

Robust Synthesis for Linear Parameter Varying Systems Using Integral Quadratic Constraints

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Abstract—A robust synthesis algorithm is proposed for a class of uncertain linear parameter varying (LPV) systems. The uncertain system is described as an interconnection of a nominal (not-uncertain) LPV system and an uncertainty whose input/output behavior is described by an integral quadratic constraint (IQC). The proposed algorithm is a coordinate-wise ascent that is similar to the well-known DK iteration for μ -synthesis. In the first step, a nominal controller is designed for the LPV system without uncertainties. In the second step, the robustness of the designed controller is evaluated and a new scaled plant for the next synthesis step is created. The robust performance condition used in the analysis step is formulated as a dissipation inequality that incorporates the IQC and generalizes the Bounded Real Lemma like condition for performance of nominal LPV systems. Both steps can be formulated as a semidefinite program (SDP) and efficiently solved using available optimization software. The effectiveness of the proposed method is demonstrated on a simple numerical example.

I. INTRODUCTION

This paper considers the robust synthesis problem for a class of uncertain linear parameter varying (LPV) systems. The uncertain system is described as an interconnection of a nominal (not-uncertain) LPV system and a structured perturbation. The state matrices of the nominal system are assumed to have an arbitrary dependence on the parameters. An arbitrary parameter dependence appears in many applications, e.g. aeroelastic vehicles [1] and wind turbines [2], [3], by linearization of nonlinear models. The existing analysis and synthesis results for nominal (not uncertain) LPV systems provide a rigorous framework for design of gain-scheduled controllers [4], [5]. The input/output behavior of the perturbation is described by an integral quadratic constraint (IQC) [6]. The perturbation can include (parametric or dynamic) uncertainty and/or nonlinearities, e.g. saturation. However, this paper focuses on the case where the IQC describes the behavior of a norm-bounded uncertainty. A robust performance condition for uncertain LPV systems is formulated as a dissipation inequality that incorporates the IQC and generalizes the Bounded Real Lemma like condition for performance of nominal LPV systems [7]. A brief review of these technical results are provided in Section II.

The robust synthesis algorithm is proposed in Section III. It is a coordinate-wise ascent that is similar to the well-known DK iteration for μ -synthesis [8]. In the first step, a nominal controller is designed for the LPV system

without uncertainties. In the second step, the robustness of the designed controller is evaluated using the dissipation inequality condition. The main technical issue is that the IQC scalings must be incorporated to create a new scaled plant for further synthesis. A new controller is designed at the next step using this scaled plant. Both steps can be formulated as a semidefinite program (SDP) and efficiently solved using available optimization software. In Section IV, the effectiveness of the proposed method is demonstrated on a simple numerical example.

It should be noted that robust synthesis conditions have recently been developed for LPV systems in [9], [10]. These recent results are for the specific class of LPV systems where the state matrices have a rational (linear fractional) dependence on the parameters. The results contained in this conference paper complement the existing results in [9], [10]. In particular, this paper handles a more general class of LPV systems where the state matrices have an arbitrary dependence on the parameters. The drawback of this generality is that the SDP conditions are more complicated and difficult to solve than those obtained for LPV models with rational dependence [11], [12], [13]. Another drawback is that the proposed algorithm inherits the non-convexity of μ -synthesis. As a result, the proposed coordinate-wise iteration will not, in general, converge to a local (nor global) optima. However, the coordinate-wise algorithm provides a useful and intuitive extension of the standard DK-synthesis approach to LPV systems. A key distinction is that DK-synthesis uses a frequency-domain robust performance condition in the D step in order to compute the scalings on a frequency grid. This approach is not possible in the analysis step of the proposed approach because the (nominal) LPV system is time-varying with an arbitrary dependence on the parameters. Hence the analysis step is instead performed based on a time-domain, dissipation inequality approach.

II. BACKGROUND

A. Linear Parameter Varying Systems

LPV systems are a class of systems whose state space matrices depend on a time-varying parameter vector $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_\rho}$. An allowable parameter trajectory ρ is a continuously differentiable function of time that is restricted at each point in time to lie in a known compact set $\mathcal{P} \subset \mathbb{R}^{n_\rho}$. The set of admissible parameter trajectories is denoted as \mathcal{A} . In some applications, the parameter rates of variation $\dot{\rho}$ are assumed to be bounded. However, only the rate-unbounded case is considered here for simplicity. Most results in this

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paper generalize, but with more extensive notation, to the rate bounded case using existing results in [4], [5], [7].

