A Rank Minimization Algorithm to Enhance Semidefinite Relaxations of Optimal Power Flow

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***Abstract*— The Optimal Power Flow (OPF) problem is non-convex and, for generic network structures, is NP-hard. A recent flurry of work has explored the use of semidefinite relaxations to solve the OPF problem. For general network structures, however, this approach may fail to yield solutions that are physically meaningful, in the sense that they are *high rank* – precluding their efficient mapping back to the original feasible set. In certain cases, however, there may exist a hidden rank-one optimal solution. In this paper, an iterative linearization-minimization algorithm is proposed to uncover rank-one solutions for the relaxation. The iterates are shown to converge to a stationary point. A simple bisection method is also proposed to address problems for which the linearization-minimization procedure fails to yield a rank-one optimal solution. The algorithms are tested on representative power system examples. In many cases, the linearization-minimization procedure obtains a rank-one optimal solution where the naive semidefinite relaxation fails. Furthermore, a 14-bus example is provided for which the linearization-minimization algorithm achieves a rank-one solution with a cost strictly lower than that obtained by a conventional solver. We close by discussing some rank monotonicity properties of the proposed methodology.**

***Index Terms*— Optimization, Optimal Power Flow, Semidefinite Programming, Rank Minimization.**

I. Introduction

The Optimal Power Flow (OPF) problem is a classic problem in power systems engineering that has been studied extensively beginning with the seminal work of Carpentier [1] in 1962. OPF is generally formulated as a static optimization problem where the objective is to minimize a convex cost function subject to possibly non-convex physical and operational constraints. The cost function is typically chosen to represent either the total cost of generation, line power losses, or the sum of voltage magnitudes across transmission buses. The cost is assumed to be affine or convex quadratic. The physical constraints represent the power balance equations described by Kirchhoff's current and voltage laws, while the operational constraints reflect bounds on real and reactive power generation, branch flows, and voltage magnitudes. Commonly, the set of decision variables are comprised of a combination of bus complex power injections and voltages. Naturally, the solution to OPF is given by a set of decision variables that yield a minimal cost operating point of the power system. Although OPF is straightforward to formulate, it is in general difficult to solve.

In its most general formulation, OPF is a high dimensional, non-convex optimization problem that is NP-hard. The non-convexity arises because the feasible set has a non-convex quadratic dependency on the set of complex bus voltages. Because of this non-convexity, the OPF problem may admit several locally optimal solutions – some of which may be suboptimal. Since its origin, a variety of techniques from mathematical programming, including linear (e.g DCOPF) and quadratic programming, have been proposed to solve the OPF problem. For a comprehensive literature survey, the interested reader is referred to [2] and to the references therein. In practice, the predominant approach to solving OPF involves the implementation of nonlinear optimization routines, capable of addressing the inherent non-convexity in OPF (e.g. MATPOWER [3], PSSE). These solvers, however, do not offer any guarantees regarding the global optimality of the solution they produce.

More recently, there has been a flurry of work exploring the use of semidefinite relaxations to solve the OPF problem – a novel approach first proposed by Bai et al. [4] and further refined by Lavaei et al. [5]. Essentially, this convex relaxation involves first recasting OPF as non-convex quadratically constrained quadratic program (NQCQP) and then applying the standard Shor relaxation to obtain a semidefinite relaxation (SDR) [6]. Qualitatively, this relaxation entails the exact reformulation of the NQCQP as a semidefinite program with a rank-one equality constraint on the set of feasible matrices. The SDR is obtained by removing the rank-one equality constraint. The relaxation is said to be *exact* if its optimal solution set contains a rank-one matrix – a condition which is difficult to verify in practice. Certain realizations of OPF yield semidefinite relaxations with optimal solutions of rank no greater than one. It has, however, been observed in practice that many instances of OPF yield semidefinite relaxations with optimal solutions of high rank – even though rank-one optimal solutions may exist. This raises several interesting questions. For instance, when is the minimal rank of the optimal solution set of the SDR strictly greater than one? Alternatively, in situations where the optimal solution set contains matrices of multiple rank, how might one uncover the *hidden* rank-one optimal solutions when they exist? These questions naturally point in the direction of solution methodologies involving rank minimization, or approximations therein – the approach taken in this paper.

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A. Related Work

In a similar vein, Lavaei and Low [5] propose solving the dual relaxation of the NQCQP and provide a sufficient condition under which the solution to the relaxed problem will be *globally optimal* for the original non-convex problem. Their main theoretical result states that the duality gap is zero for the NQCQP if the dual multiplier corresponding to the positive semidefinitness constraint (in the SDR) has a zero eigenvalue of multiplicity two. The authors empirically observed that this condition is satisfied for many IEEE benchmark networks and claimed that many "practical systems operating under normal conditions" will also satisfy this condition. However, several examples were given in [7] that demonstrate the failure of SDRs to yield rank-one optimal solutions in the case of networks with binding line flow constraints realizing negative locational marginal prices.

Building on this work, Zhang and Tse [8] explore as to whether the relaxation is exact for certain families of networks. The authors show that for tree topologies satisfying certain constraints on the nodal power injections, the set of feasible active power injections and its convex hull have the same Pareto frontier. Therefore, the minimization of an increasing function over the convex hull of the feasible set will yield solutions on the Pareto frontier of the non-convex problem. Moreover for linear objectives, they claim that the SDR will yield a unique rank-one optimal solution. Bose et al., build on these results by showing that NQCQPs having an underlying tree structure and satisfying certain technical conditions will yield SDRs obtaining rank-one optimal solutions [9, Thm.1].

For general problem structures, however, the naive SDR may fail to yield optimal solutions that can be efficiently mapped back to the original feasible set. In fact, interior point methods for SDPs will converge to a solution of maximal rank among all optimal solutions [10], [11]. This may lead to hidden rank-one optimal solutions when there exist optimal matrices with rank strictly greater than one. This gives rise to an important question. Is it possible to efficiently uncover rank-one matrices in the optimal solution set when the standard SDR fails? One approach is to solve a rank minimization problem over the optimal solution face of the SDR. Explicit rank minimization, however, is intractable as rank is neither continuous nor convex.

The remainder of the paper is organized as follows. Section II formulates the OPF problem and the corresponding semidefinite relaxation. This section also provides geometric insight as to why a naive semidefinite relaxation may fail to yield a rank-one optimal solution – even when it exists. Section III provides the main results including a description of the linearization-minimization and alternating bisection-minimization methods for computing low rank solutions to the semidefinite relaxation. Convergence and rank monotonicity properties of the linearization-minimization algorithm are also provided. Section IV demonstrates the proposed algorithms on several numerical examples. Conclusions and ideas for future work are given in Section V. Most proofs are given in the Appendix.

II. Problem Formulation

A. Notation

Let $\mathbb{F}$ be a field of real ($\mathbb{R}$) or complex numbers ($\mathbb{C}$) and denote by e_i the i^{th} standard basis vector in $\mathbb{R}^n$. Let $\mathcal{M}_{m,n}(\mathbb{F})$ be the set of all $m\times n$ matrices over $\mathbb{F}$, and $\mathcal{M}_n(\mathbb{F})$ the subset of $n\times n$ square matrices. $\mathcal{S}^n$ is the vector space of all $n\times n$ real symmetric matrices. $\mathcal{S}^n_+$ ($\mathcal{S}^n_{++}$) is the set of all $n\times n$ real symmetric, positive semidefinite (definite) matrices. The Hermitian analogues are denoted by $\mathcal{H}^n, \mathcal{H}^n_+, \mathcal{H}^n_{++}$. To ease notation, let $\succeq$ ($\succ$) denote the Loewner partial order induced by $\mathcal{S}^n_+$ ($\mathcal{S}^n_{++}$) on $\mathcal{S}^n$. In other words, $A\succeq B$ ($A\succ B$) if and only if $A-B\in\mathcal{S}^n_+$ ($A-B\in\mathcal{S}^n_{++}$). We use the same notation for the Loewner partial order induced by $\mathcal{H}^n_+$ ($\mathcal{H}^n_{++}$) on $\mathcal{H}^n$. For $A\in\mathcal{M}_{m,n}(\mathbb{F})$, let $\text{col}(A)$ be the column space of A. Moreover, let $A^\top$ and A^* denote the transpose and the complex conjugate transpose of A, respectively. For $A\in\mathcal{H}^n$, let $\lambda_i(A)$ and $\sigma_i(A)$ be the i^{th} largest eigenvalue and singular value of A respectively. For any matrix $A\in\mathcal{M}_{m,n}(\mathbb{F})$, let $\text{rank}(A)$ denote the number of nonzero singular values of A. Let $\text{Tr}:\mathcal{M}_n(\mathbb{F})\to\mathbb{R}$ denote the trace operator. For $A\in\mathcal{M}_{m,n}(\mathbb{F})$, let $\|A\|_* := \sum_{i=1}^n \sigma_i(A)$ be the nuclear norm of A, $\|A\|_2 := \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2} = \sigma_1(X)$ the spectral norm of A, and $\|A\|_F := \sqrt{\text{Tr}(A^\top A)}$ the Frobenius norm of A. If $A\in\mathcal{H}^n_+$, $\sigma_i(A)=\lambda_i(A)\geq 0$ for all $i=1,\dots,n$ and $\|A\|_* = \text{Tr}(A)$. Finally, let $\mathcal{L}:\mathcal{M}_{m,n}(\mathbb{C})\to\mathcal{M}_{2m,2n}(\mathbb{R})$ denote the mapping from complex to real matrices.

