Integral Quadratic Constraints for Delayed Nonlinear and Parameter-Varying Systems

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Abstract

The stability and performance of nonlinear and linear parameter varying (LPV) time delayed systems are analyzed. First, the input/output behavior of the time delay operator is bounded in the frequency domain by integral quadratic constraints (IQCs). A simple geometric interpretation is used to derive new IQCs for both constant and varying delays. Second, the performance of nonlinear and LPV delayed systems is bounded using dissipation inequalities that incorporate IQCs. The nonlinear or LPV part of the system is treated directly in the analysis and not bounded by IQCs. This step makes use of recent results that show, under mild technical conditions, that an IQC has an equivalent representation as a finite-horizon time-domain constraint. A numerical example with a nonlinear delayed system is provided to demonstrate the effectiveness of the method.

1 Introduction

This paper presents an approach to analyze nonlinear or linear parameter varying (LPV) time-delayed systems. In this approach the system is separated into a nonlinear or LPV system in feedback with a time delay. Stability and performance is considered for both constant and varying delays. The analysis uses the concept of integral quadratic constraints (IQCs) (Megretski and Rantzer, 1997). Specifically, IQCs describe the behavior of a system in the frequency domain in terms of an integral constraint on the Fourier transforms of the input/output signals. Several IQCs valid for constant and varying delays have already appeared in the literature, see e.g. Megretski and Rantzer (1997); Kao and Lincoln (2004); Kao and Rantzer (2007).

The main contribution of this paper is to apply IQCs for analysis of nonlinear and LPV delayed systems. Section 3 reviews the background material on frequency-domain IQCs. This section includes new IQCs for constant and varying delays constructed using a simple Nyquist plane interpretation. The standard IQC stability theorem in Megretski and Rantzer (1997) was formulated with frequency domain conditions. This requires the "nominal" part of the interconnection to be a linear, time-invariant (LTI) system. Previous work on delayed nonlinear systems bounded the nonlinear elements of the system and the time delays by IQCs and considered this frequency domain approach to analyze a "nominal" LTI systems under IQCs, see e.g. Peet and Lall (2007). In contrast, here the nominal system is either nonlinear or LPV, which can reduce the conservatism by directly treating the nonlinearity rather than overbounding it with an IQC. This necessitates a time-domain, dissipation inequality approach. The key technical issue is to construct an equivalent time-domain interpretation for the IQC. Previous work along these lines for constant IQCs appeared in Chapter 8 of Gu et al. (2002). In fact, a large class of dynamic IQCs have an equivalent expression as a finite-horizon, time-domain integral constraint (Megretski, 2010; Seiler, 2014). Section 4 provides analysis conditions for nonlinear and delayed LPV systems that incorporate the time-domain IQC into a dissipation inequality. These analysis conditions can be efficiently solved as sum-of-squares optimizations (Parrilo, 2000) and semidefinite programs (SDPs) (Boyd et al., 1994) for nonlinear and LPV delayed systems, respectively. Section 5 gives a numerical example using this approach to analyze a nonlinear delayed system.

There is a large body of literature on time-delayed systems as summarized in Gu et al. (2002); Briat (2014). Space precludes a full review of all related results. Briefly, the approaches roughly split into two categories. Lyapunov theory (Gu et al., 2002; Gu, 1997; Fridman and Shaked, 2002) can be used to determine internal stability of delayed systems using Lyapunov-Krasovskii or Lyapunov-Razumikhin functionals. Stability conditions for nonlinear (Papachristodoulou, 2004; Papachristodoulou et al., 2009) and LPV (Zhang et al., 2002) delayed systems have been developed in the Lyapunov framework. Alternatively, input-output stability conditions for delayed systems can be developed using small-gain conditions. The IQC framework

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used here yields an input-output stability condition. The most closely related work (Fu et al., 1997; Megretski and Rantzer, 1997; Kao and Lincoln, 2004; Kao and Rantzer, 2007) uses IQCs to derive stability conditions for LTI systems with constant or varying delays. As noted above, the contribution of this paper is to extend these results to nonlinear and LPV delayed systems using dynamic IQCs, where the nonlinear part is treated directly in the analysis and not bounded by IQCs. It is important to note that the Lyapunov-type results do not require the additional well-posedness assumptions that appear in the IQC framework. In addition, there are powerful necessary and sufficient analysis conditions using Lyapunov functionals (Bliman, 2002). The paper will focus on sufficient conditions to bound the performance of uncertain, delayed systems using IQCs.

2 Problem Formulation

Consider the time-delay system given by the interconnection of a nonlinear, time-invariant system $\tilde{G}$ and a constant delay $\mathcal{D}_\tau$. The delay $\tilde{w} = \mathcal{D}_\tau(v)$ is defined by $\tilde{w}(t) = v(t-\tau)$ where τ specifies the delay. It will be more convenient to express the system in terms of the deviation between the delayed and (nominal) undelayed signal, $\mathcal{S}_\tau(v) := \mathcal{D}_\tau(v) - v$. A loop transformation can be used to express the delayed system as the interconnection $F_u(G, \mathcal{S}_\tau)$ shown in Fig. 1. This loop-shift amounts to the replacement $\tilde{w} = w + v$ where $w := \mathcal{S}_\tau(v)$. The system G is assumed to be given by:

$$\begin{aligned} \dot{x}_G &= f(x_G, w, d) \\ \left[\begin{smallmatrix} z \\ e \end{smallmatrix}\right] &= h(x_G, w, d) \end{aligned} \tag{1}$$

where $x_G \in \mathbb{R}^{n_G}$, $d \in \mathbb{R}^{n_d}$, $e \in \mathbb{R}^{n_e}$, and $w, v \in \mathbb{R}^{n_v}$.

