. The continuous differentiability of A, B, R^{-1}, and P implies that A_{cl} is continuously differentiable. Hence the linearization of the closed loop system around the origin is given by $\dot{x} = A_{cl}(0)x$. By Lyapunov's indirect method (Theorem 3.7 in [11]), the origin is locally asymptotically stable.

Unfortunately, the SDRE controller provides no guarantees for global asymptotic stability. Define the region of attraction:

$$\mathcal{R} := \{x_0 \in \mathbb{R}^n \ : \ x(t) \to 0 \text{ as } t \to \infty \text{ if } x(0) = x_0\} \tag{8}$$

This is the set of initial conditions whose state trajectory converges asymptotically to the origin. Global asymptotic stability is equivalent to $\mathcal{R} = \mathbb{R}^n$. On the other hand, if $\mathcal{R}$ is a 'small' set, then we should not be too confident that the SDRE controller will stabilize the system when actually implemented. Thus the region of attraction provides some measure of the stabilizing properties of the SDRE controller. In principle, Zubov's theorem [11] provides a method to find $\mathcal{R}$, but there are two drawbacks. First, Zubov's method requires a partial differential equation to be solved. Second, the SDRE typically cannot be solved in closed form. Hence $P(x)$ is usually not available as an explicit function of x and the closed loop takes the implicit form:

$$\begin{aligned} &\dot{x} = A_{cl}(x, P)x \\ &H(x, P) = 0 \\ &P \geq 0 \end{aligned} \tag{9}$$

where we dropped P's dependence on x and explicitly noted A_{cl}'s dependence on P. Equation 9 emphasizes that P is not known explicitly as a function of x.

Given these difficulties, we will instead seek an estimate the region of attraction. Theorem 1, stated below, provides a construction for an estimate, $\hat{\mathcal{R}} \subset \mathcal{R}$. This theorem requires a Lyapunov function, $V(x)$, which proves the local asymptotic stability of the origin. Mathematically, let $D \subset \mathbb{R}^n$ be a neighborhood of the origin and let $V(x)$ be a continuously differentiable function satisfying:

$$\begin{aligned} &V(0) = 0 \text{ and } V(x) > 0 \ \forall x \in \mathbb{R}^n \backslash \{0\} \\ &\dot{V}(x) < 0 \ \forall x \in D \backslash \{0\} \end{aligned} \tag{10}$$

where $\dot{V}(x)$ is evaluated along trajectories of the closed loop system (Equation 9). In the next section, we turn Theorem 1 into a simple computational algorithm for stability region estimation.

Before proceeding, we need to introduce some notation. $Tr[M] := \sum_{k=1}^{n} m_{kk}$ is the trace of the matrix $M \in \mathcal{S}^{n\times n}$. We also define the leading principle minors of $P \in \mathcal{S}^{n\times n}$:

$$\Delta_k(P) := \det \begin{bmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \ddots & \vdots \\ p_{k1} & \cdots & p_{kk} \end{bmatrix}, \quad k = 1, \ldots, n$$

Finally, we use $V(x)$ to define two sets. Given $\gamma \in \mathbb{R}$, define the set $\hat{\mathcal{R}}_\gamma := \{x \in \mathbb{R}^n : V(x) < \gamma\}$. Also define the closure of this set: $cl(\hat{\mathcal{R}}) := \{x \in \mathbb{R}^n : V(x) \leq \gamma\}$. We are now prepared to state the main technical result.

Theorem 1 *If there exists $\gamma \in \mathbb{R}$, functions $r_1, s_1, \ldots, s_n : \mathbb{R}^n \times \mathcal{S}^{n\times n} \to \mathbb{R}$ and a function $R_2 : \mathbb{R}^n \times \mathcal{S}^{n\times n} \to \mathcal{S}^{n\times n}$ such that Equations 11 and 12 hold $\forall (x, P) \in \mathbb{R}^n \times \mathcal{S}^{n\times n}$, then $\hat{\mathcal{R}}_\gamma \subset \mathcal{R}$.*

$$s_k(x, P) \geq 0 \text{ for } k = 1, \ldots, n \tag{11}$$

$$\begin{aligned} &(V(x) - \gamma)\, x^T x + r_1(x, P)\frac{\partial V}{\partial x}(x) A_{cl}(x, P)x \\ &+ Tr\left[R_2(x, P)H(x, P)\right] - \sum_{k=1}^{n} s_k(x, P)\Delta_k(P) \geq 0 \end{aligned} \tag{12}$$