$$\mathcal{L}(A) = \begin{bmatrix}\text{Re}(A) & -\text{Im}(A)\\ \text{Im}(A) & \text{Re}(A)\end{bmatrix} \tag{1}$$

In addition, $A\succeq 0$ if and only if $\mathcal{L}(A)\succeq 0$. In other words, $\mathcal{L}$ maps the complex positive semidefinite cone $\mathcal{H}^n_+$ to the real positive semidefinite cone $\mathcal{S}^{2n}_+$.

B. Classical OPF Formulation

Consider a simple graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ where $\mathcal{V}=\{1,\dots,n\}$ is the set of nodes representing the network buses, and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the set of branches (transmission lines) connecting ordered pairs of buses (i,j). We denote by $\mathcal{V}_G\subseteq\mathcal{V}$ the set of buses connected to generators. Let $Y_{\text{bus}}\in\mathcal{S}^n$ be the network admittance matrix and denote by Y^{ij}_{bus} the (i,j) entry of Y_{bus}. Denote by $V=\begin{bmatrix}V_1,\dots,V_n\end{bmatrix}^\top\in\mathbb{C}^n$ the vector of complex bus voltages. In addition, the vectors of real and reactive power generation across the network buses are denoted by $P_G=\begin{bmatrix}P_{G1},\dots,P_{Gn}\end{bmatrix}^\top\in\mathbb{R}^n$ and $Q_G=\begin{bmatrix}Q_{G1},\dots,Q_{Gn}\end{bmatrix}^\top\in\mathbb{R}^n$, respectively, where $P_{Gi}=Q_{Gi}=0$ for all $i\notin\mathcal{V}_G$. Similarly, denote the vectors of real and reactive power demand as $P_D,Q_D\in\mathbb{R}^n$, respectively. Denote the complex power injection at bus i as $S_i=P_i+jQ_i$, where P_i and Q_i denote the real and reactive power respectively. It follows that the power balance equations at each bus $i\in\mathcal{V}$ satisfy

$$P_i = P_{Gi}-P_{Di} \quad\text{and}\quad Q_i = Q_{Gi}-Q_{Di}. \tag{2}$$

$$A_i(t_i, X) := \begin{bmatrix} c_{i1}(\text{Tr}[\Phi_i X] + P_{Di}) + c_{i0} - t_i & \sqrt{c_{i2}}(\text{Tr}[\Phi_i X] + P_{Di}) \\ \sqrt{c_{i2}}(\text{Tr}[\Phi_i X] + P_{Di}) & -1 \end{bmatrix} \quad B_{ij}(X) := \begin{bmatrix} -(S_{ij}^{\max})^2 & \text{Tr}[\Phi_{ij} X] & \text{Tr}[\Psi_{ij} X] \\ \text{Tr}[\Phi_{ij} X] & -1 & 0 \\ \text{Tr}[\Psi_{ij} X] & 0 & -1 \end{bmatrix} \quad (6)$$

Let $P,\ Q \in \mathbb{R}^n$ denote the vectors of real and reactive power injections, respectively. Finally, define the apparent power flow from bus i to j as $S_{ij} = P_{ij} + jQ_{ij}$. The operation of a power system must respect operational limits on bus voltages, generator injection capacities, and line flow capacities. We denote said upper and lower limits by superscripts max and min respectively.

The aim of OPF is to identify a feasible operating point which minimizes the total cost of generation. We consider quadratic objective functions, $f_i : \mathbb{R} \to \mathbb{R}$, of the form

$$f_i(P_{Gi}) = c_{i2} P_{Gi}^2 + c_{i1} P_{Gi} + c_{i0} \quad (3)$$

where $i \in \mathcal{V}_G$ and $c_{ik} \in \mathbb{R}$ for all $k = 0, 1, 2$. Specifically, the classical OPF problem is formulated as follows.

$$\begin{aligned} \underset{P_G, Q_G, V}{\text{minimize}} \quad & \sum_{i \in \mathcal{V}_G} f_i(P_{Gi}) \\ \text{subject to} \quad & P_{Gi}^{\min} \le P_{Gi} \le P_{Gi}^{\max} && \forall i \in \mathcal{V}_G && \text{(4a)} \\ & Q_{Gi}^{\min} \le Q_{Gi} \le Q_{Gi}^{\max} && \forall i \in \mathcal{V}_G && \text{(4b)} \\ & V_i^{\min} \le |V_i| \le V_i^{\max} && \forall i \in \mathcal{V} && \text{(4c)} \\ & |P_{ij}| \le P_{ij}^{\max} && \forall (i,j) \in \mathcal{E} && \text{(4d)} \\ & P_{ij}^2 + Q_{ij}^2 \le (S_{ij}^{\max})^2 && \forall (i,j) \in \mathcal{E} && \text{(4e)} \\ & P_i - jQ_i = V_i^* \sum_{j=1}^{n} Y_{\text{bus}}^{ij} V_j && \forall i \in \mathcal{V}_G && \text{(4f)} \end{aligned}$$

C. Semidefinite Relaxation

This section reviews the SDR of the OPF problem (4). The relaxation entails the reformulation of problem (4) as a quadratic program in $V \in \mathbb{C}^n$ which admits a rank relaxation to a semidefinite program. The reader is refered to [4], [5], [9] for more details. First, define the following matrices:

$$\begin{aligned} Y_i &:= e_i e_i^\top Y_{\text{bus}} && \forall i \in \mathcal{V} \\ Y_{ij} &:= \left(j\frac{b_{ij}}{2} - Y_{\text{bus}}^{ij} \right) e_i e_i^\top + Y_{\text{bus}}^{ij} e_i e_j^\top && \forall (i,j) \in \mathcal{E} \end{aligned}$$

where b_{ij} is the total shunt charging susceptance of branch (i, j). For all $i \in \mathcal{V}$, $(i, j) \in \mathcal{E}$ define weighting matrices as

$$\begin{aligned} \Phi_i &:= \frac{Y_i^* + Y_i}{2} \quad \Phi_{ij} := \frac{Y_{ij}^* + Y_{ij}}{2} \quad M_i := e_i e_i^\top \\ \Psi_i &:= \frac{Y_i^* - Y_i}{2j} \quad \Psi_{ij} := \frac{Y_{ij}^* - Y_{ij}}{2j} \end{aligned} \quad (5)$$

where, $\Phi_i, \Psi_i, \Phi_{ij}, \Psi_{ij} \in \mathcal{H}^n$, and $M_i \in \mathcal{S}^n$ such that

$$\begin{aligned} P_i &= \text{Tr}[\Phi_i VV^*] \quad Q_i = \text{Tr}[\Psi_i VV^*] \quad V_i^2 = \text{Tr}[M_i VV^*] \\ P_{ij} &= \text{Tr}[\Phi_{ij} VV^*] \quad Q_{ij} = \text{Tr}[\Psi_{ij} VV^*]. \end{aligned}$$