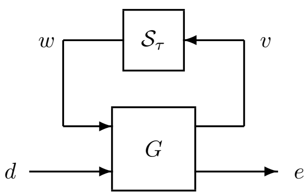

Fig. 1. Feedback interconnection with time delay $\mathcal{D}_\tau$

An input-output approach is used to analyze the time-delayed system. For a given delay τ, the induced L_2 gain from d to e is defined as:

$$\|F_u(G, \mathcal{S}_\tau)\| := \sup_{0 \neq d \in L_2^{n_d}[0,\infty),\; x_G(0)=0} \frac{\|e\|}{\|d\|} \tag{2}$$

The restriction to time $t \geq 0$ implicitly assumes zero initial conditions for both $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$. Specifically, $\tilde{w} = \mathcal{D}_\tau(v)$ is more precisely defined on $L_2[0,\infty)$ by $\tilde{w}(t) = 0$ for $t \in [0,\tau)$ and $\tilde{w}(t) = v(t-\tau)$ for $t \geq \tau$. Similarly, $w = \mathcal{S}_\tau(v)$ is defined on $L_2[0,\infty)$ by $w(t) = -v(t)$ for $t \in [0,\tau)$ and $w(t) = v(t-\tau) - v(t)$ for $t \geq \tau$. The notion of finite gain stability used in this paper is defined next.

Definition 1. The feedback interconnection of G and $\mathcal{S}_\tau$ is stable if the interconnection is well-posed and if the mapping from d to e has finite L_2 gain.

The *delay margin* is largest $\bar{\tau}$ such that the system is stable $\forall \tau \in [0, \bar{\tau}]$. The results in this paper can be used to lower bound $\bar{\tau}$ and to upper bound $\|F_u(G, \mathcal{S}_\tau)\|$ for a given $\tau \leq \bar{\tau}$.

3 Frequency Domain Inequalities

$\mathcal{D}_\tau$ is an LTI system and hence constant delays have a well-known frequency domain representation, e.g see Dullerud and Paganini (1999). Specifically, $w = \mathcal{D}_\tau(v)$ can be expressed in the frequency domain as $\hat{w}(j\omega) = \hat{\mathcal{D}}_\tau(j\omega)\hat{v}(j\omega)$ where $\hat{\mathcal{D}}_\tau(j\omega) := e^{-j\omega\tau}$. Similarly, $\hat{\mathcal{S}}_\tau(j\omega) = e^{-j\omega\tau} - 1$ is the frequency response of $\mathcal{S}_\tau$. This leads to useful frequency domain constraints on constant delays (Skogestad and Postlethwaite, 2005; Megretski and Rantzer, 1997; Gu et al., 2002). For example, a weight ϕ can be chosen so that $\hat{\mathcal{S}}_\tau \in \{\Delta : |\Delta(j\omega)| \leq |\phi(j\omega)| \;\forall\omega\}$. This frequency weighted uncertainty set has a geometric interpretation as a frequency-dependent circle in the Nyquist plane (Fig. 2). $\hat{\mathcal{S}}_\tau(j\omega)$ follows the dashed circle centered at -1 with radius 1. At each frequency $\hat{\mathcal{S}}_\tau(j\omega)$ lies within the shaded circle of radius $|\phi(j\omega)|$ centered at the origin.

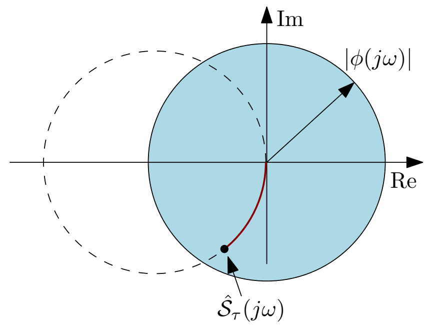

Fig. 2. Circle Interpretation for $|\hat{\mathcal{S}}_\tau(j\omega)| \leq |\phi(j\omega)|$

An algebraic interpretation is given by the following quadratic constraint on any input/output pair $w = \mathcal{S}_\tau(v)$:

$$\begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \begin{bmatrix} |\phi(j\omega)|^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} \geq 0 \quad \forall\omega \tag{3}$$

Integral quadratic constraints (IQCs) (Megretski and Rantzer, 1997) can be used to define more general frequency-domain constraints on delays based on this algebraic interpretation.

Definition 2. Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)}$ be a Hermitian-valued function. Two signals $v \in L_2^{m_1}[0,\infty)$ and $w \in L_2^{m_2}[0,\infty)$ satisfy the IQC defined by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{4}$$

where $\hat{v}(j\omega)$ and $\hat{w}(j\omega)$ are Fourier transforms of v and w, respectively. A bounded, causal operator $\Delta : L_{2e}^{m_1}[0,\infty) \to L_{2e}^{m_2}[0,\infty)$ satisfies the IQC defined by Π, denoted $\Delta \in \text{IQC}(\Pi)$, if (4) holds for all $v \in L_2^{m_1}[0,\infty)$ and $w = \Delta(v)$.

Multiple IQCs can be combined to obtain new IQCs. If the operator Δ satisfies the IQCs defined by $\{\Pi_k\}_{k=1}^N$ then Δ also satisfies the IQC defined by $\Pi(\lambda) := \sum_{k=1}^N \lambda_k \Pi_k$ for any real, non-negative numbers $\{\lambda_k\}_{k=1}^N$. $\Pi(\lambda)$ is called a conic combination of the multipliers $\{\Pi_k\}_{k=1}^N$. This fact enables many IQCs on Δ to be incorporated into an analysis.

3.1 Application to Constant Time Delays

A variety of IQCs exist for $\mathcal{S}_\tau$, e.g. see Megretski and Rantzer (1997). For clarity, the IQC multipliers are given for SISO $\mathcal{S}_\tau$. One standard multiplier is $\Pi_1 := \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$. Π_1 does not depend on the value of the delay τ and hence this multiplier is conservative. A second standard IQC multiplier is $\Pi_2(j\omega) := \begin{bmatrix} |\phi(j\omega)|^2 & 0 \\ 0 & -1 \end{bmatrix}$ where ϕ satisfies $|\hat{\mathcal{S}}_\tau(j\omega)| \le \phi(j\omega)$. This is the multiplier described previously. The use of Π_2 typically yields less conservative results because ϕ is chosen based on τ.