The state-space matrices of an LPV system are continuous functions of the parameter: $A : \mathcal{P} \rightarrow \mathbb{R}^{n_G \times n_G}$, $B : \mathcal{P} \rightarrow \mathbb{R}^{n_G \times n_d}$, $C : \mathcal{P} \rightarrow \mathbb{R}^{n_e \times n_G}$ and $D : \mathcal{P} \rightarrow \mathbb{R}^{n_e \times n_d}$. An n_G^{th} order LPV system, G_ρ , is defined by

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B(\rho(t)) \\ C(\rho(t)) & D(\rho(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} \quad (1)$$

The state matrices at time t depend on the parameter vector at time t . Hence, LPV systems represent a special class of time-varying systems. Throughout the remainder of the paper the explicit dependence on t is occasionally suppressed to shorten the notation. Moreover, it is important to emphasize that the state matrices are allowed to have an arbitrary dependence on the parameters.

B. Induced L_2 Control for LPV systems

The performance of an LPV system G_ρ can be specified in terms of its induced L_2 gain from input d to output e . The induced L_2 norm is defined by

$$\|G_\rho\| := \sup_{d \neq 0, d \in L_2, \rho \in \mathcal{A}, x(0)=0} \frac{\|e\|}{\|d\|}. \quad (2)$$

In words, this is the largest input/output gain over all possible inputs $d \in L_2$ and allowable trajectories $\rho \in \mathcal{A}$.

This norm forms the basis for the induced L_2 norm controller synthesis in [4], [5]. The results in [4], [5] are briefly summarized for the rate unbounded case. Consider an open loop LPV system G_ρ as

$$\begin{bmatrix} \dot{x} \\ e \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{22}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ d \\ u \end{bmatrix} \quad (3)$$

where $x \in \mathbb{R}^{n_G}$, $d \in \mathbb{R}^{n_d}$, $e \in \mathbb{R}^{n_e}$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$. The goal is to synthesize an LPV controller K_ρ of the form:

$$\begin{bmatrix} \dot{x}_K \\ u \end{bmatrix} = \begin{bmatrix} A_K(\rho) & B_K(\rho) \\ C_K(\rho) & D_K(\rho) \end{bmatrix} \begin{bmatrix} x_K \\ y \end{bmatrix}. \quad (4)$$

The controller generates the control input u . It has a linear dependence on the measurement y but an arbitrary dependence on the (measurable) parameter ρ . The closed-loop interconnection of G_ρ and K_ρ is given by a lower linear fractional transformation (LFT) and is denoted $\mathcal{F}_l(G_\rho, K_\rho)$. The objective is to synthesize a controller K_ρ of the specified form to minimize the closed-loop induced L_2 gain from disturbances d to errors e :

$$\min_{K_\rho} \|\mathcal{F}_l(G_\rho, K_\rho)\|. \quad (5)$$

A simple, necessary and sufficient condition does not exist to evaluate the induced L_2 norm of an LPV system. However, there are bounded-real type linear matrix inequality (LMI) conditions that are sufficient to upper bound the gain of an LPV system (Lemma 3.1 in [5]). This sufficient condition forms the basis for the synthesis result in Theorem 1 below. The notation for the synthesis result is greatly simplified by assuming the feedthrough matrices satisfy $D_{11}(\rho) = 0$,

$D_{22}(\rho) = 0$ and $D_{12}(\rho)^T = [0, I_{n_u}]$, $D_{21}(\rho) = [0, I_{n_y}]$. Under some technical rank assumptions, this normalized form can be achieved through a combination of loop-shifting and scaling [4], [14]. The input matrix is partitioned as $B_1(\rho) := [B_{11}(\rho) \ B_{12}(\rho)]$ compatibly with the normalized form of D_{21} . Similarly, the output matrix is partitioned as $C_1^T(\rho) := [C_{11}^T(\rho) \ C_{12}^T(\rho)]$ compatibly with D_{12} . Given these simplifying assumptions, the solution to the induced L_2 control synthesis problem is stated in the next theorem.

