Computation of a lower bound for the induced $\mathcal{L}_2$ norm of LPV systems

Tamas Peni[1] and Peter Seiler[2]

***Abstract*— Determining the induced $\mathcal{L}_2$ norm of a linear, parameter-varying (LPV) system is an integral part of many analysis and robust control design procedures. In general, this norm cannot be determined explicitly. Most prior work has focused on efficiently computing upper bounds for the induced $\mathcal{L}_2$ norm. This paper presents a complementary algorithm to compute lower bounds for this norm. The proposed approach is based on restricting the parameter trajectory to be a periodic signal. This restriction enables the use of recent results for exact calculation of the $\mathcal{L}_2$ norm for a periodic time varying system. The proposed lower bound algorithm has two benefits. First, the lower bound complements standard upper bound techniques. Specifically, a small gap between the bounds indicates that further computation, e.g. upper bounds with more complex Lyapunov functions, is unnecessary. Second, the lower bound algorithm returns a worst-case parameter trajectory for the LPV system that can be further analyzed to provide insight into the system performance. Numerical examples are provided to demonstrate the applicability of the proposed approach.**

I. INTRODUCTION

Determining the induced $\mathcal{L}_2$ norm of a linear, parameter-varying (LPV) system is an integral part of many analysis and robust control design procedures. In general, this norm cannot be determined exactly. Most prior work focuses on the approximation of the upper bound of the induced $\mathcal{L}_2$ norm. The method used to compute the upper bound depends primarily on the structure of the LPV system. For LFT-type LPV systems, where the system matrices are rational functions of the parameter, the upper bound can be computed using scaled small gain theorems with multipliers and the full block S-procedure [1], [6]. For LPV systems where the parameter dependence is arbitrary, the upper bound is computed using a dissipation inequality evaluated over a finite set of parameter grid points [8], [9], [5].

This paper addresses the complementary problem, i.e. the approximation of the lower bound of the $\mathcal{L}_2$ norm. The simplest way to compute a lower bound is to consider the LTI systems obtained at frozen parameter values and to take the maximum of these point-wise $\mathcal{L}_2$ norms. Unfortunately, this approach produces very conservative results in many cases as it neglects the variation of the scheduling parameter. To compute a better estimate, the $\mathcal{L}_2$ norm has to be evaluated over time varying parameter trajectories. This concept induces a complex optimization problem, where the evaluation of the cost function requires the computation of the $\mathcal{L}_2$ norm of LTV systems. Since this problem is still numerically demanding a different approach is proposed in the paper. Our approach is to restrict the scheduling trajectories to periodic signals, since in this case recent results for exact calculation of the $\mathcal{L}_2$ norm of periodic, linear time-varying (PLTV) systems can be applied. It will be shown, by numerical simulations, that an efficient algorithm can be constructed, which produces improved estimate for the lower bound of the $\mathcal{L}_2$ gain of the LPV system.

The method presented in this paper is based mainly on the algorithm proposed by Cantoni and Sandberg [2] to compute the exact $\mathcal{L}_2$ norm of continuous-time, PLTV systems. The characterisation of the $\mathcal{L}_2$ gain for PLTV systems has been well-known for some time [3]. However, the accurate computation of the $\mathcal{L}_2$ gain is numerically demanding, particularly when the period is large ([7]). Therefore, most available methods are approximation-based approaches. The most relevant methods involve the skewed truncation of a frequency domain operator in [13], [12] and the fast-sampling approach in [11]. The accuracy of skewed truncation and fast sampling methods is improved as the truncation order and number of samples, respectively, is increased. In both cases, the improved accuracy comes with increasing computational cost. In contrast to prior works, the algorithm of Cantoni and Sandberg transforms a condition for the $\mathcal{L}_2$ gain of the PLTV system into an equivalent condition for the ℓ_2 gain of a finite-dimensional, discrete-time, linear, time-invariant system. The state-space representation for the discrete-time system can be constructed from the point solutions of well-behaved matrix Riccati differential equations. The algorithm can be reliably implemented by using standard numerical methods which makes it promising for the application to the LPV lower bound problem.