Semidefinite programs require both linear objective and constraints. While the objective function in problem (4) is quadratic, it can be reformulated in its epigraph form by letting, $f_i(P_{Gi}) \le t_i$, where $t_i \in \mathbb{R}_+$. Using a change of variables defined by equation (2) and the Schur complement formula, one can readily verify that

$$f_i(P_{Gi}) \le t_i \iff A_i(t_i, VV^*) \preceq 0,$$

where $A_i(\cdot, \cdot)$ is defined in (6). Similarly, the quadratic constraint in (4e) is equivalent to $B_{ij}(VV^*) \preceq 0$, where $B_{ij}(\cdot)$ is defined in (6). Lastly, for all $i \in \mathcal{V}$, use equation (2) to define,

$$\begin{aligned} P_i^{\min} &:= P_{Gi}^{\min} - P_{Di} & P_i^{\max} &:= P_{Gi}^{\max} - P_{Di} \\ Q_i^{\min} &:= Q_{Gi}^{\min} - Q_{Di} & Q_i^{\max} &:= Q_{Gi}^{\max} - Q_{Di} \end{aligned}$$

The preceding transformations lead to an equivalent quadratic formulation of the classical OPF problem (4). It can be equivalently reformulated as a rank-one constrained SDP through a change of variables $X := VV^*$ with an additional constraint on the positive semidefiniteness of X. Finally, the standard semidefinite relaxation entails the removal of the rank-one constraint – the only source of non-convexity. We now present the semidefinite relaxation in both its polar and rectangular forms.

Leveraging on the preceding development, it's straightforward to cast the semidefinite relaxation of OPF in its native polar coordinates $X \in \mathcal{H}_+^n$ as problem (7).

$$\begin{aligned} \underset{t \in \mathbb{R}_+^d,\ X \succeq 0}{\text{minimize}} \quad & \mathbf{1}^\top t && && (7) \\ \text{subject to} \quad & P_i^{\min} \le \text{Tr}(\Phi_i X) \le P_i^{\max} && \forall i \in \mathcal{V} \\ & Q_i^{\min} \le \text{Tr}(\Psi_i X) \le Q_i^{\max} && \forall i \in \mathcal{V} \\ & (V_i^{\min})^2 \le \text{Tr}(M_i X) \le (V_i^{\max})^2 && \forall i \in \mathcal{V} \\ & \text{Tr}[\Phi_{ij} X] \le P_{ij}^{\max} && \forall (i,j) \in \mathcal{E} \\ & A_i(t_i, X) \preceq 0 && \forall i \in \mathcal{V}_G \\ & B_{ij}(X) \preceq 0 && \forall (i,j) \in \mathcal{E} \end{aligned}$$

where $d := |\mathcal{V}_G|$ denotes the number of buses connected to generators.

Rectangular Coordinates: An equivalent relaxation in rectangular coordinates follows from the application of the transformation $\mathcal{L} : \mathcal{M}_{m,n}(\mathbb{C}) \to \mathcal{M}_{2m,2n}(\mathbb{R})$, defined in (1), to the complex weighting matrices in (5) to obtain the corresponding real weighting matrices. For clarity in exposition, we employ the shorthand notation $W^r := \mathcal{L}(W)$ to denote the real transformation for any $W \in \mathcal{M}_{m,n}(\mathbb{C})$. The *semidefinite relaxation in rectangular coordinates* (8) is obtained with a change of variables $X := V^r (V^r)^\top \in \mathcal{S}_+^{2n}$. We note that the matrices $A_i(\cdot, \cdot)$ and $B_{ij}(\cdot)$ are now implicitly defined in terms of the real weighting matrices $(\Phi_i^r, \Phi_{ij}^r, \Psi_{ij}^r)$.

$$
\begin{aligned}
\underset{t\in\mathbb{R}_+^d,\ X\succeq 0}{\text{minimize}} \quad & \mathbf{1}^\top t && && (8)\\
\text{subject to} \quad & P_i^{\min} \le \mathrm{Tr}\,(\Phi_i^r X) \le P_i^{\max} && \forall i \in \mathcal{V}\\
& Q_i^{\min} \le \mathrm{Tr}\,(\Psi_i^r X) \le Q_i^{\max} && \forall i \in \mathcal{V}\\
& (V_i^{\min})^2 \le \mathrm{Tr}(M_i^r X) \le (V_i^{\max})^2 && \forall i \in \mathcal{V}\\
& \mathrm{Tr}\,(\Phi_i^r X) \le P_{ij}^{\max} && \forall (i,j) \in \mathcal{E}\\
& A_i(t_i, X) \preceq 0 && \forall i \in \mathcal{V}_G\\
& B_{ij}(X) \preceq 0 && \forall (i,j) \in \mathcal{E}
\end{aligned}
$$

For the remainder of the paper, we will restrict our attention to the semidefinite relaxation in rectangular form (8). Moving forward, we let the set $\mathcal{D} \subseteq \mathbb{R}_+^d \times \mathcal{S}_+^n$ denote the set of *feasible solutions* to problem (8) and $J(t, X)$ denote the *cost* incurred for any pair $(t, X) \in \mathcal{D}$. Clearly, any pair $(t^\circ, X^\circ) \in \mathcal{D}$ achieving the minimum of (8) yields a lower bound,

$$J^\circ := J(t^\circ, X^\circ) = \underset{(t,X)\in\mathcal{D}}{\text{minimize}}\ J(t, X), \tag{9}$$

on the minimum value of the original non-convex OPF problem (4). This evokes a pair of interesting questions. For what family of OPF problems is the lower bound J° achieved? And, in such cases, how might one efficiently construct a point in the original non-convex feasible set achieving the lower bound J°?

In the following section, we reinterpret these questions in terms of the optimal facial structure of the semidefinite relaxation of OPF. In particular, we leverage existing results from the semidefinite programming literature to provide geometric insight as to why the naive semidefinite relaxation may fail to yield a rank-one optimal solution – even when it exists. Moreover, we discuss how one might numerically verify the nonexistence of rank-one optimal solutions to the semidefinite relaxation of OPF using sum of squares (SOS) programming.

D. Insight

Most commercial solvers implementing semidefinite programs (SDP) rely on primal-dual interior point methods. Of relevance to the discussion at hand, are the convergence properties of such numerical methods. Namely, interior point methods are guaranteed to converge to a primal-dual optimal solution pair of maximal rank for nondegenerate SDPs [10], [11]. More precisely, let $\mathcal{D}$ denote the *feasible spectrahedron* for a given SDP. And, denote by $\mathcal{F} \subseteq \mathcal{D}$ and $\mathrm{ri}(\mathcal{F})$ the *primal optimal face* and its *relative interior*, respectively. The following result from [11] establishes that points belonging to the relative interior of the optimal face have maximum rank among all optimal solutions of the semidefinite program and that interior point methods are guaranteed to converge to optimal solutions in the relative interior of the optimal face.

Theorem 2.1: [11, Lemma 3.1, 4.2] For any $X \in \mathcal{F}$ and $Y \in \mathrm{ri}(\mathcal{F})$, $\mathrm{col}(X) \subseteq \mathrm{col}(Y)$. In other words,

$$\mathrm{rank}(Y) = \max\{\mathrm{rank}(X) : X \in \mathcal{F}\} \quad \forall \ \ Y \in \mathrm{ri}(\mathcal{F}).$$

Moreover, interior point methods for semidefinite programs converge to an optimal solution $Y \in \mathrm{ri}(\mathcal{F})$.

The implication of Theorem 2.1 is that *a naive semidefinite relaxation of the OPF problem will fail* to yield an optimal solution that can be efficiently mapped back to the original feasible solution set (i.e. rank-one solutions) if the optimal face of the semidefinite relaxation contains points with rank strictly greater than one. With the aim of quantifying the role of optimal facial structure in either realizing or obfuscating efficiency of the semidefinite relaxation, we delineate the following three categories of optimal facial geometries.

C1. The maximal rank of the optimal face is one.
C2. The minimal rank of the optimal face is strictly greater than one.
C3. The minimal rank of the optimal face is one, while the maximal rank is strictly greater than one.