Theorem 1 ([4], [5]): Let \mathcal{P} be a given compact set and G_ρ an LPV system (Equation 3) that satisfies the normalizing assumptions above. There exists a controller K_ρ as in Equation 4 such that $\|\mathcal{F}_l(G_\rho, K_\rho)\| \leq \gamma$ if there exist matrices $P = P^T > 0$ and $Q = Q^T > 0$ such that $\forall \rho \in \mathcal{P}$

$$\begin{bmatrix} P & I_{n_x} \\ I_{n_x} & Q \end{bmatrix} \geq 0 \quad (6)$$

$$\begin{bmatrix} Q\hat{A}(\rho)^T + \hat{A}(\rho)Q - \gamma B_2(\rho)B_2(\rho)^T & QC_{11}(\rho)^T & B_1(\rho) \\ C_{11}(\rho)^T Q & -\gamma I_{n_{e1}} & 0 \\ B_1(\rho)^T & 0 & -\gamma I_{n_d} \end{bmatrix} < 0 \quad (7)$$

$$\begin{bmatrix} \tilde{A}(\rho)^T P + P\tilde{A}(\rho) - C_2(\rho)^T C_2(\rho) & PB_{11}(\rho) & C_1(\rho)^T \\ B_{11}(\rho)^T P & -\gamma I_{n_{d1}} & 0 \\ C_1(\rho) & 0 & -\gamma I_{n_e} \end{bmatrix} < 0 \quad (8)$$

where $\hat{A}(\rho) := A(\rho) - B_2(\rho)C_{12}(\rho)$ and $\tilde{A}(\rho) := A(\rho) - B_{12}(\rho)C_2(\rho)$.

Proof: The proof uses a matrix elimination argument similar to that used in the LMI approach to H_∞ synthesis for LTI systems [15]. Moreover, if the conditions are satisfied then an LPV controller ($A_K(\rho)$, $B_K(\rho)$, $C_K(\rho)$, $D_K(\rho)$) can be constructed from the open loop plant matrices and the feasible values of P , Q , and γ . The controller reconstruction formulation is given in [4], [5]. ■

C. Robustness Analysis of LPV Systems

Integral quadratic constraints (IQCs) [6] provide a framework for robustness analysis. The IQC and the stability condition in [6] are all specified in frequency domain. A related stability theorem can be formulated using dissipation theory, and a time-domain IQC [7]. The time domain approach is preferred for robust analysis of LPV systems. A time domain IQC is defined by a symmetric matrix $M = M^T \in \mathbb{R}^{n_z \times n_z}$ and a stable linear system $\Psi \in \mathbb{RH}_\infty^{n_z \times (n_v + n_w)}$. Ψ is denoted as

$$\begin{bmatrix} \dot{x}_\psi(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A_\psi & B_{\psi 1} & B_{\psi 2} \\ C_\psi & D_{\psi 1} & D_{\psi 2} \end{bmatrix} \begin{bmatrix} x_\psi(t) \\ v(t) \\ w(t) \end{bmatrix} \quad (9)$$

The initial condition for Ψ is always taken as $x_\psi(0) = 0$.

Definition 1: A bounded, causal operator $\Delta : L_{2e}^{n_v} \rightarrow L_{2e}^{n_w}$ satisfies an IQC defined by (Ψ, M) if the following inequality holds for all $v \in L_2^{n_v}[0, \infty)$, $w = \Delta(v)$ and $T \geq 0$:

$$\int_0^T z(t)^T M z(t) dt \geq 0 \quad (10)$$

where z is the output of the linear system Ψ as defined in Equation 9. The notation $\Delta \in \text{IQC}(\Psi, M)$ is used if Δ satisfies the IQC defined by (Ψ, M) .

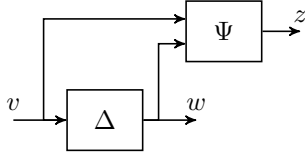


Fig. 1. Graphical interpretation of the IQC

Fig. 1 provides a graphical interpretation of the IQC. The input and output signals of Δ are filtered through Ψ . If $\Delta \in \text{IQC}(\Psi, M)$ then the output signal z satisfies the (time-domain) constraint in Equation 10 for any finite-horizon $T \geq 0$. IQCs can be used to model a variety of nonlinearities and uncertainties (Cite Ranzer). However, this paper will focus on the special case where Δ is a norm-bounded uncertainty. Specifically, the results in this paper hold for uncertainties Δ that satisfy the following assumptions:

Assumption 1: Δ satisfies $\text{IQC}(\Psi, M_b)$ where $M_b := \begin{bmatrix} b^2 I_{n_v} & 0 \\ 0 & -I_{n_w} \end{bmatrix}$, $\Psi := \begin{bmatrix} \Psi_{11} & 0 \\ 0 & \Psi_{22} \end{bmatrix}$, $\Psi_{11} \in \mathbb{RH}_{\infty}^{n_v \times n_v}$ and $\Psi_{22}, \Psi_{22}^{-1} \in \mathbb{RH}_{\infty}^{n_w \times n_w}$.