The paper is organised as follows: in the next section the recent results for computing $\mathcal{L}_2$ gain of periodic systems are reviewed. Our approach for computing lower bounds on $\mathcal{L}_2$ norm of LPV systems is presented in section III. Section IV is devoted to the numerical simulations and analysis. The conclusions are drawn in section V.

II. COMPUTING THE INDUCED $\mathcal{L}_2$ NORM OF PERIODIC LTV SYSTEMS

This section recalls the main elements of the algorithm proposed by Cantoni and Sandberg [2] for computing the induced $\mathcal{L}_2$ norm of continuous-time, periodic linear time-varying (PLTV) systems. The state-space matrices of a PLTV system are assumed to be piecewise continuous functions of the time: $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, $B : \mathbb{R}_+ \to \mathbb{R}^{n\times p}$, $C : \mathbb{R}_+ \to \mathbb{R}^{q\times n}$ and $D : \mathbb{R}_+ \to \mathbb{R}^{q\times p}$. In addition, these state matrices are assumed to have a period h, i.e.

$$\begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} = \begin{pmatrix} A(t+h) & B(t+h) \\ C(t+h) & D(t+h) \end{pmatrix} \ \forall t$$

An n_x^{th} order PLTV system, G, is then defined by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{1}$$

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The induced L_2 norm of this system is defined in the standard way as

$$\|G\| = \sup_{u \neq 0, u \in \mathcal{L}_2} \frac{\|y\|}{\|u\|}, \tag{2}$$

where $\|\cdot\|$ represents the signal $\mathcal{L}_2$-norm, i.e. $\|u\| = (\int_0^\infty u(t)^T u(t) \quad dt)^{\frac{1}{2}}$. The initial condition is assumed to be $x(0) = 0$. Throughout the remainder of the paper the explicit dependence on t is occasionally suppressed to shorten the notation.

In the algorithm presented in [2] the following three differential Riccati equations play the key role:

$$\dot{Z} = -(H^T Z + ZH + \gamma Z B R^{-1} B^T Z + \gamma C^T S^{-1} C) \tag{3a}$$

$$\dot{X} = (H + \gamma B R^{-1} B^T Z) X \tag{3b}$$

$$\dot{Y} = HY + YH^T + \gamma Y C^T S^{-1} C Y + \gamma B R^{-1} B^T \tag{3c}$$

where

$$H := A + BR^{-1} D^T C$$
$$R := \gamma^2 I - D^T D$$
$$S := \gamma^2 I - D D^T$$

and the time dependence is supressed in these equations. The basic algorithm in [2] involves a bisection on γ. At each step a value for γ is selected and then the following steps are used to determine if γ is an upper or lower bound on $\|G\|$. A necessary condition for $\|G\| > \gamma$ is given by the inequality $\gamma^2 I - D(t)^T D(t) > 0$ for all $t \in [0, h]$. Assuming that this condition is satisfied then first solve the Riccati differential equation in Z over the interval $[0, h]$ with boundary condition $Z(h) = 0$. It is shown in [2] that if $Z(t)$ fails to have a bounded solution on $[0, h]$ then $\|G\| > \gamma$. If the solution $Z(t)$ exists and is uniformly bounded on $[0, h]$ then further calculations are needed to determine if γ is an upper or lower bound on $\|G\|$. For this, the remaining differential equations in X and Y have to be solved with boundary conditions $X(0) = I$ and $Y(0) = 0$, respectively. The point solutions $X(h)$ and $Y(h)$ are used to construct the following discrete-time system:

$$\begin{aligned} \xi_{k+1} &= A_\gamma \xi_k + B_\gamma \mu_k \\ \zeta_k &= C_\gamma \xi_k \end{aligned} \tag{4}$$

where $A_\gamma := X(h)$ and B_γ, C_γ are chosen to satisfy $Z(0) = C_\gamma^T C_\gamma$ and $Y(h) = B_\gamma B_\gamma^T$. Let $\|A_\gamma, B_\gamma, C_\gamma\|_\infty$ denote the induced ℓ_2 norm of (4). The next theorem follows from the results in [2]