Proof: Take any $\bar{x}$ such that $\dot{V}(\bar{x}) = 0$. First we show that either $\bar{x} = 0$ or $V(\bar{x}) \geq \gamma$. At $\bar{x}$, there exists a unique, positive semidefinite matrix, $\bar{P}$, that satisfies the SDRE: $H(\bar{x}, \bar{P}) = 0$. Thus $Tr\left[R_2(\bar{x}, \bar{P})H(\bar{x}, \bar{P})\right] = 0$ and hence Equation 12 evaluated at $(\bar{x}, \bar{P})$ is:

$$(V(\bar{x}) - \gamma)\, \bar{x}^T \bar{x} \geq \sum_{k=1}^{n} s_k(\bar{x}, \bar{P})\Delta_k(\bar{P}) \tag{13}$$

For $k = 1, \ldots, n$, Equation 11 implies $s_k(\bar{x}, \bar{P}) \geq 0$ and $\bar{P} \geq 0$ implies $\Delta_k(\bar{P}) \geq 0$. It follows from Equation 13 that $(V(\bar{x}) - \gamma)\, \bar{x}^T \bar{x} \geq 0$. Therefore, $\dot{V}(\bar{x}) = 0$ implies that either $\bar{x} = 0$ or $V(\bar{x}) \geq \gamma$.

This result can be restated as: $\dot{V}(x) \neq 0 \ \forall x \in \hat{\mathcal{R}}_\gamma \backslash \{0\}$. By assumption, $\dot{V}(x) < 0 \ \forall x$ in a neighborhood of the origin and $V(x)$ is continuously differentiable. We conclude that $\dot{V}(x) < 0\ \forall x \in \hat{\mathcal{R}}_\gamma \backslash \{0\}$.

The proof is completed by a standard Lyapunov argument. Take any $x_0 \in \hat{\mathcal{R}}_\gamma$. Then for some $\epsilon > 0$, $x_0 \in cl(\hat{\mathcal{R}}_{\gamma-\epsilon})$. Since $cl(\hat{\mathcal{R}}_{\gamma-\epsilon}) \subset \hat{\mathcal{R}}_\gamma$, $\dot{V}(x) < 0$ $\forall x \in cl(\hat{\mathcal{R}}_{\gamma-\epsilon})$. It follows from the proof of Theorem 3.1 in [11] that if $x(0) = x_0$ then $x(t) \to 0$ as $t \to \infty$. Hence $x_0 \in \mathcal{R}$ and thus $\hat{\mathcal{R}}_\gamma \subset \mathcal{R}$. ∎

4 Sum of Squares Optimization

In the first subsection, we review the required results on sum of squares (SOS) optimization. Most of this material can be found in [14] and [15]. In the second subsection, we apply these results to obtain an SDP which can be used to estimate the region of attraction.

4.1 Sum of Squares Polynomials

First we introduce notation pertaining to polynomials of many variables. $\mathbb{N}$ denotes the set of nonnegative integers, $\{0, 1, \ldots\}$, and $\mathbb{N}^n$ is the set of n-dimensional vectors with entries in $\mathbb{N}$. For $\alpha \in \mathbb{N}^n$, a monomial in variables $\{x_i\}_{i=1}^n$ is given by $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Thus, a monomial is simply the product of powers of variables. The degree of a monomial is defined as $\deg x^\alpha := \sum_{i=1}^n \alpha_i$. A polynomial in $\{x_i\}_{i=1}^n$ is a finite linear combination of monomials:

$$p(x) := \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha = \sum_{\alpha \in \mathcal{A}} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \tag{14}$$

where $c_\alpha \in \mathbb{R}$ and $\mathcal{A}$ is a finite set of vectors in $\mathbb{N}^n$. Using the definition of deg for a monomial, the degree of $p(x)$ is defined as $\deg p(x) := \max_{\alpha \in \mathcal{A}}\ [\deg x^\alpha]$. In words, the degree of a polynomial is the largest degree of its monomials.