Norm-bounded nonlinear operators can be modeled using an IQC that satisfies Assumption 1. As an example, let Δ be a casual (SISO), LTI uncertainty with norm bound $\|\Delta\| \leq b$. Then Δ satisfies the IQC defined by M_b and $\Psi(s) := \begin{bmatrix} D(s) & 0 \\ 0 & D(s) \end{bmatrix}$ where $D(s)$ is any stable, minimum phase transfer function. This is equivalent to the use of frequency-dependent D -scales in μ -analysis. The causality of Δ can be used to show that the time-domain constraint (Equation 10) holds over all finite time horizons [7]. MIMO LTI uncertainties also satisfy an IQC of the form given in Assumption 1.

Given this assumption, an uncertain LPV system is described by the interconnection of an LPV system G_{ρ} and an uncertainty Δ , as depicted in Fig. 2. This interconnection represents an upper LFT denoted $\mathcal{F}_u(G_{\rho}, \Delta)$. The filter Ψ has been included in Fig. 2 as it is used for the analysis.

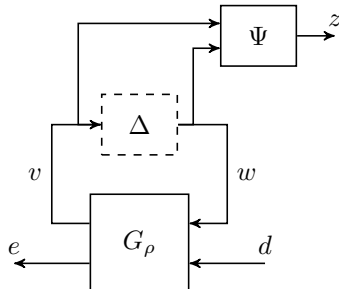


Fig. 2. Analysis Interconnection

The dynamics of the analysis interconnection in Fig. 2 are described by $w = \Delta(v)$ and

$$\begin{bmatrix} \dot{x} \\ z \\ e \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ d \end{bmatrix} \quad (11)$$

where the state vector is $x = [x_G; x_{\psi}] \in \mathbb{R}^{n_G+n_{\psi}}$ with x_G and x_{ψ} being the state vectors of the LPV system G_{ρ} and the filter Ψ respectively. The uncertainty Δ is shown in the dashed box in order to signify that it is removed for the analysis. The signal w is treated as an external signal subject to the constraint 10. This effectively replaces the precise relation $w = \Delta(v)$ with the quadratic constraint on z .

A natural performance metric for this uncertain LPV system is the worst-case gain over all $\|\Delta\| \leq 1$:

$$\sup_{\Delta \in \text{IQC}(\Psi, M_{b=1}), \rho \in \mathcal{A}} \|\mathcal{F}_u(G_{\rho}, \Delta)\| \quad (12)$$

This metric is inconvenient for robust synthesis as the controller must achieve both robust stability and minimal input/output gain. Thus it is standard, e.g. in DK-synthesis, to instead use a robust performance metric that simultaneously scales both the uncertainty level and the system gain. This metric used for robust synthesis is formally defined as

Definition 2: $\mathcal{F}_u(G_{\rho}, \Delta)$ achieves Robust Performance (RP) of level γ if

$$\sup_{\Delta \in \text{IQC}(\Psi, M_{\frac{1}{\gamma}}), \rho \in \mathcal{A}} \|\mathcal{F}_u(G_{\rho}, \Delta)\| \leq \gamma \quad (13)$$

Let $r_{\Delta}[G_{\rho}]$ denote the smallest level of RP achievable by G_{ρ} .

In other words, RP of level γ is achieved if the worst-case induced L_2 gain from the input d to the output e is $\leq \gamma$ over all admissible trajectories $\rho \in \mathcal{A}$ and uncertainties $\|\Delta\| \leq \frac{1}{\gamma}$. The next theorem provides a (sufficient) dissipation-inequality condition to upper bound the robust performance. This theorem is a minor modification of the worst-case gain condition in [7].

Theorem 2: Assume $\mathcal{F}_u(G_{\rho}, \Delta)$ is well posed for all $\Delta \in \text{IQC}(\Psi, M_{\frac{1}{\gamma}})$. Then $\mathcal{F}_u(G_{\rho}, \Delta)$ achieves RP of level γ if there exists a matrix $P = P^T \in \mathbb{R}^{(n_G+n_{\psi}) \times (n_G+n_{\psi})}$ and a scalar $\lambda \geq 0$ such that $P \geq 0$ and $\forall \rho \in \mathcal{P}$

$$\begin{bmatrix} PA + A^T P & PB_1 & PB_2 \\ B_1^T P & 0 & 0 \\ B_2^T P & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} C_1^T \\ D_{11}^T \\ D_{12}^T \end{bmatrix} M_{\frac{1}{\gamma}} [C_1 \ D_{11} \ D_{12}] \\ + \frac{1}{\gamma^2} \begin{bmatrix} C_2^T \\ D_{21}^T \\ D_{22}^T \end{bmatrix} [C_2 \ D_{21} \ D_{22}] < 0 \quad (14)$$

In Equation 14 the dependence of the state matrices on ρ has been omitted.