IQCs defined by Π_1 and Π_2 both represent circle constraints on $\hat{\mathcal{S}}_\tau$ at each frequency. Π_1 is a circle centered at -1 with radius 1 and Π_2 a circle centered at the origin with radius $|\phi(j\omega)|$. A smaller circle constraint can be constructed for $\mathcal{S}_\tau$. The midpoint of the segment connecting $\hat{\mathcal{S}}_\tau(j\omega)$ and the origin is given by $\frac{1}{2}\hat{\mathcal{S}}_\tau(j\omega)$. The following multiplier Π_3 defines a circle centered at this midpoint with radius equal to the absolute value of this midpoint as shown in Fig. 3.

$$\Pi_3(j\omega) := \begin{bmatrix} 0 & \frac{1}{2}\hat{\mathcal{S}}_\tau(j\omega) \\ \frac{1}{2}\hat{\mathcal{S}}_\tau(j\omega) & -1 \end{bmatrix} \tag{5}$$

Π_3 requires a rational function fit of $\hat{\mathcal{S}}_\tau(j\omega)$ so that state-space numerical methods can be applied. Moreover, the IQCs on $\mathcal{S}_\tau$ can be converted, if needed, into equivalent IQCs on $\mathcal{D}_\tau$ by reversing the loop-transformation, i.e. by replacing $w = \tilde{w} - v$ in the IQC.

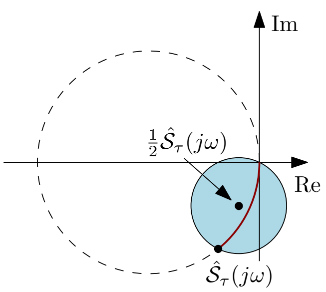

Fig. 3. "Small" Circle Constraint on $\mathcal{S}_\tau$ described by Π_3

3.2 Application to Time-Varying Delays

The results for constant delays can be extended to time-varying delays. The varying delay $\tilde{w} = \mathcal{D}_{\bar{\tau},r}(v)$ is defined by $\tilde{w}(t) = v(t - \tau(t))$ where $\tau(t)$ is the delay at time t. The subscripts $\bar{\tau}$ and r denote that the delay satisfies $\tau(t) \in [0,\bar{\tau}]$ and $|\dot{\tau}(t)| \le r \ \forall t \ge 0$. If $r = 0$ then $\mathcal{D}_{\bar{\tau},r}$ corresponds to a constant delay with value $\tau \in [0,\bar{\tau}]$. In addition, define $w = \mathcal{S}_{\bar{\tau},r}(v)$ by $w = \mathcal{D}_{\bar{\tau},r}(v) - v$, i.e. $\mathcal{S}_{\bar{\tau},r}$ is the deviation from the undelayed signal. A varying delay does not have a valid frequency domain interpretation but the frequency-domain intuition still yields useful constraints.

The basic IQCs for time-varying delays arise from two simple norm bounds. First, if $r < 1$ then $\|\mathcal{D}_{\bar{\tau},r}\| \le \frac{1}{\sqrt{1-r}}$ (Section 3.2 in Gu et al. (2002) and Lemma 1 in Kao and Rantzer (2007)). Second, let $\mathcal{S}_{\bar{\tau},r} \circ \frac{1}{s}$ denote $\mathcal{S}_{\bar{\tau},r}$ composed with an integrator at the input. Then this combined system is bounded by $\|\mathcal{S}_{\bar{\tau},r} \circ \frac{1}{s}\| \le \bar{\tau}$ (Lemma 1 in Kao and Lincoln (2004)). These two bounds are tight in the sense that the gain is achieved for some input v and varying delay $\tau(t)$ that satisfies the bounds $\bar{\tau}$ and r (Lemma 1 in Kao and Rantzer (2007)).

Three IQCs are now given for time-varying delays. For clarity the multipliers are expressed for SISO $\mathcal{S}_{\bar{\tau},r}$. First, if $r < 1$ then $\Pi_4 := \begin{bmatrix} \frac{r}{1-r} & -1 \\ -1 & -1 \end{bmatrix}$ is valid for $\mathcal{S}_{\bar{\tau},r}$. This is analogous to the multiplier Π_1 for constant delays. Π_4 depends on the rate of variation r but does not depend on the maximum delay $\bar{\tau}$. Proposition 2 in Kao and Rantzer (2007) gives a delay-dependent IQC that can be used to reduce the conservatism. The IQC in Kao and Rantzer (2007) depends on a rational bounded transfer function $\phi_5(s)$ that satisfies:

$$|\phi_5(j\omega)| > \begin{cases} \bar{\tau}|\omega| & \text{if } \bar{\tau}|\omega| \le 1 + \frac{1}{\sqrt{1-r}} \\ 1 + \frac{1}{\sqrt{1-r}} & \text{if } \bar{\tau}|\omega| > 1 + \frac{1}{\sqrt{1-r}} \end{cases} \tag{6}$$

If $r < 1$ then $\mathcal{S}_{\bar{\tau},r}$ satisfies the IQC defined by $\Pi_5(j\omega) := \begin{bmatrix} |\phi_5(j\omega)|^2 & 0 \\ 0 & -1 \end{bmatrix}$. Note that for $r \ge 1$ this IQC is not well-posed. The multiplier Π_5 is analogous to Π_2 from the previous section. The bound on $|\phi_5|$ effectively increases the radius of the circle constraint defined by Π_5 at high frequencies to account for the time-varying delay. Proposition 3 in Kao and Rantzer (2007) gives a similar IQC multiplier that is valid for $r < 2$. Finally, recall that Π_3 defined a smaller circle than the multipliers Π_1 and Π_2. This frequency domain intuition can be used to derive a new, related IQC for varying delays.