Given a polynomial, $p(x)$, consider the following question: Is $p(x) \geq 0\ \forall x \in \mathbb{R}^n$? Checking the global nonnegativity of a general polynomial is computationally difficult [1]. A simple sufficient condition for a polynomial, $p(x)$, to be globally nonnegative is the existence of polynomials $\{f_i\}_{i=1}^m$ such that $p(x) = \sum_{i=1}^m f_i^2(x)$. If such a decomposition exists, then we say that $p(x)$ is a sum of squares (SOS).

Surprisingly, the set of SOS polynomials is easy to characterize due to the close ties with positive semidefinite matrices. For example, consider the SOS polynomial $p(x) = x_1^2 + 2x_1^2 x_2^2 + 3x_2^2$. The vector of all monomials of degree ≤ 2 is $z(x) = [1\ x_1\ x_2\ x_1^2\ x_1x_2\ x_2^2]^T$. For any $\lambda \in \mathbb{R}$, $p(x)$ can be decomposed as:

$$p(x) = z(x)^T \left[\begin{array}{c|ccc} 0_3 & & & \\ \hline & 1 & 0 & -\lambda \\ & 0 & 2+2\lambda & 0 \\ & -\lambda & 0 & 3 \end{array}\right] z(x)$$

For some values of λ, Q is sign indefinite (e.g. $\lambda = 10$) while for other values $Q \geq 0$ (e.g. $\lambda = 0$). Theorem 2, stated below, clarifies these ideas. The theorem, introduced as a "Gram matrix" method by Choi, Lam, and Reznick [3], is enlightening and it is included for completeness. Note that a polynomial of odd degree cannot be globally nonnegative, hence the theorem restricts the polynomial degree to be even.

Theorem 2 *Let $p(x)$ be a polynomial of degree $2d$ in n variables. Let $z(x)$ be a vector of all monomials of degree $\leq d$ in n variables. The length of $z(x)$ is $l_z := \binom{n+d}{d}$.*

$p(x)$ is a SOS if and only if there exists $Q \in \mathbb{R}^{l_z \times l_z}$, $Q \geq 0$ such that $p(x) = z(x)^T Q z(x)$.

Proof: ($\Rightarrow$) If $p(x)$ is a SOS, then there exist polynomials $\{f_i\}_{i=1}^m$ such that $p(x) = \sum_{i=1}^m f_i^2(x)$. Since each $f_i(x)$ is a finite linear combination of monomials, there exists a vector, $a_i \in \mathbb{R}^{l_z}$, such that $f_i(x) = z(x)^T a_i$. Form the matrix, $A \in \mathbb{R}^{l_z \times m}$, whose columns are a_i and define $Q = AA^T \geq 0$. Then $p(x) = z(x)^T Q z(x)$.

($\Leftarrow$) Assume there exists $Q \in \mathbb{R}^{l_z \times l_z}$, $Q \geq 0$ such that $p(x) = z(x)^T Q z(x)$. If $rank(Q) = m$, then there exists a matrix $A \in \mathbb{R}^{l_z \times m}$ such that $Q = AA^T$. Let a_i denote the i^{th} column of A and define the polynomial $f_i = z(x)^T a_i$. Then $p(x) = \sum_{i=1}^m f_i^2(x)$. ∎

The key point is that checking if $p(x)$ is a SOS can be done with a SDP feasibility problem:

$$\textit{Does } \exists Q \geq 0 \textit{ such that } p(x) = z(x)^T Q z(x)?$$

Given a basis for polynomials of degree d, the polynomial equality constraint is nothing more than equality constraints on the entries of Q. SOSTOOLS [16] handles the conversion of this problem into an SDP feasibility problem. Used in conjunction with the MATLAB's symbolic toolbox, this allows the user to easily specify SOS constraints. More importantly, if $p(x, \lambda)$ is affine in λ then the following can be solved as a SDP problem:

$$\begin{gathered} \min_\lambda c^T \lambda \ \text{ s.t. } p(x, \lambda) \text{ is a SOS} \\ F(\lambda) \geq 0 \end{gathered} \tag{15}$$

[1] A discussion of the computational complexity of checking global nonnegativity is given in [14]. For a general function this task is undecidable and for a general polynomial it is NP hard.

where $F(\lambda) \geq 0$ is a linear matrix inequality constraint on λ. Equation 15 is an example of an SOS optimization. As a very simple example, we can find the minimum λ such that $x_1^2 + \lambda x_1^2 x_2^2 + 3x_2^2$ is a SOS.