Proof: The proof is based on defining a storage function $V : \mathbb{R}^{n_G+n_{\psi}} \rightarrow \mathbb{R}^+$ by $V(x) := x^T P x$. Left and right multiply Equation 14 by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ to

show that V satisfies the dissipation inequality:

$$\dot{V}(t) + \lambda z(t)^T M_{\frac{1}{\gamma}} z(t) \leq d(t)^T d(t) - \gamma^{-2} e(t)^T e(t) \quad (15)$$

The dissipation inequality (Equation 15) can be integrated from $t = 0$ to $t = T$ with the initial condition $x(0) = 0$. The IQC condition (Equation 10) along with $\lambda \geq 0$ and $P \geq 0$ imply $\|e\| \leq \gamma \|d\|$. Details are in [7]. ■

III. ROBUST SYNTHESIS

A. Problem Formulation

Consider the robust synthesis problem for uncertain LPV systems as shown in Fig 3. The uncertain LPV system is described by the interconnection of an open loop LPV system G_ρ and a perturbation Δ . G_ρ is defined as

$$\begin{bmatrix} \dot{x} \\ v \\ e \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) & B_3(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) & D_{13}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) & D_{23}(\rho) \\ C_3(\rho) & D_{31}(\rho) & D_{32}(\rho) & D_{33}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ d \\ u \end{bmatrix} \quad (16)$$

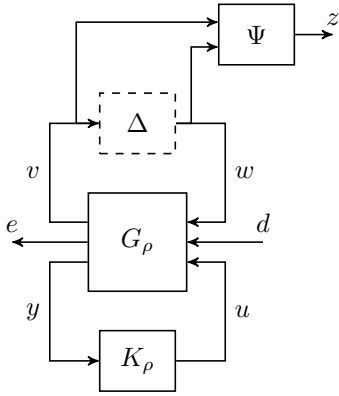


Fig. 3. LPV Robust Synthesis

The objective is to synthesize an LPV controller K_ρ as in Equation 4 to minimize the robust performance of the closed-loop system:

$$\inf_{K_\rho \text{ stabilizing}} r_\Delta [F_L(G_\rho, K_\rho)] \quad (17)$$

The condition in Theorem 2 (Equation 14) provides an upper bound on the closed-loop robust performance condition. This results in a matrix inequality that is bilinear in the state matrices for K_ρ and the storage matrix $P \geq 0$. This leads to a non-convex optimization. In addition, the scaling for the IQC $\lambda \geq 0$ also leads to non-convexity in the optimization. A standard coordinate-wise approach is used to decouple the design into a controller synthesis step (for K_ρ) and an uncertainty analysis step (for P and λ). As with DK-synthesis, there are no guarantees that the coordinate-wise iteration will lead to a local optima let alone a global optima. However, it is a useful heuristic that will enable the robust synthesis to extended naturally from LTI to LPV systems. The control synthesis step for K_ρ is described in Section III-B. The analysis step for λ and P is described in Section III-C. Finally, the detailed description of the coordinate-wise iteration using these two steps is given in Section III-D.

B. Synthesis Step

Assume that the IQC filter $\Psi = \begin{bmatrix} \Psi_{11} & 0 \\ 0 & \Psi_{22} \end{bmatrix}$ has been computed from the previous step of the coordinate-wise iteration. Define the following scaled system:

$$\tilde{G}_\rho := \begin{bmatrix} \Psi_{11} & 0 \\ 0 & I_{n_e+n_y} \end{bmatrix} G_\rho \begin{bmatrix} \Psi_{22}^{-1} & 0 \\ 0 & I_{n_d+n_u} \end{bmatrix} \quad (18)$$

In the synthesis step, the objective is to synthesize a controller K_ρ to minimize the induced L_2 gain of the closed-loop system shown in Figure 4 below. This synthesis step is completed by applying Theorem 1 in Section II-B. The synthesis step with the scaled system ensures the original, unscaled system achieved a specified level of robust performance as stated in the following Lemma.