Theorem. Assume $\gamma > 0$ is such that $\gamma^2 I - D(t)^T D(t) > 0$ for all $t \in [0, h]$ and (3a) has a solution $Z(t)$ bounded over $[0, h]$. Then $\|G\| < \gamma$ if and only if $\|A_\gamma, B_\gamma, C_\gamma\|_\infty < 1$.

This result forms the basis for the following bisection algorihm in [2]. Select $\underline{\gamma} > 0$ such that $\underline{\gamma}^2 I - D(t)^T D(t) > 0$ for all $t \in [0, h]$ and $\bar{\gamma}$ such that $\|G\| < \bar{\gamma}$. The bisection is summarized as:

1) Stop if $\bar{\gamma} - \underline{\gamma}$ is within a desired stopping error ϵ. Otherwise, select $\gamma := \frac{1}{2}(\bar{\gamma} + \underline{\gamma})$.
2) Integrate (3a) starting from the boundary condition $Z(h) = 0$. If the solution goes unbounded then set $\underline{\gamma} := \gamma$ and return to step 1. If the solution is bounded on $[0, h]$ then continue to step 3.
3) Solve (3b) and (3c) for $X(t)$ and $Y(t)$ starting from $X(0) = I$ and $Y(0) = 0$. Form the discrete-time system as described above. If $\|A_\gamma, B_\gamma, C_\gamma\|_\infty < 1$ then set $\bar{\gamma} := \gamma$. Otherwise set $\underline{\gamma} := \gamma$. Continue to step 1.

The output of the algorithm are the tight bounds $[\underline{\gamma}, \overline{\gamma}]$ for the induced gain: $\underline{\gamma} \leq \|G\| \leq \overline{\gamma}$ and $\overline{\gamma} - \underline{\gamma} \leq \epsilon$.

III. Computation of a lower bound for the induced $\mathcal{L}_2$ norm of LPV systems

This section considers the computation of a lower bound on the induced $\mathcal{L}_2$ norm for a linear parameter varying (LPV) system. The LPV system is assumed to be given in state-space form as follows:

$$G: \quad \begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))w(t) \\ z(t) &= C(\rho(t))x(t) + D(\rho(t))w(t) \end{aligned} \tag{5}$$

where the system matrices are continuous functions of the parameter ρ. In addition, $\rho(\cdot)$ is a piecewise continuous function of time, $\rho : \mathbb{R}^+ \to \mathbb{R}^m$, that is assumed to satisfy the known bounds

$$\underline{\rho_i} \leq \rho_i(t) \leq \overline{\rho}_i, \quad \underline{\mu_i} \leq \dot{\rho}_i(t) \leq \overline{\mu}_i, \ \forall t, \ 1 \leq i \leq m \tag{6}$$

The set of parameter vectors $\rho \in \mathbb{R}^m$ satisfying the magnitude constraints in (6) is denoted by $\mathcal{P}$ and the set of admissible trajectories containing all piecewise continuously differentiable trajectories that satisfy both the magnitude and the rate constraints in (6) is denoted by $\mathcal{A}$. The performance of G can be specified in terms of its induced $\mathcal{L}_2$ gain from input w to output z assuming $x(0) = 0$:

$$\|G\| = \sup_{0 \neq w \in \mathcal{L}_2(\mathbb{R}^p), \rho(\cdot) \in \mathcal{A}} \frac{\|z\|}{\|w\|}$$

The class of LPV system given above has an arbitrary dependence on the parameter. For this class of systems there are known linear matrix inequality (LMI) conditions to efficiently compute an upper bound on the induced $\mathcal{L}_2$ gain [10], [9].