We make two remarks concerning SOS polynomials. First, not all globally nonnegative polynomials are SOS. The 'size' of the set of globally nonnegative polynomials that are not SOS is unknown. Second, a common relaxation used in control theory is the S-procedure [2]. Many problems leading to an SOS optimization can be viewed as generalizations of the S-procedure.

4.2 Application to Stability Region Estimation

To apply the results of the previous subsection, we need to make additional assumptions on the nonlinear regulation problem. Specifically, we assume that $A(x)$, $B(x)$, $R(x)$ and $Q(x)$ are polynomial functions of x. We temporarily assume that $R(x)$ is unimodular: $\det(R(x)) = c$, where c is a nonzero constant independent of x. This assumption will be removed below.

We can use the linearization of the closed loop system, $A_{cl}(0)$, to find the Lyapunov function required by Theorem 1. Given any $Q > 0$, the Lyapunov equation ($A_{cl}^T(0)M + MA_{cl}(0) = -Q$) has a positive definite solution $M \in \mathbb{R}^{n \times n}$. It follows from the proof of Lyapunov indirect method (Theorem 3.7 in [11]) that $V(x) = x^T M x$ satisfies the conditions in Equation 10.

Consider the following optimization:

$$\max \gamma \quad \text{s.t.} \quad s_k(x,P) \text{ is a SOS for } k = 1, \ldots, n \tag{16}$$

$$\begin{aligned} &(V(x) - \gamma)\, x^T x + r_1(x,P) \frac{\partial V}{\partial x}(x) A_{cl}(x,P)x \\ &+ Tr\left[R_2(x,P)H(x,P)\right] \\ &- \sum_{k=1}^{n} s_k(x,P)\Delta_k(P) \text{ is a SOS} \end{aligned}$$

where we restrict our search to $r_1, s_1, \ldots, s_n$ that are polynomials in x and P with degree $\leq d$. Moreover, we constrain R_2 to be a matrix of polynomials in x and P with each entry having degree $\leq d$. By Theorem 1, any point (x, P, γ) that satisfies the constraints gives a stability region estimate $\hat{\mathcal{R}}_\gamma$. By maximizing γ, we obtain the largest possible stability region estimate. This has the form of Equation 15 but with multiple SOS constraints. γ and the coefficients of $r_1, R_2, s_1, \ldots, s_n$ are free to be chosen. These variables are the "λ" in Equation 15. Thus we can use SOSTOOLS to solve this optimization. In the following section, we demonstrate this method on a simple example.

Before proceeding, we note that the unimodular assumption on $R(x)$ was used to ensure that $R^{-1}(x)$ is a polynomial matrix. If $R(x)$ is not unimodular, then $H(x,P)$ is, in general, a rational function of x. However, $R(x) > 0$ implies that $\det(R(x)) > 0$ $\forall x$. Thus we can multiply both sides of Equation 12 by $\det(R(x))$ without affecting the constraint. Since $\det(R(x)) \cdot R^{-1}(x)$ is a matrix polynomial function of x, this turns the second constraint in the SOS optimization above into a polynomial constraint.

5 Example

The following example is from [8]. The system is given in state-dependent coefficient form:

$$\dot{x} = \begin{bmatrix} -1 & 2x_1^2 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{17}$$

The regulation cost is given with matrices $Q = I_2$ and $R = 1$. For this system, $A_{cl}(0) = \left[\begin{smallmatrix} -1 & 0 \\ 0 & \sqrt{2} \end{smallmatrix}\right]$ and solving $A_{cl}^T(0)M + MA_{cl}(0) = -I_2$ yields $M = \left[\begin{smallmatrix} 0.50 & 0 \\ 0 & \sqrt{2}/4 \end{smallmatrix}\right]$. We use the Lyapunov function $V(x) = x^T M x$ in the SOS optimization (Equation 16).