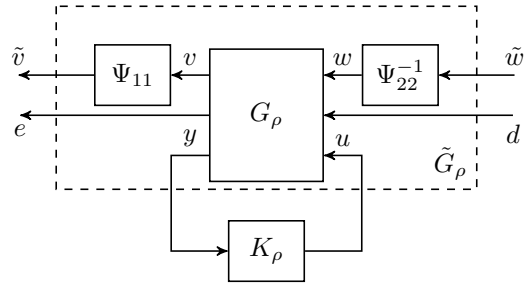


Fig. 4. LPV Synthesis

Lemma 1: If $\sup_{\rho \in \mathcal{A}} \|\mathcal{F}_l(\tilde{G}_\rho, K_\rho)\| \leq \gamma$ then

$$\sup_{\Delta \in \text{IQC}(\Psi, M_{\frac{1}{\gamma}}), \rho \in \mathcal{A}} \|\mathcal{F}_u(\mathcal{F}_l(G_\rho, K_\rho), \Delta)\| \leq \gamma \quad (19)$$

Proof: The proof requires the transformation from the original interconnection (Figure 3) to the scaled interconnection shown in Figure 5 below. In the transformed interconnection, \tilde{G}_ρ is the scaled LPV system. It can be easily shown that $\Delta \in \text{IQC}(\Psi, M_{\frac{1}{\gamma}})$ if and only if $\tilde{\Delta} \in \text{IQC}(I_{n_v+n_w}, M_{\frac{1}{\gamma}})$. Thus the original interconnection achieves RP of level γ if and only if the scaled interconnection achieves RP of level γ .

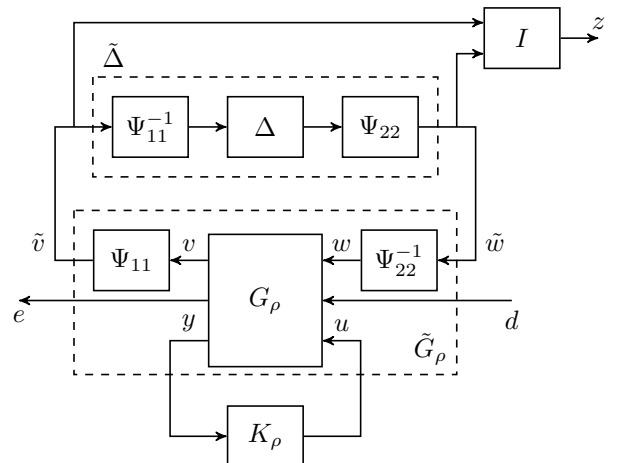


Fig. 5. Block Diagram Transformation

The proof is thus concluded by showing that $\sup_{\rho \in \mathcal{A}} \|\mathcal{F}_l(\tilde{G}_\rho, K_\rho)\| \leq \gamma$ implies the scaled interconnection achieves RP of level γ . Let $d \in L_2$, $\rho \in \mathcal{A}$, $\tilde{\Delta}$ denote any input, admissible parameter trajectory, and uncertainty satisfying $IQC(I_{n_v+n_w}, M_{\frac{1}{\gamma}})$. Let e , \tilde{v} , and \tilde{w} denote the resulting signals of the scaled interconnection with zero initial conditions. By assumption $\|\mathcal{F}_l(\tilde{G}_\rho, K_\rho)\| \leq \gamma$ and hence $\|\tilde{v}\|^2 + \|e\|^2 \leq \gamma^2(\|\tilde{w}\|^2 + \|d\|^2)$. Moreover, $\gamma^2 \|\tilde{w}\|^2 \leq \|\tilde{v}\|^2$ since $\tilde{\Delta} \in IQC(I_{n_v+n_w}, M_{\frac{1}{\gamma}})$. Combining these two inequalities yields $\|e\|^2 \leq \gamma^2 \|d\|^2$. Thus the scaled interconnection (and hence the original interconnection) achieves RP of level γ . ■

C. Analysis Step

The analysis step assumes that a controller K_ρ has been computed from the previous step of the coordinate-wise iteration. Define the following scaled system $N_\rho := F_l(G_\rho, K_\rho)$. The objective at this step is to analyze the robust performance of $F_u(N_\rho, \Delta)$. Theorem 2 can be applied to bound the robust performance of this system. However, several issues must be addressed in order to apply the matrix inequality condition in this theorem.