Category C1 will have *only rank-one optimal solutions*,

$$\max\{\mathrm{rank}(X) : X \in \mathcal{F}\} = 1.$$

In this case, the naive semidefinite relaxation will yield a rank-one optimal point that can be easily mapped (through a dyadic decomposition) to a globally optimal solution of the original non-convex OPF problem. Recent results have shown that OPF problems, satisfying certain technical conditions and defined on networks with radial topologies, yield semidefinite relaxations with at most rank-one optimal solutions [9], [8].

Category C2 corresponds to semidefinite relaxations that do not admit rank-one optimal points. Namely, the minimal rank of the optimal face is strictly greater than one

$$\min\{\mathrm{rank}(X) : X \in \mathcal{F}\} > 1,$$

which implies that the optimal value of such a semidefinite relaxation would yield a *strict lower bound* on the global minimum of the original OPF problem (4). This amounts to a non-zero optimality gap between the relaxation and the original problem. Clearly then, verifying strictness of the global lower bound given by the semidefinite relaxation amounts to verifying emptiness of the intersection between the optimal face and the set of all rank-one positive semidefinite matrices.

This condition has a natural geometric interpretation for matrices belonging to the positive semidefinite cone $\mathcal{S}_+^n$. Namely, a matrix $X \in \mathcal{S}_+^n$ is rank-one if and only if it spans an *extreme ray* of the cone [12]. Hence, the semidefinite relaxation will possess a rank-one optimal solution if and only if its optimal face $\mathcal{F}$ has a nonempty intersection with and extreme ray of $\mathcal{S}_+^n$.

Remark 1: *(Positivstellensatz).* The nonexistence of rank-one optimal solutions to the semidefinite relaxation can be verified numerically by means of a Positivstellensatz-based infeasibility certificate. Stengle's Positivstellensatz states that if a system of polynomial equations and inequalities defining a semialgebraic set is infeasible, it is always possible to find an algebraic certificate that confirms that the said semialgebraic set is empty [13]. The construction of polynomials that satisfy said identity can be accomplished through sum of squares programming (SOS) with bounded degree polynomials on the semialgebraic set defined by the intersection of the optimal face of the semidefinite relaxation with the rank-one algebraic variety. One drawback of this approach, however, is that the computational complexity required to implement such SOS methods grows rapidly as a function of the number of constraints, variables, and degree of polynomials. □

Remark 2: *(A 3-bus system with no rank-one solutions).* There exist exceedingly simple power systems whose OPF semidefinite relaxation does not admit a rank-one optimal solution. Consider, for example, the three bus system examined in [7]. One can readily verify, through exhaustive search of the non-convex feasible set, that the optimal value of the semidefinite relaxation is a strict lower bound on the global optimum of the OPF problem. This implies the nonexistence of a rank-one optimal solution to the relaxation. Moreover, this example gives pause, as it reveals the potential fragility of such relaxations. Further theoretical work is required to provide general sufficient conditions under which the semidefinite relaxation of OPF is guaranteed to fail. □

Category C3 refers to the family of semidefinite relaxations possessing both high rank and *hidden* rank-one optimal solutions. More precisely,

$$\min\{\text{rank}(X) : X \in \mathcal{F}\} = 1 \quad \text{and}$$
$$\max\{\text{rank}(X) : X \in \mathcal{F}\} > 1.$$

We refer to the rank-one solutions as *hidden*, given the propensity of interior point methods to converge to optimal points of maximal rank (c.f. Theorem 2.1). A solution to a semidefinite relaxation belonging to this family will fail to yield useful information regarding the potential optimality gap induced by the relaxation. In certain cases, (e.g. when the dual multiplier corresponding to the constraint on positive-semidefinitness in the relaxed problem has a zero eigenvalue of multiplicity two [5]), the solution to the semidefinite relaxation can be efficiently mapped back to feasible set of the OPF problem without loss of optimality. In general, however, mapping a high-rank solution to the semidefinite relaxation back to the original feasible set is NP-hard.

This inspires the exploration of methodologies capable of uncovering *hidden* rank-one optimal solutions to the semidefinite relaxation, when they exist. Qualitatively, this amounts to identifying matrices of minimal rank among all matrices belonging to the optimal face of the semidefinite program (8). The optimal face is defined as

$$\mathcal{F} = \{(t, X) \in \mathcal{D} \;:\; J(t, X) \le J^\circ\}, \tag{10}$$

where $J(t, X)$ denotes the cost incurred by any feasible pair $(t, X) \in \mathcal{D}$ and J° denotes the optimal value of (8). Essentially, computing an optimal point of minimal rank entails the solution of a rank minimization problem restricted to the optimal face of the semidefinite relaxation.

$$\begin{aligned} \underset{t,\, X}{\text{minimize}} \quad & \text{rank}(X) \\ \text{subject to} \quad & (t, X) \in \mathcal{F} \end{aligned} \tag{11}$$

Remark 3: *(OPF as Rank Minimization).* In the event that the optimal face of the semidefinite relaxation possesses a rank-one matrix, problem (11) reveals that OPF can be equivalently reformulated as a problem of rank minimization over a spectrahedral set. Explicit rank minimization, however, is known to be computationally NP-hard in general. □

As a tractable alternative, one might naturally solve an approximation to the rank minimization problem through suitable choice of a convex surrogate for rank, which is neither continuous nor convex.

In [14], Fazel et al. prove that the nuclear norm is the convex envelope of rank on spectral norm balls. This property fails to hold, however, for general convex sets. While the nuclear norm has been shown to be an effective surrogate for rank over certain affine equality constrained sets satisfying a restricted isometry property [15], it can behave quite poorly over more general spectrahedral sets. In fact, when optimizing over the feasible spectrahedron derived from the semidefinite relaxation of the OPF problem, one can show that naive nuclear norm minimization will frequently fail to find *low-rank* feasible solutions – even when they exist. This behavior derives from the near invariance of nuclear norm over the feasible spectrahedron – an observation also made by the authors in [16]. More precisely, for any feasible pair $(t, X) \in \mathcal{D}$, one can readily derive the following lower and upper bounds on the nuclear norm of X.

$$\sum_{i=1}^{n} (V_i^{\min})^2 \;\le\; \|X\|_* \;\le\; \sum_{i=1}^{n} (V_i^{\max})^2 \tag{12}$$

In practice, the lower and upper bounds on bus voltage magnitude – $V_i^{\min}$ and $V_i^{\max}$, respectively – are chosen to be close to 1 per unit (p.u) for all buses i, because of strict requirements on power quality. This suggests that all feasible solutions to (8) have nearly equal nuclear norm, which reveals why naive nuclear norm regularization may fail to distinguish between *low* and *high-rank* solutions.

As an alternative to nuclear norm minimization, we analyze in Section III the behavior of an algorithm that involves solving a sequence of weighted trace minimization problems, where the weighting matrices are recursively chosen to drive small (but non-zero) eigenvalues of the successive solution iterates to zero – an approach which derives largely from the work in [17].

III. Main Results

Faced with an intractable rank minimization problem (11), we take the approach of approximating rank with a continuously differentiable, strictly concave function $g : \mathcal{S}_+^n \to \mathbb{R}$. With g acting as a surrogate for rank, we instead propose to solve the alternative problem

$$\begin{aligned} \text{minimize} \quad & g(X) \\ \text{subject to} \quad & X \in \mathcal{C} \end{aligned} \tag{13}$$

where $\mathcal{C} \subset \mathcal{S}_+^n$ is a convex, compact subset of the positive semidefinite cone. To address the non-convexity of problem (13), in Section III-A we describe a standard *iterative linearization-minimization algorithm* to obtain a sequence of convex differentiable problems, whose optimal solutions are guaranteed to converge to a local minimum of g on $\mathcal{C}$.

In Section III-B, we focus our attention on specific instances of g belonging to the *log-det family*. Namely, we consider

$$g(X) = \log\det(f(X) + \delta I),$$

where the underlying parameterization (in the regularization constant $\delta > 0$ and mapping $f : \mathcal{S}_+^n \to \mathcal{S}_+^n$) controls the quality of g's approximation to rank. Notice, that for $f(X) = X$, we recover the classical log-det heuristic [17], [18]. Working with rank surrogates of this form, we employ a gradient descent method to compute a local minimum of g – with the aim of recovering a rank-one matrix belonging to the optimal face $\mathcal{F}$ of the semidefinite relaxation for OPF (c.f. Equation (10)). In the event that we fail to recover a rank-one point in the optimal face, we suggest in Section III-C a simple bisection algorithm to iteratively relax the set of feasible points until a rank-one feasible point is obtained. Finally, in Section III-D we explore how one might iteratively choose a sequence of regularization parameters $\{\delta_k\}$, so that the resulting solution iterates satisfy certain *rank monotonicity* properties.