Theorem 1. Let $\phi_6(s)$ be a transfer function satisfying:

$$|\phi_6(j\omega)| > \begin{cases} \frac{1}{2}\bar{\tau}|\omega| & \text{if } \frac{1}{2}\bar{\tau}|\omega| \le 1 + \frac{1}{\sqrt{1-r}} \\ 1 + \frac{1}{\sqrt{1-r}} & \text{if } \frac{1}{2}\bar{\tau}|\omega| > 1 + \frac{1}{\sqrt{1-r}} \end{cases} \tag{7}$$

If $r < 1$ then $\mathcal{S}_{\bar{\tau},r}$ satisfies the IQC defined by:

$$\Pi_6(j\omega) := \begin{bmatrix} |\phi_6(j\omega)|^2 - \frac{1}{4}|\hat{\mathcal{S}}_{\bar{\tau}}(j\omega)|^2 & \frac{1}{2}\hat{\mathcal{S}}_{\bar{\tau}}(j\omega) \\ \frac{1}{2}\hat{\mathcal{S}}_{\bar{\tau}}(j\omega) & -1 \end{bmatrix} \tag{8}$$

Proof. First show $\|\Delta\| \leq 1$ where $\Delta := (\mathcal{S}_{\bar{\tau},r} - \frac{1}{2}\mathcal{S}_{\bar{\tau}}) \circ \phi_6^{-1}$. The proof is only sketched as it is similar to that given for Proposition 2 in Kao and Rantzer (2007). Let $v \in L_2$ be an input signal and $\hat{v} := \mathcal{F}(v)$ its corresponding Fourier Transform. Decompose v as $v_L + v_H$ where v_L and v_H are the low and high frequency components, respectively. Specifically, the low-frequency content is defined in the frequency domain by $\hat{v}_L(j\omega) := \hat{v}(j\omega)$ if $|\omega| \leq \frac{2}{\bar{\tau}}\left(1 + \frac{1}{\sqrt{1-r}}\right)$ and $\hat{v}_L(j\omega) := 0$ otherwise. The high-frequency content is defined similarly. Then using the linearity of Δ and the triangle inequality yields $\|\Delta v\| \leq \|\Delta v_L\| + \|\Delta v_H\|$. Lemmas 1 and 2 in Appendix A bound the gains on the high and low frequency components by $\|\Delta v_H\| \leq \|v_H\|$ and $\|\Delta v_L\| \leq \|v_L\|$. Thus $\|\Delta v\| \leq \|v_L\| + \|v_H\| = \|v\|$, i.e. $\|\Delta\| \leq 1$. The bound on Δ can be equivalently expressed as a quadratic, frequency-domain constraint on the input/output signals of $\mathcal{S}_{\bar{\tau},r}$. It follows that $\mathcal{S}_{\bar{\tau},r}$ satisfies the IQC defined by Π_6. □

To show the relationship between Π_3 and Π_6, consider the Taylor series expansion for $\hat{\mathcal{S}}_{\bar{\tau}}$ which is $\bar{\tau}\omega + \mathcal{O}(\omega^2)$. Hence Π_6 is, by proper choice of ϕ_6, equivalent to the constant delay multiplier Π_3 at low frequencies. Again, Π_6 requires a bounded rational function fit of $\hat{\mathcal{S}}_{\tau}(j\omega)$ so that state-space numerical methods can be applied. Theorem 1 demonstrates the benefit of the frequency domain intuition even for varying delays. Note that ultimately all multipliers in this paper use bounded rational function fits. In Jönsson (1996) approaches are given that allow adding unbounded multipliers, e.g. Popov multipliers, to the analysis.

4 Time Domain Stability Analysis

This section shows that, under some mild technical conditions, the frequency domain IQCs from the previous section have an equivalent time domain representation (Section 4.1). This is used to derive stability conditions for delayed nonlinear and parameter varying systems (Sections 4.2 and 4.3).

4.1 Time Domain IQCs

Let Π be an IQC multiplier that is a rational and uniformly bounded function of $j\omega$, i.e. $\Pi \in \mathbb{RL}_\infty^{(m_1+m_2)\times(m_1+m_2)}$. The time domain interpretation is based on factorizing the multiplier as $\Pi = \Psi^\sim M \Psi$ where $M = M^T \in \mathbb{R}^{n_z \times n_z}$ and $\Psi \in \mathbb{RH}_\infty^{n_z \times (m_1+m_2)}$. The restriction to rational, bounded multipliers Π ensures that such factorizations can be numerically computed via transfer function or state-space methods Youla (1961); Scherer and Wieland (2004). Such factorizations are not unique and two specific factorizations are provided in Appendix B.

Next, let (v, w) be a pair of signals that satisfy the IQC in (4) and define $\hat{z}(j\omega) := \Psi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}$. Then the IQC can be written as: $\int_{-\infty}^{\infty} \hat{z}(j\omega)^* M \hat{z}(j\omega) d\omega \geq 0$. By Parseval's theorem (Zhou et al., 1996), this frequency-domain inequality can be equivalently expressed in the time-domain as:

$$\int_0^\infty z(t)^T M z(t)\, dt \geq 0 \tag{9}$$

where z is the output of the LTI system Ψ:

$$\begin{aligned} \dot{\psi}(t) &= A_\psi \psi(t) + B_{\psi 1} v(t) + B_{\psi 2} w(t), \quad \psi(0) = 0 \\ z(t) &= C_\psi \psi(t) + D_{\psi 1} v(t) + D_{\psi 2} w(t) \end{aligned} \tag{10}$$

Thus $v \in L_2^{m_1}[0,\infty)$ and $w \in L_2^{m_2}[0,\infty)$ satisfy the IQC defined by $\Pi = \Psi^\sim M \Psi$ if and only if the filtered signal $z = \Psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$ satisfies the time domain constraint in (9). Similarly, a bounded, causal system Δ satisfies the IQC defined by $\Pi = \Psi^\sim M \Psi$ if and only if (9) holds for all $v \in L_2^{m_1}[0,\infty)$ and $w = \Delta(v)$. To simplify notation, $\Delta \in$ IQC(Π) will also be denoted by $\Delta \in$ IQC(Ψ, M). Fig. 4 provides a graphical interpretation for $\Delta \in$ IQC(Ψ, M). The input/output signals of Δ are filtered by Ψ and the output z satisfies (9).