Our aim is to use the results of the previous section to determine a lower bound for the induced $\mathcal{L}_2$ gain of the LPV system above. This is done by restricting the scheduling parameter trajectories to a linear combination of periodic basis functions. Specifically, let $\phi_k : \mathbb{R} \to \mathbb{R}$ $(k = 1, \ldots, R)$ denote continuously differentiable, periodic basis functions of period 1, i.e. $\phi_k(t+1) = \phi_k(t)$ $\forall t$. Let then the set of admissible periodic trajectories be defined as follows:

$$\mathcal{A}_p = \{\rho(\cdot, c) \mid \rho(\cdot, c) \in \mathcal{A}\}$$

where

$$\rho(t, c) = \sum_{k=1}^{R} c_k \phi_k(t/h),$$
$$c := [c_1^T, \ldots, c_R^T, h]^T \in \mathbb{R}^{mR+1}, c_k \in \mathbb{R}^m. \tag{7}$$

Note that the parameter vector c contains the period h, so this is also tuned by the algorithm described below. We can also define the set of admissible parameters $\mathcal{C}_p$ as $\mathcal{C}_p = \{c \in \mathbb{R}^{mR+1} | \rho(\cdot, c) \in \mathcal{A}_p\}$. The magnitude and rate constraints (6), which ensure the relation $\mathcal{A}_p \subseteq \mathcal{A}$, give infinite number of constraints for c. By defining a suitable dense grid $\{t_i\}_{i=0}^T$ over $[0, 1]$, i.e. $0 = t_0 < t_1 < \ldots < t_T = 1$ then these infinite constraints can be 'approximated' by a finite set of linear constraints in the following form:

$$\begin{bmatrix} \phi_1(t_i)I & \ldots & \phi_N(t_i)I & 0 \\ -\phi_1(t_i)I & \ldots & -\phi_N(t_i)I & 0 \\ \dot{\phi}_1(t_i)I & \ldots & \dot{\phi}_N(t_i)I & -\overline{\mu} \\ -\dot{\phi}_1(t_i)I & \ldots & -\dot{\phi}_N(t_i)I & \underline{\mu} \end{bmatrix} \cdot c \leq \begin{bmatrix} \overline{\rho} \\ -\underline{\rho} \\ 0 \\ 0 \end{bmatrix} \quad (8)$$

Since the constraints are linear, $\mathcal{C}_p$ is a polytope and it is characterized as $\mathcal{C}_p = \{c \in \mathbb{R}^{mR+1} \mid c \text{ satisfies } (8)\}$. Note that (8) guarantees the satisfaction of the magnitude and rate constraints only at the grid points $t_0, t_1, \ldots, t_T$. Therefore the admissibility of every parameter trajectory obtained by using this 'approximation' has to be checked before the trajectory is accepted as a potential parameter variation of (5). This analysis can be done e.g. by checking the inequalities above over a denser time grid having break points in every interval (t_i, t_{i+1}), $i = 0, \ldots, T-1$.

Let $G_{\rho(\cdot,c)}$ denote the PLTV system obtained by evaluating the LPV system G along the periodic trajectory specified by $\rho(\cdot, c)$. The method described in the previous section can thus be used to evaluate the gain $\|G_{\rho(\cdot,c)}\|$. It follows immediately from $\mathcal{A}_\phi \subset \mathcal{A}$ that $\|G_{\rho(\cdot,c)}\| \leq \|G\|$ for all $\rho(\cdot, c) \in \mathcal{A}_p$. The lower bound on the induced $\mathcal{L}_2$ gain of G is thus defined as follows:

$$\gamma_{lb} = \sup_{\rho(\cdot,c) \in \mathcal{A}_p} \|G_{\rho(\cdot,c)}\| = \sup_{c \in \mathcal{C}_p} \|G_{\rho(\cdot,c)}\| \quad (9)$$

This is a finite dimensional optimization with linear programming constraints and nonlinear objective function that can be evaluated (within some tolerance) using the bisection algorithm summarized in the previous section. This problem is non-convex in general. However any nonlinear, optimization algorithm can be applied to this problem since a local maxima still yields a lower bound on $\|G\|$. One issue is that a single, accurate evaluation of $\|G_{\rho(\cdot,c)}\|$ requires many bisection steps and, as a consequence the matrix differential equations (Equations (3b), (3c) and (3a)) must be integrated many times for a single evaluation of the objective function. Thus the evaluation of $\|G_{\rho(\cdot,c)}\|$ is computationally costly.