A. Iterative Linearization-Minimization

In this section, we introduce the iterative linearization–minimization algorithm and we discuss its convergence properties. We work in a general framework where we consider arbitrary strictly concave functions and arbitrary convex compact subsets of the positive semidefinite cone.

Let $\{g_k\}_{k\in\mathbb{N}}$ be a sequence of smooth, strictly concave functions, converging pointwise to g over a convex, compact set $\mathcal{C} \subset \mathcal{S}_+^n$. Moreover, assume that the sequence is monotonic nonincreasing. Namely,

$$g_{k+1}(X) \le g_k(X) \quad \forall \;\; X \in \mathcal{C} \;\; \text{and} \;\; k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, we define the linearization of $g_k(X)$ around $Y \in \mathcal{C}$ as

$$\Lambda_k(X, Y) \;=\; g_k(Y) + \mathrm{Tr}[\nabla g_k(Y)^\top (X - Y)],$$

from which we readily derive the *iterative linearization-minimization algorithm* as follows.

$$\begin{aligned} X_{k+1} &\in \underset{X\in\mathcal{C}}{\operatorname{argmin}} \;\; \Lambda_{k+1}(X, X_k) \\ &= \underset{X\in\mathcal{C}}{\operatorname{argmin}} \;\; \mathrm{Tr}\left[\nabla g_{k+1}(X_k)^\top X\right], \end{aligned} \tag{14}$$

where $\nabla g : \mathcal{S}_+^n \to \mathcal{S}^n$ is the gradient of g. The algorithm can be initialized at any point $X_0 \in \mathcal{S}_+^n$. Before presenting the result on convergence, we have the following useful Lemma.

Lemma 3.1: Consider a sequence of iterates $\{X_k\}$ generated by the recurrence relation (14). We have that

$$g_{k+1}(X_{k+1}) < g_k(X_k) \tag{15}$$

for all $k \in \mathbb{N}$ such that $X_k \neq X_{k+1}$. And

$$\lim_{k\to\infty} \; \mathrm{Tr}\left[\nabla g_{k+1}(X_k)^\top (X_{k+1} - X_k)\right] = 0. \tag{16}$$

Proof: First consider the proof of (15). By strict concavity, we have that $g_k(X)$ is strictly less than it's linearization around $Y \in \mathcal{C}$ for all $X \neq Y$. In particular, for $X_k \neq X_{k+1}$, we have that

$$g_{k+1}(X_{k+1}) < g_{k+1}(X_k) + \mathrm{Tr}[\nabla g_{k+1}(X_k)^\top (X_{k+1} - X_k)].$$

And by optimality of X_{k+1} according to (14), we arrive at $g_{k+1}(X_{k+1}) < g_{k+1}(X_k)$. The desired result follows immediately from monotonicity of the sequence $\{g_k\}$.

Consider now the proof of (16). The sequence $\{g_k(X_k)\}$ of real numbers is bounded from below by continuity of the limit function g over a compact set $\mathcal{C}$. Hence, it follows from (15) and the monotone convergence theorem that the sequence $\{g_k(X_k)\}$ has a finite limit. The desired result follows from the fact that the quantity $g_{k+1}(X_k) + \mathrm{Tr}[\nabla g_{k+1}(X_k)^\top (X_{k+1} - X_k)]$ is sandwiched from above and below by $g_k(X_k)$ and $g_{k+1}(X_{k+1})$, respectively. ■

We now discuss the convergence properties of the proposed algorithm. First, we provide the definition of a stationary point.

Definition 3.2: Let $h : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ be a continuously differentiable function defined on the set $\mathcal{K} \subseteq \mathcal{M}_n(\mathbb{R})$. A matrix $X \in \mathcal{K}$ satisfying

$$\mathrm{Tr}\left[\nabla h(X)^\top (Y - X)\right] \ge 0 \quad \forall \;\; Y \in \mathcal{K}$$

is a said to be a *stationary point* of h over $\mathcal{K}$.

Theorem 3.3: Consider a sequence of iterates $\{X_k\}$ generated by the recurrence relation (14). We have the following convergence properties.

(a) The sequence $\{X_k\}$ satisfies $\|X_{k+1} - X_k\|_F \to 0$.

(b) Every limit point of $\{X_k\}$ is a stationary point.

While the proof of Theorem 3.3 (which can be found in Appendix I) follows largely from arguments in [19], it is included for completeness, as it ameliorates a minor gap in the proof of a similar result (Theorem II.2) appearing in [18].

B. A Rank Minimization Heuristic

In a similar spirit with previous work [17], [18], we now consider a surrogate family for rank of the *log-det* type. More precisely, we define the sequence of surrogates $\{g_k\}$ as

$$g_k(X) = \log\det(f(X) + \delta_k I), \quad k = 1, 2, \ldots \tag{17}$$

where the sequence of regularization parameters $\{\delta_k\}$ is assumed to be monotonic nonincreasing with a finite limit $\delta > 0$. Moreover, we restrict $f : \mathcal{S}_+^n \to \mathcal{S}_+^n$ to a family of mappings that preserve *strict concavity* and *continuous* *differentiability* of g_k on a convex, compact subset $\mathcal{C}$ of the positive semidefinite cone for all k. It follows readily, by the monotonic convergence of $\{\delta_k\} \to \delta > 0$, that $\{g_k\}$ is a monotonic sequence of functions satisfying

$$\lim_{k\to\infty} g_k(X) = g(X) := \log\det(f(X) + \delta I)$$

for every in $X \in \mathcal{C}$. The gradient is easily computed as

$$\nabla g_k(X) = (f(X) + \delta_k I)^{-1} \nabla f(X).$$

An iterative linearization-minimization of the functions $\{g_k\}$ in (17) will converge, by Theorem 3.3, to a stationary point of g on some compact set. Section III-D discusses the selection of regularization coefficients $\{\delta_k\}$ to ensure certain rank monotonicty properties of the iterates $\{X_k\}$.

Remark 4: For the *identity mapping* $f(X) = X$, we recover the classical log-det heuristic [17]. Other natural candidates for f include the *quadratic*, $f(X) = X^\top X$, or *exponential mappings*, $f(X) = I - \exp(-\tau X)$, where $\tau > 0$ is a regularization constant controlling the concavity of f. We remark that there exists a broad literature quantifying, both analytically and empirically, the behavior of a much larger family of rank surrogates that go beyond the log-det family. However, such a discussion is beyond the scope of this paper and we refer the reader to [15], [20], [21] for a partial cross section of relevant literature. A careful analysis exploring the appropriate choice of surrogates for rank under semidefinite relaxations for OPF is left for future work. □

Recall from Section II-D, our objective of efficiently extracting *hidden* rank-one matrices belonging to the optimal face $\mathcal{F}$ of the OPF semidefinite relaxation. Leveraging on the preceding development, we now offer a simple iterative heuristic in Table I with the aim of doing precisely that. Given a high rank (>1) solution $(t^\circ, X^\circ) \in \mathcal{F}$ to the semidefinite relaxation (9), we initialize the iterative linearization-minimization algorithm with a feasible set restricted to the optimal face $\mathcal{F}$, and initial condition (t°, X°). For notational brevity, we denote the iterative linearization-minimization algorithm in Table I as the mapping

$$(\bar{t}, \overline{X}) \;=\; \Gamma(\mathcal{F}, t^\circ, X^\circ),$$

where $(\bar{t}, \overline{X}) \in \mathcal{F}$ denotes the converged value (within a prescribed tolerance) of the gradient descent method.