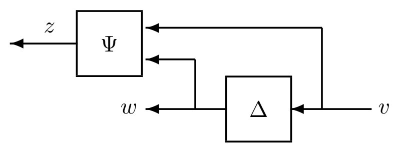

Fig. 4. Interpretation of the IQC defined by $\Pi = \Psi^\sim M \Psi$

The time domain constraint (9) holds, in general, only over infinite time intervals. The term hard IQC was introduced in Megretski and Rantzer (1997) for the following more restrictive property: Δ satisfies the IQC defined by Π and $\int_0^T z(t)^T M z(t)\, dt \geq 0$ holds for all $T \geq 0$, $v \in L_{2e}^{m_1}[0,\infty)$ and $w = \Delta(v)$. By contrast, IQCs for which the time domain constraint need not hold over all finite time intervals are called soft IQCs. Hard and soft IQCs were later generalized in Megretski et al. (2010) to include the effect of initial conditions and the terms were renamed complete and conditional IQCs, respectively. The hard/soft terminology will be used here. The validity of the constraint over finite-horizons (rather than infinite-horizons) is significant as it enables the constraint to be used within the dissipation inequality framework, see Section 4.2. An issue is that the factorization of Π as $\Psi^\sim M \Psi$ is not unique. As a result, the terms hard and soft are not inherent to the multiplier Π but instead depend on the factorization (Ψ, M) as defined next.

Definition 3. Let Π be factorized as $\Psi^\sim M \Psi$ with Ψ stable. Then (Ψ, M) is a hard IQC factorization of Π if for any bounded, causal operator $\Delta \in \text{IQC}(\Pi)$ the following time-domain inequality holds

$$\int_0^T z(t)^T M z(t)\, dt \geq 0 \tag{11}$$

for all $T \geq 0$, $v \in L_{2e}^{m_1}[0,\infty)$, $w = \Delta(v)$, and $z = \Psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$.

It was shown in Megretski (2010) that a broad class of multipliers have a hard factorization. The proof uses a new min/max theorem to lower bound $\int_0^T z(t)^T M z(t)\, dt$. A similar factorization result was obtained in Seiler (2014) using a game-theoretic interpretation. The next theorem summarizes the main factorization required to incorporate IQCs into a dissipation inequality.

Theorem 2. Let $\Pi = \Pi^\sim \in \mathbb{RL}_\infty^{(m_1+m_2)\times(m_1+m_2)}$ be partitioned as $\left[\begin{smallmatrix} \Pi_{11} & \Pi_{21}^\sim \\ \Pi_{21} & \Pi_{22} \end{smallmatrix}\right]$ where $\Pi_{11} \in \mathbb{RL}_\infty^{m_1 \times m_1}$ and $\Pi_{22} \in \mathbb{RL}_\infty^{m_2 \times m_2}$. Assume $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$. Then Π has a hard factorization (Ψ, M).

Proof. The sign definite conditions on Π_{11} and Π_{22} ensure that Π has a factorization (Ψ, M) where Ψ is square and both Ψ, Ψ^{-1} are stable. This follows from Lemmas 3 and 4 in Appendix B. Moreover, Appendix B provides a numerical algorithm to compute this special (J-spectral) factorization using state-space methods. The conclusion that (Ψ, M) is a hard factorization follows from Theorem 2.4 in Megretski (2010). □

4.2 *Analysis of Nonlinear Delayed Systems*

This section derives analysis conditions for the nonlinear delayed system $F_u(G, \mathcal{S}_\tau)$ shown in Fig. 1. For concreteness the discussion focuses on constant delays $\mathcal{S}_\tau$ but the results also hold using IQCs for varying delays $\mathcal{S}_{\bar{\tau},r}$. Assume $\mathcal{S}_\tau$ satisfies the IQC defined by Π and, in addition, Π has a hard factorization (Ψ, M). The delayed system is analyzed by appending Ψ to $\mathcal{S}_\tau$ as shown in Fig. 5. The interconnection in Fig. 5 involves extended dynamics of the form:

$$\begin{aligned} \dot{x} &:= F(x, w, d) \\ \left[\begin{smallmatrix} z \\ e \end{smallmatrix}\right] &:= H(x, w, d) \end{aligned} \tag{12}$$

$x := \left[\begin{smallmatrix} x_G \\ \psi \end{smallmatrix}\right] \in \mathbb{R}^{n_G + n_\psi}$ is the extended state. The functions F and H can be easily determined from the dynamics of G and Ψ defined in (1) and (10). The theorem below provides a sufficient condition for $\|F_u(G, \mathcal{S}_\tau)\| \leq \gamma$. The main condition is a dissipation inequality that uses both the hard IQC satisfied by $\mathcal{S}_\tau$ and a storage function V defined on the extended state x. The system $\mathcal{S}_\tau$ is shown as a dashed box in Fig. 5 because the analysis essentially replaces the precise relation $w = \mathcal{S}_\tau(v)$ with the hard IQC constraint on z that specifies the signals (v, w) that are consistent with $\mathcal{S}_\tau$.