A significant reduction in computation time can be achieved as follows. Let γ_{ub} denote an upper bound on the gain of the LPV system, i.e. $\gamma_{ub} \geq \|G\|$. Such an upper bound can be determined by using standard methods based on dissipativity relation (e.g. Bounded Real type LMI conditions) [10], [9]. For any $\rho(\cdot, c) \in \mathcal{A}_p$, the γ_{ub} is also an upper bound on the resulting PLTV system $G_{\rho(\cdot,c)}$. As a result, the matrix differential equations (3) will have a well-defined, bounded solution. Moreover, let $(A_{\gamma_{ub},\rho(\cdot,c)}, B_{\gamma_{ub},\rho(\cdot,c)}, C_{\gamma_{ub},\rho(\cdot,c)})$ denote the discrete-time system computed from the PLTV system $G_{\rho(\cdot,c)}$ with the value γ_{ub}. Let $\nu(c, \gamma_{ub})$ denote the induced ℓ_2 (standard $\mathcal{H}_\infty$) norm of this discrete-time system. The bound $\gamma_{ub} \geq \|G_{\rho(\cdot,c)}\|$ implies that $\nu(c, \gamma_{ub}) < 1$. It is reasonable to maximize $\nu(c, \gamma_{ub})$ in c, i.e. to bring its value closer to 1, since in this case the lower bound, corresponding to the trajectory assigned by the optimal c^* value, will be "large" and thus it provides better estimate for the lower bound on the $\mathcal{L}_2$ norm of the LPV system. Formally, this maximization can be defined by the following optimization problem:

$$c^* = arg \sup_{c \in \mathcal{C}_p} \nu(c, \gamma_{ub}) \quad (10)$$

The objective function of this optimization only requires a single integration of the matrix differential equations. Thus $\nu(c, \gamma_{ub})$ can be evaluated with significantly less computation than $\|G_{\rho(\cdot,c)}\|$. Solving this related optmization yields a parameter vector c^* that achieves a (local or global) maxima. After this optimization, the bisection algorithm of Cantoni and Sandberg is run once to compute $\|G_{\rho(\cdot,c^*)}\|$ and this yields a lower bound γ_{lb} on $\|G\|$.

IV. Numerical examples

In this section two numerical examples are presented to demonstrate the applicability of the proposed method. To initialize our algorithms we need to determine an upper bound γ_{ub} for the induced $\mathcal{L}_2$ gain. In all examples γ_{ub} is computed by solving the following optimization problem:

$$\begin{gathered} \min_{V(x,\rho)} \gamma \\ V(x, \rho) > 0, \dot{V}(x, \rho, \dot{\rho}) \leq \gamma^2 w^T w - z^T z \end{gathered} \quad (11)$$

where the Lyapunov (storage) function $V(x, \rho)$ was chosen to be quadratic: $V(x, \rho) = x^T P(\rho) x$, $P(\rho) = P(\rho, P_0, P_1, \ldots, P_M)$ and P_i-s denote the free (matrix) variables to be found. The infinite LMI constraints obtained were transformed to a finite set by choosing a suitable dense grid Γ over the parameter domain $\mathcal{P}$ and only the inequalities evaluated at the grid points are considered [8]. (By using this grid-based relaxation, we implicitly assume that the γ_{opt} value obtained from (11) converges to an upper bound of the induced norm as the density of the grid increses.)