TABLE I: Iterative Linearization-Minimization Algorithm

Algorithm 1: $(\bar{t}, \overline{X}) \;=\; \Gamma(\mathcal{C}, t_0, X_0)$

Given a convex, compact set $\mathcal{C} \subset \mathcal{D}$, an initial condition (t_0, X_0), a stopping tolerance $\varepsilon > 0$, and maximum number of iterations $\overline{k}$
Initialize $k = 0$
Repeat
1. *Compute.* $(t_{k+1}, X_{k+1}) \in \underset{(t,X)\in\mathcal{C}}{\operatorname{argmin}} \operatorname{Tr}[\nabla g_{k+1}(X_k)^\top X]$
2. *Update.* $k = k + 1$

Until $\|X_k - X_{k-1}\|_F < \varepsilon$ or $k = \overline{k}$
Output $(\bar{t}, \overline{X}) = (t_k, X_k)$

C. An Alternating Bisection-Minimization Method

In the event that the rank minimization heuristic fails to yield a rank-one solution in $\mathcal{F}$ (i.e. $\operatorname{rank}(\overline{X}) > 1$), one of two motives could be at play. Firstly, there may not exist a rank-one point belonging to the optimal face $\mathcal{F}$ (c.f. category C2). Secondly, while there may exist a rank-one point in $\mathcal{F}$, the heuristic may fail to recover it, as we have provided no guarantee on the algorithm's ability to recover a minimum rank solution. In either case, we offer in Table II a simple bisection method to iteratively relax the set of feasible points until a rank-one feasible point is obtained. And naturally, there is no guarantee as to whether the resulting rank-one point is globally optimal for the original OPF problem (11), unless the global lower bound J° is achieved.

The iterative relaxation of the feasible set obeys a simple *bisection rule* described as follows. First, let $\overline{J}$ denote a global upper bound on the optimal cost of the OPF problem – a quantity that most commercial solvers can readily provide. If the rank minimization heuristic (c.f. Table I) fails to recover a rank-one point on the optimal face $\mathcal{F}$, i.e.

$$\operatorname{rank}(\overline{X}_0) > 1, \quad \text{where} \;\; (\bar{t}_0, \overline{X}_0) = \Gamma(\mathcal{F}, t^\circ, X^\circ),$$

we enlarge the feasible set to include points incurring a cost no greater than than the bisection point, $J_1 := J^\circ + 0.5(\overline{J} - J^\circ)$ in the interval $[J^\circ, \overline{J}]$. The expanded feasible set is

$$\mathcal{F}_1 = \{(t, X) \in \mathcal{D} \;:\; J(t, X) \le J_1\},$$

and apply the rank minimization heuristic over the new initial condition $(\bar{t}_0, \overline{X}_0)$ and feasible set $\mathcal{F}_1$ to obtain an updated solution $(\bar{t}_1, \overline{X}_1) = \Gamma(\mathcal{F}_1, \bar{t}_0, \overline{X}_0)$. The subsequent decision to bisect from above or below J_1, at the following time step, depends on the rank of the current solution $\overline{X}_1$. This alternation between bisection and optimization repeats ad nauseum until the bisection points converge to within a prescribed tolerance of one another. We refer the reader to Table II for a precise description of said method.

Remark 5: We mention two caveats. First, for certain realizations of the OPF problem, one may not be able to efficiently obtain a global upper bound, $\overline{J}$, through which to parameterize the bisection algorithm, as finding a point belonging to the non-convex feasible set of OPF is, in general, NP-hard. Second, the bisection algorithm's ability to recover a rank-one solution may be sensitive to the recursive choice of initial condition for the rank minimization algorithm at each bisection step. We have suggested one possible recursion, where the solution at the previous bisection step, initializes the rank minimization algorithm at the current step. One can imagine many variations in said scheme. □

D. Rank Monotonicity

Success of the iterative rank minimization algorithm (I) hinges on its convergence to a rank-one point belonging to

TABLE II: Alternating Bisection-Minimization Algorithm

Algorithm 2: *Alternating Bisection-Minimization*

Given bounds (ℓ_0, u_0), an initial condition $(\bar{t}_0, \overline{X}_0)$, and stopping tolerance $\varepsilon > 0$
1. *Bisect.* $J_1 = \ell_0 + \frac{1}{2}(u_0 - \ell_0)$
2. *Set.* $k = 1$

Repeat

7. *Update set.* $\mathcal{F}_k = \{(t, X) \in \mathcal{D} : J(t, X) \leq J_k\}$
8. *Call Algorithm 1.* $(\bar{t}_k, \overline{X}_k) = \Gamma(\mathcal{F}_k, \bar{t}_{k-1}, \overline{X}_{k-1})$
9. **if** rank$(\overline{X}_k) > 1$
   1. *Bisect from above.* $J_{k+1} = J_k + \frac{1}{2}(u_k - J_k)$
   2. *Update bounds.* $\ell_{k+1} = J_k,\ \ u_{k+1} = u_k,$
10. **else if** rank$(\overline{X}_k) = 1$
    1. *Bisect from below.* $J_{k+1} = \ell_k + \frac{1}{2}(J_k - \ell_k)$
    2. *Update bounds.* $\ell_{k+1} = \ell_k,\ \ u_{k+1} = J_k,$
11. *Update time.* $k = k + 1$

Until $|J_k - J_{k-1}| < \varepsilon$
Output $J_k,\ \bar{t}_{k-1},\ \overline{X}_{k-1}$

the optimal face $\mathcal{F}$. As such, it's natural to ask as to whether the iterates $\{X_k\}$ are monotonic in rank? Namely, can one guarantee that the rank$(X_{k+1}) \leq$ rank(X_k) for all k? This is a nuanced question, as the practical evaluation of rank requires approximation.

The rank of a matrix is equal to the number of non-zero singular values of the matrix. This fact is useful for theoretical analyses but it raises subtle issues when performing numerical computations with floating point numbers. In particular, the finite precision of floating point arithmetic implies that nonzero singular values cannot be distinguished from zero if their magnitude is sufficiently small. Conversely, numerical errors that arise in floating point computations can cause a matrix to have spurious non-zero singular values. As a consequence a threshold tolerance is typically used to determine the number of non-zero singular values and hence the rank of a matrix. To be precise, the numerical results generated in this paper calculate the rank of a matrix as the number of singular values that exceed a certain threshold ε. These numerical issues related to the matrix rank raise interesting questions that should be addressed by any practical semidefinite programming algorithm.

In light of the preceding discussion, we introduce a notion of *near low rank*, which is meant to capture matrices that are well approximated by low rank matrices. More precisely, we have the following definition.

Definition 3.4: A matrix $X \in \mathcal{M}_n(\mathbb{R})$ is defined to be *ε-near rank-p* if X satisfies

$$X \ = \ M + N, \qquad M, N \in \mathcal{M}_n(\mathbb{R})$$

where rank$(M) = p$ and $\|N\|_F \leq \varepsilon$.

Equivalently, a matrix is said to be ε-near rank-p if it lives within a ε-radius ball centered around a rank-p matrix.

We now explore certain rank monotonicity properties of the matrix iterates $\{X_k\}$ generated by the the rank minimization heuristic (14) under the sequence of surrogates $g_k(X) = \log\det(X + \delta_k I)$.

Theorem 3.5 (Near rank monotonicity): Let rank$(X_k) = p \geq 1$. Then X_{k+1} is ε-near rank-r (where $r \leq p$), if

$$\delta_{k+1} \ \leq \ \frac{\varepsilon}{p}.$$

Proof: The proof of the Theorem is omitted due to the lack of space. ■

Remark 6: *(Approximate constraint satisfaction).* Using this notion of near low rank, one can pose interesting questions regarding *approximate constraint satisfaction*. For example, consider an ε-near rank-one matrix $X = M + N$ (where rank$(M) = 1$ and $\|N\|_F \leq \varepsilon$) belonging to the optimal face $\mathcal{F}$ of the OPF semidefinite relaxation. While the naive rank-one approximation $X \approx M$ may result in a violation of constraints (i.e. $M \notin \mathcal{F}$), the violation will be mild. And for practical engineering problems such as OPF, minor constraint violations may be tolerable. It's therefore natural to ask as to *when the optimal face $\mathcal{F}$ possesses nearly rank-one matrices that can be efficiently computed?* Conversely, for semidefinite relaxations which do not possess rank-one optimal solutions, can one systematically and efficiently construct a *mild relaxation* of the optimal face $\mathcal{F}' \supset \mathcal{F}$ such that $\mathcal{F}'$ admits a rank-one matrix? □

IV. Numerical Studies

The primary objective of this section is to present a cross section of numerical results on the performance of the linearization-minimization and alternating bisection-minimization algorithms. A number of representative power system examples are presented for which the naive semidefinite relaxation fails. In Section IV-A, examples are provided for which the linearization-minimization algorithm succeeds in finding *hidden* rank-one optimal solutions that are also globally optimal for the original OPF problem. In the event that said algorithm fails to find a rank-one matrix on the optimal face, the alternating bisection-minimization method can be applied. In Section IV-B, this alternating bisection-minimization method is used to find a rank-one feasible solution that yields a cost no larger than that obtained via a conventional non-convex solver. Throughout this section X_0, $\overline{X}_0$ and $\overline{X}_k$ denote, respectively, the optimal solution to the naive semidefinite relaxation, the rank minimization heuristic over $\mathcal{F}$, and the k^{th} step of the alternating bisection-minimization method. In Appendix II, we list the values of the parameters used in our studies.