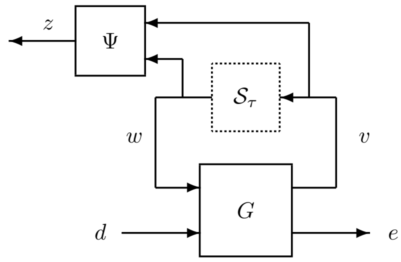

Fig. 5. Analysis Interconnection Structure

Theorem 3. Assume $F_u(G, \mathcal{S}_\tau)$ is well-posed and $\mathcal{S}_\tau$ satisfies the hard IQC defined by (Ψ, M). Then $\|F_u(G, \mathcal{S}_\tau)\| \leq \gamma$ if there exists a scalar $\lambda \geq 0$ and a continuously differentiable storage function $V : \mathbb{R}^{n_G + n_\psi} \to \mathbb{R}$ such that:

i) $V(0) = 0$,
ii) $V(x) \geq 0 \;\; \forall x \in \mathbb{R}^{n_G + n_\psi}$,
iii) The following dissipation inequality holds for all $x \in \mathbb{R}^{n_G + n_\psi}, w \in \mathbb{R}^{n_v}, d \in \mathbb{R}^{n_d}$

$$\lambda z^T M z + \nabla V(x)\; F(x, w, d) \leq \gamma^2 d^T d - e^T e \tag{13}$$

where z and e are functions of (x, w, d) as defined by H in Equation 12.

Proof. Let $d \in L_2^{n_d}[0,\infty)$ be any input signal. From well-posedness, the interconnection $F_u(G, \mathcal{S}_\tau)$ has a solution that satisfies the dynamics in (12). The dissipation inequality (13) can be integrated from $t = 0$ to $t = T$ with the initial condition $x(0) = 0$ to yield:

$$\begin{aligned} &\lambda \int_0^T z(t)^T M z(t)\, dt + V\left(x(T)\right) \leq \\ &\gamma^2 \int_0^T d(t)^T d(t)\, dt - \int_0^T e(t)^T e(t)\, dt \end{aligned} \tag{14}$$

Apply the hard IQC condition, $\lambda \geq 0$, and $V \geq 0$ to show (14) implies $\int_0^T e(t)^T e(t)\, dt \leq \gamma^2 \int_0^T d(t)^T d(t)\, dt$. □

It is important to recall that soft IQCs only hold, in general, over infinite time horizons and they require the signals (v, w) to be in L_2. Hence they cannot be used in the dissipation inequality proof since we don't know, a priori, that (v, w) are in L_2. On the other hand, hard IQCs hold over finite time horizons and for all signals (v, w) in the extended space L_{2e}. Hence inequality 14 can be used at all finite times to demonstrate finite gain from d to e. It is also notable that the dissipation inequality (13) is an algebraic constraint on variables (x, w, d). The dissipation inequality only depends on $\mathcal{S}_\tau$ via the term $z^T M z$ and hence the delay value τ only appears through the choice of the multiplier Π. Specifically, Π typically depends on the value of τ, e.g. Π_2 and Π_3 defined previously. The delay τ is selected and then the multiplier Π and its hard factorization (Ψ, M) are constructed. Thus for a given delay τ, Theorem 3 provides convex conditions on V, λ, and γ that are sufficient to upper bound $\|F_u(G, \mathcal{S}_\tau)\|$.

This leads to a useful numerical procedure under additional assumptions. If the dynamics of G in (1) are described by polynomial vector fields then the functions F and H in the extended system (12) are also polynomials. If the storage function V is also restricted to be polynomial then the dissipation inequality (13) and non-negativity condition $V \geq 0$ are simply global polynomial constraints. In this case the search for a feasible storage function V and scalars λ, γ can be formulated as a sum-of-squares (SOS) optimization Parrilo (2000, 2003); Lasserre (2001). For fixed delay τ this yields a convex optimization to upper bound $\|F_u(G, \mathcal{S}_\tau)\|$. In addition, bisection can be used to find the largest delay $\bar{\tau}$ such that $\|F_u(G, \mathcal{S}_\tau)\|$ remains finite. If the multiplier Π covers $\mathcal{S}_\tau$ for all $\tau \in [0, \bar{\tau}]$ then $\bar{\tau}$ is a lower bound on the true delay margin. It is a lower bound because the dissipation inequality is only a sufficient condition. An example of this SOS method is given in Section 5. The computation for this SOS approach grows rapidly with the degree and number of variables contained in the polynomial constraint. This currently limits the approach to situations where the extended system roughly involves a cubic vector field and state dimension $\leq 7 - 10$.

It should be noted that multiple IQCs can be used in the analysis. Specifically, assume $\mathcal{S}_\tau$ satisfies the hard IQCs defined by (Ψ_k, M_k) for $k = 1, \cdots, N$. Each Ψ_k can be appended to $\mathcal{S}_\tau$ to yield a filtered output z_k. Theorem 3 remains valid if the dissipation inequality (13) is modified to include the term $\sum_{k=1}^{N} \lambda_k z_k^T M_k z_k$ for any constants $\lambda_k \geq 0$. In this case the extended system includes the dynamics of G and the dynamics of each Ψ_k ($k = 1, \cdots, N$). The analysis consists of a search for the storage V, gain bound γ, and constants λ_k that lead to feasibility of the three conditions in Theorem 3.