A. LPV system with gain-scheduled PI controller

The first example, taken from [4] is a feedback interconnection of a first-order LPV system with a gain-scheduled proportional-integral controller (see Fig. 1). The state-space matrices of the closed-loop system are as follows

$$\begin{aligned} A(\rho) &:= \begin{bmatrix} -\frac{1}{\tau(\rho)}(1 + K_p(\rho)K(\rho)) & \frac{1}{\tau(\rho)} \\ -K_i(\rho)K(\rho) & 0 \end{bmatrix} \\ B(\rho) &:= \begin{bmatrix} \frac{1}{\tau(\rho)} K_p(\rho) \\ K_i(\rho) \end{bmatrix}, \\ C(\rho) &:= [-K(\rho) \ 0], \\ D &:= 1 \end{aligned} \quad (12)$$

where $\tau(\rho) := \sqrt{133.6 - 16.8\rho}$, $K(\rho) := \sqrt{4.8\rho - 8.6}$ and

$$K_p(\rho) = \frac{2\xi_{cl}\omega_{cl}\tau(\rho) - 1}{K(\rho)}, \quad K_i(\rho) = \frac{\omega_{cl}^2 \tau(\rho)}{K(\rho)}, \quad \xi_{cl} = 0.7, \quad \omega_{cl} = 0.25.$$

The scheduling parameter is assumed to vary in the interval $[2, 7]$ and $\dot{\rho} \in [-1, \ 1]$. We are interested in the induced $\mathcal{L}_2$ norm between $r(t)$ and $e(t)$. By performing the optimization

(11) with

$$V(x,\rho) = x^T \left(P_0 + \sum_{k=1}^{6} \rho^k P_k + \frac{1}{\rho} P_7 + \frac{1}{\rho^2} P_8 + \frac{1}{\rho^3} P_9 \right) x$$
$$\Gamma = \{\rho_1 = 2, \ldots, \rho_{100} = 7\}, \quad \rho_{k+1} - \rho_k = 5/99$$

we got $\gamma_{ub} = 2.964$ for the upper bound. Using the parameter values in Γ the frozen lower bound was also computed as $\gamma_{lb,fr} = \max_k \|G_{\rho_k}\|$, $\rho_k \in \Gamma$, where G_{ρ_k} denotes the LTI system obtained by substituting $\rho(t) = \rho_k$ for all t. The lower bound we obtained is $\gamma_{lb,fr} = 1.1066$. To compute the lower bound by using the algorithm introduced in the previous section, we chose the following structure for the periodic trajectories:

$$\rho(t,c) = \sum_{k=1}^{9} c_k \cos(kt/h)$$

and we let the algorithm tune the period in the interval $[h_{min} \;\; h_{max}] = [12 \;\; 20]$. The lower bound we obtained is $\gamma_{lb} = 2.5862$, at $h = 13.8$. The nonlinear optimization (10) was solved by the *patternsearch* solver of MATLAB. The worst case scheduling trajectory can be seen in Fig. 2. Note that, γ_{lb} is very close to γ_{ub}: the difference is only $\gamma_{ub} - \gamma_{lb} = 0.3838$. This means that we have very tight bounds on the norm of G: $2.5862 = \gamma_{lb} \leq \|G\| \leq \gamma_{ub} = 2.964$.

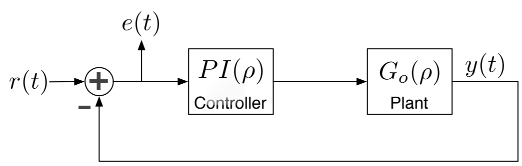

Fig. 1. The closed loop interconnection of the parameter-varying plant and the gain-scheduled PI controller in example IV-A.