For this example, we have the following polynomial functions (bulleted entry can be inferred from matrix symmetry):

$$\begin{aligned} A_{cl}(x,P)x &= \begin{bmatrix} -x_1 + 2*x_1^2*x_2 \\ -p_{12}*x_1 + (-1-p_{22})*x_2 \end{bmatrix} \\ H(x,P) &= \begin{bmatrix} -2*p_{11}+1-p_{12}^2 & -2*p_{12}+2*x_1^2*p_{11}-p_{12}*p_{22} \\ \bullet & 4*x_1^2*p_{12}-2*p_{22}+1-p_{22}^2 \end{bmatrix} \\ \Delta_1(P) &= p_{11}, \quad \Delta_2(P) = \det\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \end{aligned}$$

Note that we don't have to solve the Riccati equation, we only need to express the equality constraints. This can be easily done with symbolic software.

Restricting the functions s_1, s_2, r_1 and R_2 to have degree ≤ 2 and solving this optimization with SOSTOOLS [16] gives the optimal value of $\gamma_{opt} = 0.88$. This optimization took 15 seconds on a 1.4 GHz processor. The shaded ellipse in Figure 2 is the estimated stability region, $\hat{\mathcal{R}}_{\gamma_{opt}}$. The union of the white rectangle and diamond is the stability region estimate obtained using vector norms [8]. The figure also shows the state trajectories (solid lines). The two solid dots are points where $\dot{V}(x) = 0$: $(x_1, x_2) = (1.19, 0.69)$ and $(x_1, x_2) = (-1.19, -0.69)$. The state trajectories are tangent to the level set $V(x) = \gamma_{opt}$ at these points. It is interesting that $V(x) = 0.88 = \gamma_{opt}$ at these points. Hence the optimization has found the largest possible level set satisfying $\dot{V} \neq 0$ over its interior.

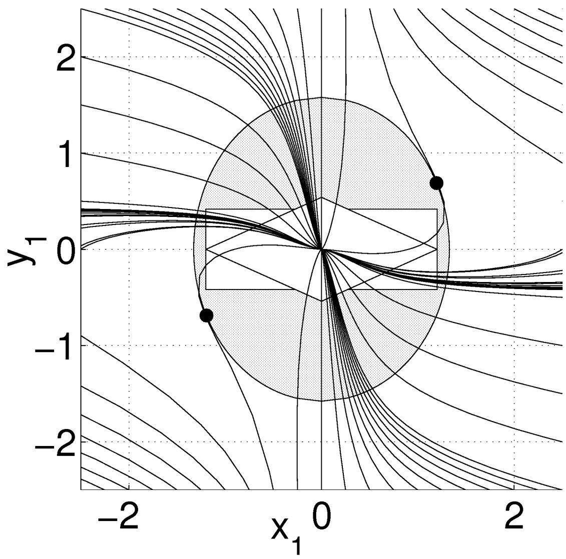

Figure 2: Stability Region Estimation: Solid lines are state trajectories. Shaded ellipse is the estimate from the SOS optimization. Union of white rectangle and diamond is the estimate from [8]. Black dots are points where $\dot{V}(x) = 0$.

6 Conclusions

In this paper, we investigated the closed loop stability of SDRE controlled systems. The stability analysis of such systems is typically complicated because the closed loop is not known in a closed form. We presented a theorem that turned stability region estimation into a functional search problem. We then turned this search into a SDP problem using results on sum of squares polynomials.

We comment on two aspects which make this method conservative: First, the stability region estimate relies on the choice of the Lyapunov function. We could solve the SOS optimization for many Lyapunov functions and take the union of the estimates. It would preferable to modify the algorithm to search simultaneously for the Lyapunov function and the multiplier functions. This is not possible in the current formulation because the term involving $r_1 \frac{\partial V}{\partial x}$ would be bilinear. Second, if $\dot{V}(x) = 0$ then this formulation prevents x from being in the stability region estimate. However, $\dot{V}(x) = 0$ does not necessarily imply that $x \notin \mathcal{R}$. Perhaps it is possible to reduce the conservativeness of the current algorithm by applying LaSalle's invariance principle.

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