4.3 Analysis of LPV Delayed Systems

An LPV system is a linear system whose state space matrices depend on a time-varying parameter vector $\rho : \mathbb{R}^+ \to \mathbb{R}^{n_\rho}$. The parameter is assumed to be a continuously differentiable function of time and admissible trajectories are restricted to a known compact set $\mathcal{P} \subset \mathbb{R}^{n_\rho}$. The state matrices are continuous functions of ρ, e.g. $A_G : \mathcal{P} \to \mathbb{R}^{n_x \times n_x}$. Define the LPV system G_ρ with inputs (w, d) and outputs (v, e) as:

$$\begin{aligned} \dot{x}_G(t) &= A_G(\rho(t))\, x_G(t) + B_G(\rho(t)) \begin{bmatrix} w(t) \\ d(t) \end{bmatrix} \\ \begin{bmatrix} v(t) \\ e(t) \end{bmatrix} &= C_G(\rho(t))\, x_G(t) + D_G(\rho(t)) \begin{bmatrix} w(t) \\ d(t) \end{bmatrix} \end{aligned} \tag{15}$$

The state matrices at time t depend on the parameter vector at time t. Hence, LPV systems represent a special class of time-varying systems. The explicit dependence on t is occasionally suppressed to shorten the notation.

By loop-shifting, a delayed LPV system can be modeled as $F_u(G_\rho, \mathcal{S}_\tau)$ where $w = \mathcal{S}_\tau(v)$. This is similar to the interconnection in Fig. 1 but with G_ρ as the "nominal" system. As a slight abuse of notation, $\|F_u(G_\rho, \mathcal{S}_\tau)\|$ will denote the worst-case L_2 gain over all allowable parameter trajectories:

$$\|F_u(G_\rho, \mathcal{S}_\tau)\| = \sup_{\rho \in \mathcal{P}} \sup_{0 \neq d \in L_2^{n_d}[0,\infty),\ x_G(0)=0} \frac{\|e\|}{\|d\|} \tag{16}$$

Assume $\mathcal{S}_\tau$ satisfies the IQC defined by Π and that Π has a hard factorization (Ψ, M). Append Ψ to $\mathcal{S}_\tau$ as in Fig. 5 to yield an extended LPV system of the form:

$$\begin{aligned} \dot{x} &= A(\rho)x + B_1(\rho)w + B_2(\rho)d \\ z &= C_1(\rho)x + D_{11}(\rho)w + D_{12}(\rho)d \\ e &= C_2(\rho)x + D_{21}(\rho)w + D_{22}(\rho)d \end{aligned} \tag{17}$$

where $x := \begin{bmatrix} x_G \\ \psi \end{bmatrix} \in \mathbb{R}^{n_G + n_\psi}$ with x_G and ψ denoting the state vectors of G_ρ (15) and Ψ (10), respectively. The next theorem bounds $\|F_u(G_\rho, \mathcal{S}_\tau)\|$ using a dissipation inequality stated in the form of a linear matrix inequality. The theorem is stated assuming a single multiplier for $\mathcal{S}_\tau$ but many IQC multipliers can be included as described previously.

Theorem 4. Assume $F_u(G_\rho, \mathcal{S}_\tau)$ is well posed and $\mathcal{S}_\tau$ satisfies the hard IQC defined by (Ψ, M). Then $\|F_u(G_\rho, \mathcal{S}_\tau)\| \leq \gamma$ if there exists a scalar $\lambda \geq 0$ and a matrix $P = P^T \in \mathbb{R}^{n_x + n_\psi}$ such that $P \geq 0$ and for all $\rho \in \mathcal{P}$:

$$\begin{aligned} &\begin{bmatrix} A^T P + PA & PB_1 & PB_2 \\ B_1^T P & 0 & 0 \\ B_2^T P & 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C_2^T \\ D_{21}^T \\ D_{22}^T \end{bmatrix} \begin{bmatrix} C_2 & D_{21} & D_{22} \end{bmatrix} \\ &\qquad + \lambda \begin{bmatrix} C_1^T \\ D_{11}^T \\ D_{12}^T \end{bmatrix} M \begin{bmatrix} C_1 & D_{11} & D_{12} \end{bmatrix} < 0 \end{aligned} \tag{18}$$

In (18) the dependence of the state matrices on ρ has been omitted to shorten the notation.

Proof. Define a storage function $V : R^{n_G + n_\psi} \to \mathbb{R}^+$ by $V(x) = x^T P x$. Left and right multiply (18) by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ to show that V satisfies:

$$\lambda z(t)^T M z(t) + \dot{V}(t) \leq \gamma^2 d(t)^T d(t) - e(t)^T e(t) \tag{19}$$

The remainder of the proof follows from this dissipation inequality similar to the proof given for Theorem 3. □

Theorem 4 involves parameter dependent LMI conditions. These are infinite dimensional (one for each $\rho \in \mathcal{P}$) and they are typically approximated by finite-dimensional LMIs evaluated on a grid of parameter values. The analysis can then be performed as an SDP (Boyd et al., 1994). If the LPV system has a rational dependence on ρ then finite dimensional LMI conditions can be derived (with no gridding) using the techniques in Packard (1994); Apkarian and Gahinet (1995). In addition, Theorem 4 makes no assumptions on $\dot{\rho}$. It can be extended to include rate-bounds using parameter-dependent storage functions as in Wu et al. (1996).

5 Numerical Example: Delayed Nonlinear System

Consider the feedback system in Fig. 6 where $\mathcal{D}_{\bar{\tau},r}$ is a varying delay, Δ a norm-bounded uncertainty with $\|\Delta\| \le 0.1$ and L is the following nonlinear system:

$$\begin{aligned} \dot{x}_G &= \begin{bmatrix} -49 & 0 \\ 1 & 0 \end{bmatrix} x_G + \begin{bmatrix} 8 \\ 0 \end{bmatrix} \tilde{w} + p(x_G) \\ y &= \begin{bmatrix} -4.5 & 1.5 \end{bmatrix} x_G \end{aligned} \qquad (20)$$

where $p(x_G) := \begin{bmatrix} 2x_{G,1}^2+3x_{G,2}^2-0.2x_{G,1}^3, & -x_{G,2}^3 \end{bmatrix}^T$. The loop shift described in Section 2 brings the (nominal) feedback system into the form $F_u(G, \mathcal{S}_{\bar{\tau},r})$ as shown in Fig. 1.