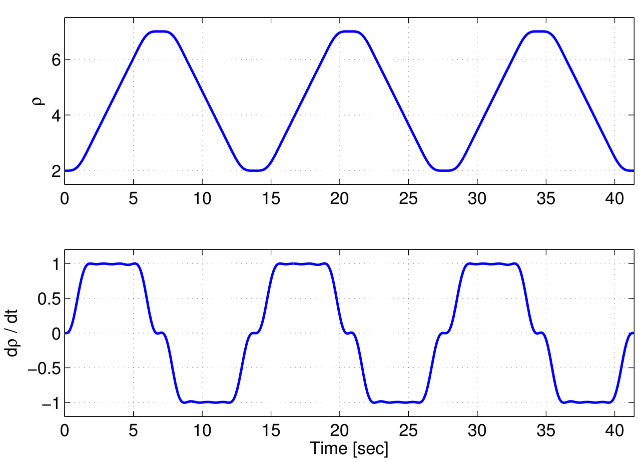

Fig. 2. The worst case parameter trajectory and its time derivative for the example in Section IV-A.

B. Input and output scaled LTI system

The next example was constructed by taking two copies of the simple LTI system $1/(s+1)$, scaling the input of the first and he output of the second by the same time varying parameter and computing the difference of the two outputs (see Fig. 3). The system matrices of the LPV system obtained are as follows:

$$A := \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B(\rho) := \begin{bmatrix} 1 \\ \rho \end{bmatrix}$$
$$C(\rho) := \begin{bmatrix} \rho & -1 \end{bmatrix}, \qquad D := 0;$$

We assumed that $\rho \in [-1 \;\; 1]$ and $\dot{\rho} \in [-\overline{\mu} \;\; \overline{\mu}]$. The bounds on the induced norm between $w(t)$ and $z(t)$ was computed for different values of $\overline{\mu}$. It follows from the structure of the system that if ρ is constant then the difference between the input and the output scaled systems is 0. This implies that $\gamma_{lb,fr} = 0$. Next, the upper bound γ_{ub} was computed by using (11) with the following strorage function and parameter grid:

$$V(x,\rho) = x^T \left(P_0 + \sum_{k=1}^{10} \rho^k P_k \right) x$$
$$\Gamma = \{\rho_1 = -1, \ldots, \rho_{100} = 1\}, \quad \rho_{k+1} - \rho_k = 2/99$$

The upper bounds obtained are collected in Table I. To compute the lower bound we constructed the periodic trajectories as a sum of *sinus* functions:

$$\rho(t,c) = \sum_{k=1}^{9} c_k \sin(kt/h)$$

During the optimization the period h was constrained to be smaller than 20. The nonlinear optimization (10) was solved again by the *patternsearch* solver of MATLAB. The computation time was in each case approx. 200 sec. The lower bounds and the period of the worst-case scheduling trajectories obtained for the different $\overline{\mu}$ values are collected in Table I. The lower and upper bounds are also plotted in Fig. 4. It can be seen that the upper and lower bounds are very close to each other, which means that by using the upper and lower bound algorithms together we could precisely compute the norm of this LPV system. In two particular cases, $\overline{\mu} = 1$ and $\overline{\mu} = 2$, we plotted also the worst case scheduling trajectories in Fig. 5 and Fig 6, repectively.

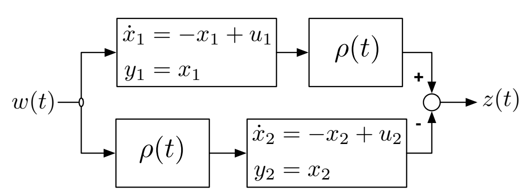

Fig. 3. The LPV system in Section IV-B.

V. Conclusion

In the paper a numerical method is proposed for computing the lower bound of the induced $\mathcal{L}_2$-gain of continuous-time, LPV systems. The algorithm finds this bound by using nonlinear optimization over periodic scheduling parameter trajectories. Restricting the domain of parameter trajectories to periodic signals enables to use the recent results for exact calculation of the $\mathcal{L}_2$ norm for a periodic time varying system. It was shown that the proposed algorithm can be reliably implemented by standard numerical tools and provides suitable precise approximation for the $\mathcal{L}_2$ bound.