A. Linearization-Minimization Iteration

Table III summarizes the power system networks used to test the linearization-minimization algorithm. For each example, the naive semidefinite relaxation fails to return a rank-one solution (Column 2). In each case, the linearization-minimization algorithm successfully converges to a rank-one

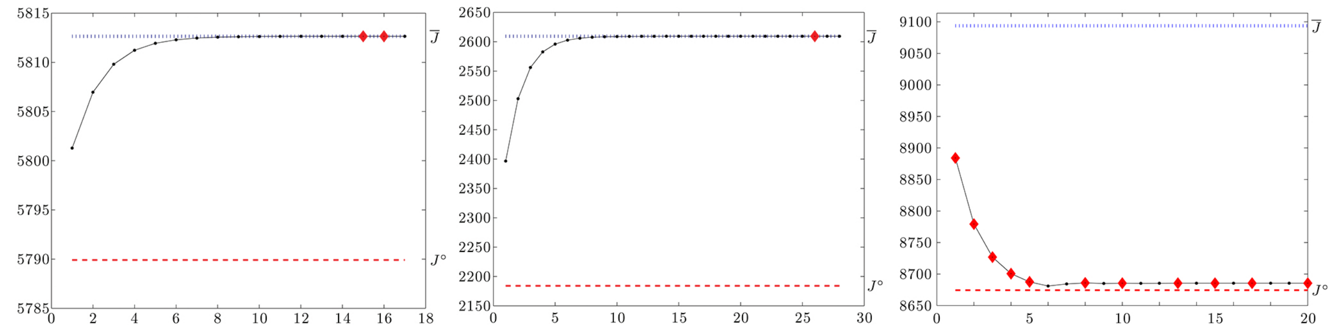

Fig. 1: $J(\overline{X}_k)$ vs # of Iterations (Bisection Method) (a) 3-Bus Example (b) 5-Bus Example, (c) Modified IEEE 14 Bus Example (14B)

optimal point (Column 3) typically in a small number of iterations (Column 4). Thus the optimal cost for the semidefinite relaxation, J°, is in fact equal to the optimal cost of the OPF problem. Moreover, the rank-one solution returned from the linearization-minimization algorithm can be used to construct an optimal solution for the non-convex OPF problem. These results verify that primal/dual solvers will fail to return rank-one optimal solutions for the naive semidefinite relaxation even when such solutions exist (c.f. Theorem 2.1). The values of $\overline{J}$ in the last column denote the upper bound on the optimal cost of the OPF problem given by the non-convex solver MATPOWER [3]. The last result in Table III is of particular interest. This example is a modified IEEE 14 Bus system (14A) for which the linearization-minimization algorithm yields a rank-one globally optimal solution with a cost 12.4% lower than the sub-optimal solution obtained with MATPOWER. This example was constructed from the standard IEEE 14 Bus test case [22] by tightening a subset of the line capacity constraints. A precise description can be found in [23].

TABLE III: Power system examples with hidden rank-one optimal solutions. Precise systems descriptions can be obtained from (9 bus [24]), (30 bus [25]) (118 bus [22]), (14A bus [23]).

Syst.	rank(X_0)	rank($\overline{X}_0$)	Iter.	J°	$\overline{J}$
9	8	1	3	5296.7	5296.7
30	9	1	3	576.9	576.9
118	236	1	100	129661	129661
14A	26	1	3	**8092.8**	**9093.8**

B. Alternating-Bisection Method

For certain problems, the linearization-minimization algorithm fails to uncover a rank-one point in $\mathcal{F}$ – i.e. rank($\overline{X}_0$) > 1. In such cases, one of two scenarios could be at play. Either the optimal face $\mathcal{F}$ of the semidefinite relaxation does not possess a rank-one matrix or the rank minimization heuristic may simply fail in recovering a rank-one points in $\mathcal{F}$ when they do in fact exist. Table IV provides three representative examples of such cases. For each example, the rank minimization heuristic is able to find a *lower rank matrix* (on $\mathcal{F}$) than that achieved by the naive semidefinite relaxation. However, the iteration does not converge to a rank-one solution. In each case there is a non-zero gap between the cost achieved for the semidefinite relaxation, J°, and the MATPOWER upper bound obtained for the original OPF problem, $\overline{J}$.

The alternating bisection-minimization method is applied to the cases in Table IV. Figure 1 depicts the cost of a feasible point produced at every step of the bisection for the examples considered in Table IV. The red diamonds denote the iterates achieving rank-one feasible points, while the black circles denote iterates corresponding to high rank feasible points. We observe in Figure 1, that in the case of the three and five bus examples, the minimum cost obtained by a rank-one feasible point through bisection coincides with the cost produced by MATPOWER. This may lead one to believe that the optimal face $\mathcal{F}$ of the semidefinite relaxation may not admit a rank-one feasible point. On the other hand, for the modified IEEE 14 Bus example (14B), the proposed bisection-minimization heuristic obtains a rank-one feasible point that yields a substantially lower cost than the upper bound $\overline{J}$ obtained from MATPOWER. More precisely, the minimum cost rank-one point derived from the alternating bisection-minimization method is within 0.1266% of the relaxed lower bound J°, as compared to 4.8326% for the MATPOWER solution. We refer the reader to Remark 6 for a discussion on the role of mild constraint relaxations in deriving *nearly optimal rank-one solutions*.

To summarize, we observe that in many cases the iterative linearization-minimization algorithm successfully uncovers a hidden rank-one point that is also *globally* optimal for the original OPF problem. If the rank minimization algorithm fails to uncover a rank-one optimal point, then the alternating bisection-minimization method can be applied. In this case, a rank-one feasible solution is obtained that yields a cost that is no greater than that achieved by MATPOWER – and for certain systems, achieves a substantially lower cost than MATPOWER.

V. Conclusion and Future Directions

This paper considered the non-convex Optimal Power Flow (OPF) problem and the corresponding semidefinite relaxation. For certain power systems and cost structures, the naive semidefinite relaxation may fail to yield low rank

TABLE IV: Failing to uncover rank-one solutions in $\mathcal{F}$. Descriptions of the test cases considered can be found in (3 bus [7]), (5 bus [26]), and (14B bus [23]).

Syst.	rank(X_0)	rank($\overline{X}_0$)	Iter.	J°	$\overline{J}$
3	4	2	3	5789.9	5812.6
5	6	6	40	2184.0	2609.3
14B	6	5	30	8674.5	9093.8

solutions that can be efficiently mapped to the original non-convex feasible set for OPF. In part, this derives from the propensity of interior point methods for semidefinte programs to converge to points of maximal rank on the optimal solution face. This inspires the exploration of methodologies capable of uncovering *hidden* rank-one optimal solutions to the semidefinite relaxation, when they exist. Essentially, this amounts to solving a rank minimization problem over the optimal face of the semidefinite relaxation.

In the paper, two rank minimization heuristics were proposed to compute rank-one solutions for the relaxation. The algorithms were tested on multiple representative power system examples. In many cases, the rank minimization heuristic obtains a hidden rank-one solution, where the naive semidefinite relaxation fails. Moreover, a simple 14-bus example was provided for which the rank minimization heuristic obtains a rank-one solution with a strictly lower cost than that obtained by a conventional solver.