First, the standard DK-synthesis (for LTI Δ) uses frequency gridding to construct D -scales with an arbitrary frequency dependence. This freedom is lost in the proposed dissipation inequality approach because the LPV system is time-varying and does not have a frequency response. Instead, several basis functions $\{\Psi_k\}_{k=1}^N$ can be selected to describe frequency-dependent constraints on Δ . Each Ψ_k can be appended to the inputs/outputs of Δ to yield a filtered output z_k . Theorem 2 remains valid if the matrix inequality condition (Equation 14) is modified to include the term

$$\sum_{k=1}^N \lambda_k \begin{bmatrix} C_{1k}(\rho)^T \\ D_{11k}(\rho)^T \\ D_{12k}(\rho)^T \end{bmatrix} M_{\frac{1}{\gamma}} \begin{bmatrix} C_{1k}(\rho) & D_{11k}(\rho) & D_{12k}(\rho) \end{bmatrix} \quad (20)$$

for any constants $\lambda_k \geq 0$. In this case the extended system includes the dynamics of G_ρ as well as the dynamics of each Ψ_k ($k = 1, \dots, N$). In addition, $(C_{1k}, D_{11k}, D_{12k})$ denote the output state matrices of the extended system associated with output z_k . The robust performance analysis consists of a search for the matrix $P \geq 0$, performance bound γ , and the constants $\lambda_k \geq 0$ that lead to feasibility of the matrix inequality. This approach also enables many IQCs for Δ to be incorporated into the analysis. The matrix inequality is bilinear in γ and the scalings λ_k . However, it is quasiconvex and can be efficiently solved via a bisection on γ . This leads to optimal scalings $\lambda_{k,opt}$ and performance level γ_{opt} . The IQCs can be combined into a single frequency domain multiplier $\Pi_{opt} := \sum_{k=1}^N \lambda_{k,opt} \Psi_k^* M_{\frac{1}{\gamma_{opt}}} \Psi_k$. Since each Ψ_k is assumed to be diagonal (by Assumption 1) the multiplier Π_{opt} can be factorized as $\Pi_{opt} = \Psi_{opt}^* M_{\frac{1}{\gamma_{opt}}} \Psi_{opt}$. This factorization can be performed using standard state-space spectral factorization techniques on each diagonal block [16]. This yields a Ψ_{opt} that can be used as the IQC filter in the next synthesis step. It should be noted that there is currently

no clear guideline to select basis functions but based on experience.

D. Iteration Algorithm

The robust synthesis algorithm basically iterates between the synthesis step and analysis step as described in Sections III-B and III-C. The detailed steps including the initialization and termination are given below. The stopping criteria is specified by an absolute tolerance $\epsilon_{tol} > 0$.

- 1) *Initialization*: Set $\Psi^{(0)} := I_{n_v+n_w}$, $\gamma^{(0)} = +\infty$, $i = 1$.
- 2) *Synthesis*: Given $\Psi^{(i-1)}$, synthesize a controller $K_\rho^{(i)}$ as described in Section III-B.
- 3) *Analysis*: Given $K_\rho^{(i)}$, analyze the system as described in Section III-C to obtain an updated RP level $\gamma^{(i)}$ and filter $\Psi^{(i)}$.
- 4) *Termination*: If $\gamma^{(i-1)} - \gamma^{(i)} > \epsilon_{tol}$ then set $i = i + 1$ and return to Step 2. Otherwise terminate the iteration.

The algorithm can be easily modified to incorporate other stopping criteria, e.g. maximum number of iterations and/or relative stopping tolerances.

IV. NUMERICAL EXAMPLE

A simple example is used to demonstrate the applicability of the proposed method. The objective is to design an LPV controller K_ρ for an uncertain LPV system G_ρ . The robust synthesis interconnection is depicted in Fig. 6.

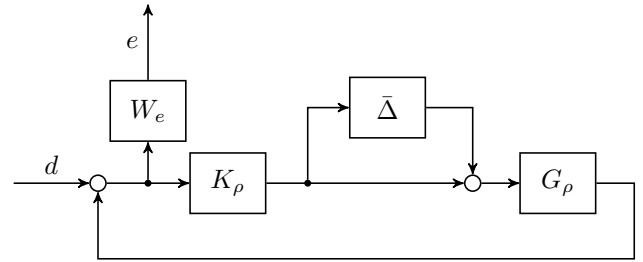


Fig. 6. Synthesis interconnection

The nominal system G_ρ , taken from [17], is a first order system with dependence on a single parameter ρ . It can be written as

$$\begin{aligned} \dot{x}_G &= -\frac{1}{\tau(\rho)}x_G + \frac{1}{\tau(\rho)}u_G \\ y &= C(\rho)x_G \end{aligned} \quad (21)$$

with the time constant $\tau(\rho)$ and output gain $C(\rho)$ depending on the scheduling parameter as follows:

$$\begin{aligned} \tau(\rho) &= \sqrt{133.6 - 16.8\rho} \\ C(\rho) &= \sqrt{4.8\rho - 8.6}. \end{aligned} \quad (22)$$