$\overline{\mu}$	γ_{lb}	γ_{ub}	$\gamma_{ub}-\gamma_{lb}$	h
0.1000	0.0825	0.1087	0.0262	19.9990
0.4000	0.3174	0.3342	0.0168	20.0008
0.7000	0.4567	0.4805	0.0238	14.1852
1.0000	0.5448	0.5766	0.0318	14.1736
1.3000	0.6143	0.6435	0.0292	9.7496
1.6000	0.6386	0.6924	0.0538	8.9974
2.0000	0.6893	0.7403	0.0510	7.2397

TABLE I
UPPER AND LOWER BOUNDS ON THE $\mathcal{L}_2$ NORM OF THE LPV SYSTEM DEFINED IN SECTION IV-B

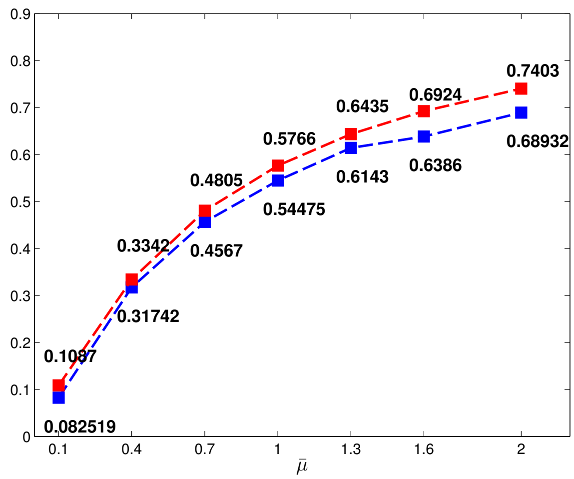

Fig. 4. Upper and lower bounds on the induced $\mathcal{L}_2$ norm computed for the LPV system in Section IV-B. The bounds are plotted as a functions of the rate limit $\overline{\mu}$.

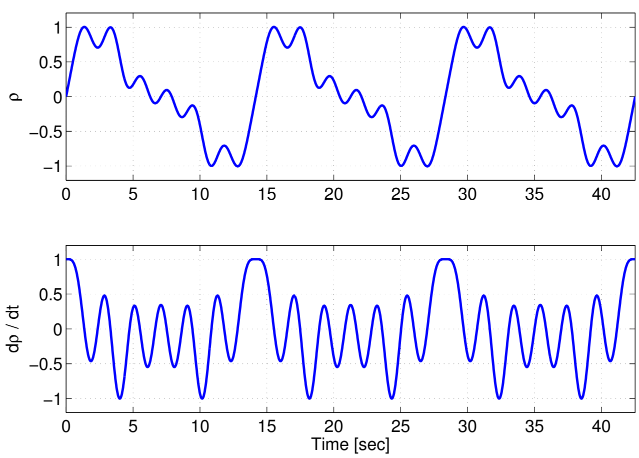

Fig. 5. The worst case parameter trajectory and its time derivative for the example in Section IV-B in case of $\overline{\mu} = 1$.

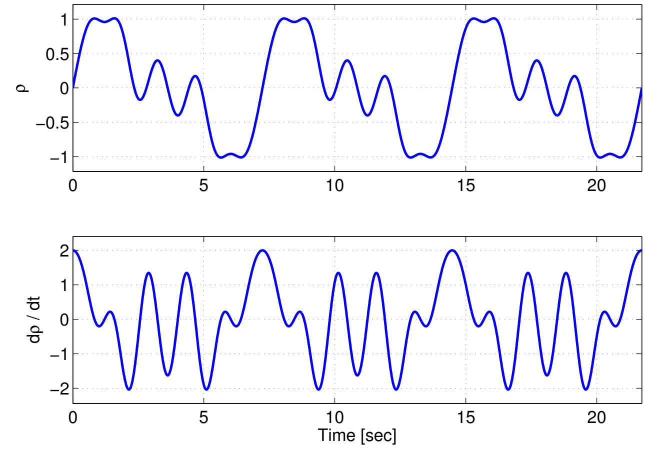

Fig. 6. The worst case parameter trajectory and its time derivative for the example in Section IV-B in case of $\overline{\mu} = 2$.

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