Future work will explore refined convergence properties for the proposed rank minimization heuristics. Of interest, is the specification of conditions under which the heuristics are guaranteed to converge to *near rank-one* optimal solutions belonging to $\mathcal{F}$. In addition, the paper discusses a simple 3-bus system for which the optimal face of the semidefinite relaxation appears to have minimal rank strictly greater than one. This raises the interesting research question of deriving general sufficient conditions for the non-existence of rank-one points in $\mathcal{F}$, as this would provide a characterization of systems for which the semidefinite relaxation is guaranteed to fail.

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APPENDIX I
PROOF OF THEOREM 3.3

While the proof of Theorem 3.3 (a) follows largely from the proof of Theorem 14.1.3 in [19], we include a concise version here for completeness. First, we define a hemivariate functional and a strongly downward sequence. We then use these definitions to show that a strictly concave function is hemivariate and that the sequence of iterates $\{X_k\}$ generated as in (14) is strongly downward in the function g_k for each k.

Definition 1.1: [19,] A functional $g : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ is said to be *hemivariate* on a set $V_0 \subset \mathcal{M}_n(\mathbb{R})$ if it is not constant on any line segment of V_0 – that is, if there does not exist distinct points $X, Y \in V_0$ such that $\theta X + (1-\theta)Y \in V_0$ and $g(\theta X + (1-\theta)Y) = g(X)$ for all $\theta \in [0, 1]$.

Definition 1.2: [19,] Let $g : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ and $\{X_k\}$ be a sequence of iterates in some subset $V_0 \subset \mathcal{M}_n(\mathbb{R})$. We say that $\{X_k\}$ is *strongly downwards* in g if:
(a) $\theta X_k + (1-\theta)X_{k+1} \in V_0$
(b) $g(X_k) \ \geq \ g(\theta X_k + (1-\theta)X_{k+1}) \ \geq \ g(X_{k+1})$
for all $\theta \in [0, 1]$

We are now ready to show that the sequence of iterates $\{X_k\}$ generated by the iterative linearization-minimization (14) is strongly downward in g_k (for each k) and that the limit function g of the sequence of the $\{g_k\}$ is hemivariate.

Lemma 1.3: Let $\{g_k\}$ be a sequence of smooth, strictly concave functions converging pointwise to a smooth, strictly concave function g over a convex, compact set $\mathcal{C} \subset \mathcal{S}^n_+$. The following statements hold.

1) The limit function g is hemivariate.
2) A sequence of iterates $\{X_k\}$ generated by the iterative linearization-minimization (14) is strongly downward in the function g_k for each k.

Proof:
1) Suppose, for the sake of contradiction, that $\exists$ distinct $X, Y \in \mathcal{C}$ such that

$$g(\theta X + (1-\theta)Y) = g(X) \quad \forall\ \theta \in [0, 1]$$

Since g is strictly concave,

$$g(X) = g(\theta X + (1-\theta)Y) > \theta g(X) + (1-\theta)g(Y)$$

for all $\theta \in [0, 1]$. By taking $\theta = 1$ we have $g(X) > g(X)$, a contradiction.

2) Let X_k, X_{k+1} be two successive iterates belonging to $\mathcal{C}$. Because $\mathcal{C}$ is convex, it follows that $X_a = \theta X_k + (1-\theta)X_{k+1} \in \mathcal{C}$ for all $\theta \in [0, 1]$. By strict concavity, we have

$$\begin{aligned} g_{k+1}(X_a) &> \theta g_{k+1}(X_k) + (1-\theta)g_{k+1}(X_{k+1}) \\ &\overset{(a)}{>} \theta g_{k+1}(X_{k+1}) + (1-\theta)g_{k+1}(X_{k+1}) \\ &= g_{k+1}(X_{k+1}) \end{aligned} \tag{18}$$

where (a) follows from the proof of Lemma 3.1. Moreover,

$$\begin{aligned} g_{k+1}(X_a) &< g_{k+1}(X_k) + \text{Tr}\left[\nabla g_{k+1}(X_k)^\top (X_a - X_k)\right] \\ &\overset{(a)}{=} g_{k+1}(X_k) + \\ &\qquad (1-\theta)\text{Tr}(\nabla g_{k+1}(X_k)^\top (X_{k+1} - X_k)) \\ &\overset{(b)}{\leq} g_{k+1}(X_k) \end{aligned} \tag{19}$$

where (a) follows from linearity of the trace operator and (b) from optimality of X_{k+1} according to (14). Inequalities (18) and (19), imply that

$$g_{k+1}(X_k) > g_{k+1}(\theta X_k + (1-\theta)X_{k+1}) > g_{k+1}(X_{k+1})$$

for all $\theta \in [0, 1]$ – from which it follows that the sequence $\{X_k\}$ is strongly downward in the function g_{k+1} for each k. ■

We now prove Theorem 3.3 (a). The interested reader is refered to Theorem 1.4.3 in [19] for more details. Suppose, for the sake of contradiction, that $\lim_{k\to\infty} \|X_k - X_{k+1}\|_F \geq \varepsilon > 0$. Without loss of generality, consider two subsequences, such that $\{X_{k_n}\}_n \to \overline{X}$ and $\{X_{k_n+1}\}_n \to \widehat{X}$. By assumption, for every $\varepsilon > 0$,

$$\|X_{k_n+1} - X_{k_n}\|_F \geq \varepsilon > 0 \qquad \forall \ \ n \geq 1.$$

Because $\mathcal{C}$ is closed it contains all its limit points. Therefore,

$$\|\overline{X} - \widehat{X}\|_F \geq \varepsilon > 0.$$

The sequence $\{g_k(X_k)\}$ is monotonic non-increasing (Lemma 3.1) and since g_k is continuous on a compact set, g_k is bounded from below for all k. It follows from the monotone convergence theorem that the sequence $\{g_k(X_k)\}$ converges, i.e.

$$\lim_{k\to\infty} (g_{k+1}(X_{k+1}) - g_k(X_k)) = 0.$$

It follows that $g(\widehat{X}) = g(\overline{X})$. And by convexity of $\mathcal{C}$, we have that $\theta\overline{X} + (1-\theta)\widehat{X} \in \mathcal{C}$ for all $\theta \in [0, 1]$. Moreover, the sequence $\{X_k\}$ is strongly downward in g_{k+1}. Therefore,

$$\begin{aligned} g_{k+1}(X_{k+1}) \leq g_{k+1}(\theta X_k + (1-\theta)X_{k+1}) &\leq g_{k+1}(X_{k+1}) \\ &\overset{(a)}{<} g_k(X_k) \end{aligned}$$

where (a) follows from Lemma 3.1. Taking limits gives,

$$g(\overline{X}) = g(\theta\overline{X} + (1-\theta)\widehat{X}) = g(\widehat{X})$$

which contradicts the fact that g is hemivariate (Lemma 1.3). Therefore, $\lim_{k\to\infty}(X_k - X_{k+1}) = 0$. This completes the proof of (a).

We now prove part (b) of Theorem 3.3. Let $\overline{X} = \lim_{k\to\infty} X_k$. Since X_{k+1} was chosen to minimize 14, it must be true that

$$\text{Tr}\left[\nabla g_{k+1}(X_k)^\top (X_{k+1} - X_k)\right] \leq \text{Tr}\left[\nabla g_{k+1}(X_k)^\top (X - X_k)\right]$$

for all $X \in \mathcal{C}$. Taking limits and applying Lemma 3.1 yields

$$0 \leq \text{Tr}\left[\nabla g(\overline{X})^\top (X - \overline{X})\right].$$

Since the limit point was chosen arbitrarily, it follows from definition (3.2) that every limit point of $\{X_k\}$ is a stationary point. This complete the proof of part (b).

Appendix II

The following table lists the parameter values used in the numerical studies.

TABLE V: Inputs to Algorithm 1 & 2

Input	Algorithm 1	Algorithm 2
ε	10^{-5}	10^{-6}
$\overline{k}$	100	–
φ	10^{-9}	10^{-9}

where $\text{rank}(X) = 1$ if $\sigma_2(X) < \varphi$.