The scheduling parameter ρ is restricted to the interval $[2, 7]$. For all the following analysis scenarios a grid of 11 points is used that span the parameter space equidistantly.

The objective of the LPV controller is to offer good tracking performance at low frequencies while being robust against uncertainties in the plant input at high frequencies.

The uncertainty $\bar{\Delta}$ is described by a norm-bounded operator Δ in conjunction with the frequency weighting

$$W_u(s) = \frac{s+1}{s+100} \quad (23)$$

such that $\bar{\Delta} = \Delta W_u$. The tracking objective is specified weighting the channel from the reference input d to the control error e with

$$W_e(s) = \frac{0.1s+10}{s+1}. \quad (24)$$

A robust LPV controller is designed using the algorithm proposed in Section III-C. The results of the control design are shown in Fig. 7. Two different cases are considered. In the first, Δ is bounded by a single IQC($\Psi_1, M_{\frac{1}{\gamma}}$) with $\Psi_1 = I$. The second one uses both IQC($\Psi_1, M_{\frac{1}{\gamma}}$) and the additional IQC($\Psi_2, M_{\frac{1}{\gamma}}$) with

$$\Psi_2(s) = \begin{bmatrix} D_2(s) & 0 \\ 0 & D_2(s) \end{bmatrix}, \quad D_2(s) = \frac{s+100}{100(s+1)}. \quad (25)$$

Both cases converge to a solution after 16 and 12 iterations respectively using as stopping criteria $\epsilon_{tol} = 0.05$. Adding more IQCs to bound Δ results in a better robust performance of the final controller, i.e. $\gamma = 3.54$ for case 1 and $\gamma = 2.75$ for case 2. Both clearly outperform the nominal control design which has a robust performance $\gamma = 6.02$.

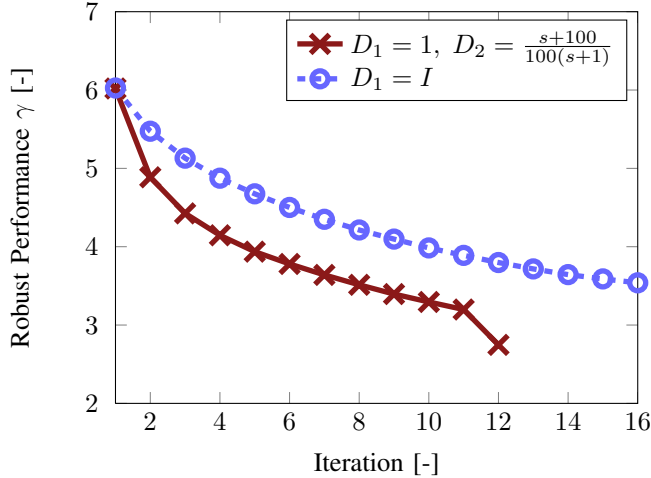


Fig. 7. Results of the Iterative Synthesis Algorithm

V. CONCLUSION

This paper proposed a robust synthesis algorithm for a class of uncertain LPV systems. The uncertain system is described as an interconnection of a nominal LPV system where the state matrices have arbitrary dependence on parameters, and a norm bounded uncertainty described by IQCs. The proposed coordinate-wise algorithm is similar to the well-known DK iteration for μ -synthesis and therefore provides a useful and intuitive extension of the standard DK-synthesis approach to LPV systems. The effectiveness of this method was shown by a simple numerical example.

Future works will consider more general uncertainties $\Delta \in \text{IQC}(\Psi, M)$ where Ψ contains full matrix dynamics and prove the convergent property of the algorithm.

VI. ACKNOWLEDGMENTS

This work was supported by the National Science Foundation under Grant No. NSF-CMMI-1254129 entitled "CA-REER: Probabilistic Tools for High Reliability Monitoring and Control of Wind Farms." This work was also supported by the University of Minnesota Institute on the Environment, IREE Grants No. RS-0039-09 and RL-0011-13. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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