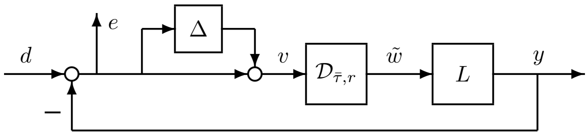

Fig. 6. Classical Feedback System

For comparison, a nominal linear analysis is first performed with Δ set to zero. Let G_{lin} denote the linearization of G around $x_G = 0$ obtained by neglecting $p(x_G)$. A delay margin estimate for $F_u(G_{lin}, \mathcal{S}_{\bar{\tau},r})$ was computed from the LMI condition in Theorem 4. Bisection was used to find the largest time delay for which the gain from d to e is finite. The LMI at each bisection step was solved using Matlab's `LMILab` toolbox. For constant delay ($r = 0$), the standard multipliers Π_1 and Π_2 yield a delay margin of 0.06sec. However, using Π_1 and the new "small" circle multiplier Π_3 yields a larger margin of 1.96sec. The exact delay margin estimated from the frequency response of G_{lin}, is 2.05 sec. For time-varying delays with $r = 0.1$, the analysis using Π_4 and Π_5 gives a delay margin of 0.04sec. Using Π_4 and the new multiplier Π_6 again results in a larger margin of 0.21sec.

Next consider the (nominal) delayed nonlinear system. A delay margin estimate for $F_u(G, \mathcal{S}_\tau)$ was computed from the dissipation inequality in Theorem 3 using a quartic storage function V. The SOSOPT toolbox (Balas et al., 2013) was used for all computations. For constant delay ($r = 0$), $\{\Pi_1, \Pi_2\}$ give a delay margin of 0.04sec while Π_1 and the new multiplier Π_3 again yield a larger margin of 1.09sec. Both estimates are degraded by the nonlinearity compared to the linear results. The computation for both analyses took $\approx$ 5sec to perform 13 bisection steps with a tolerance of 10^{-3}. For a varying delay with $r = 0.1$, the analysis was not able guarantee any nonzero margin using either $\{\Pi_4, \Pi_5\}$ or $\{\Pi_4, \Pi_6\}$. This indicates a lack of robustness to the combined nonlinearity and varying delay. If p is scaled down to $\tilde{p} = 0.001p$ then the analysis with $\{\Pi_4, \Pi_5\}$ or $\{\Pi_4, \Pi_6\}$ both recover the linear analysis results of 0.04sec and 0.21 sec. This is not surprising but confirms that it is possible to obtain non-zero margins for varying-delay nonlinear systems using Theorem 3.

The analysis was repeated again for the uncertain, delayed nonlinear system. This analysis includes the uncertainty $\|\Delta\| \le 0.1$ which is described by $\Pi = \begin{bmatrix} 0.1^2 & 0 \\ 0 & -1 \end{bmatrix}$. For constant delays ($r = 0$), the delay margin is degraded from 1.09sec (no uncertainty) to 0.32sec (with uncertainty). This analysis was performed using $\{\Pi_1, \Pi_3\}$ for the delay. This demonstrates that the proposed method can analyze nonlinear systems with combinations of delays and uncertainties.

Finally, the dissipation inequality in Theorem 3 can be used for performance analysis. Figure 7 shows the L_2 gain of the nonlinear system from d to e with (orange dashed) and without uncertainty (red dash-dot) for constant delays. The multipliers Π_1 and Π_3 were used to compute these curves. It took 7.5 sec to evaluate the gain on a grid of 20 delay values. For comparison the figure also shows the gain of the linear system $F_u(G_{lin}, \mathcal{S}_\tau)$ computed using two methods. The blue dashed curve is the gain computed using the LMI condition in Theorem 4 also with multipliers Π_1 and Π_3. The gray solid curve is the true induced L_2 gain of the linear system estimated from the frequency response of $F_u(G_{lin}, \mathcal{S}_\tau)$. The two linear results are close which provides confidence in the upper bounds computed for the nonlinear system. The figure also shows that the delayed nonlinear system has significantly larger gain as compared to the linearized system especially with the uncertainty. This again indicates that the nonlinearities and uncertainty degrade the performance.

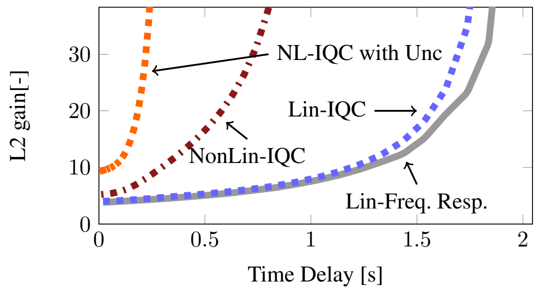

Fig. 7. Induced L_2 gain versus delay

6 Conclusions

This paper developed input/output analysis conditions for nonlinear and LPV time-delayed systems. The approach bounds the time delay using general, dynamic integral quadratic constraints (IQCs). Dissipation inequalities were provided that incorporate the IQCs into the analysis of the delayed system. This step required an equivalent time-domain interpretation for IQCs. A numerical example was provided to demonstrate the proposed methods.

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A Lemmas for proof of Theorem 1

Lemma 1. If $r < 1$ then $\|\mathcal{S}_{\bar{\tau},r} - \frac{1}{2}\mathcal{S}_{\bar{\tau}}\| \leq 1 + \frac{1}{\sqrt{1-r}}$.

Proof. The triangle inequality as well as the definitions of $\mathcal{S}_{\bar{\tau},r}$ and $\mathcal{S}_{\bar{\tau}}$ imply $\|\mathcal{S}_{\bar{\tau},r} - \frac{1}{2}\mathcal{S}_{\bar{\tau}}\| \leq \|\mathcal{D}_{\bar{\tau},r}\| + \frac{1}{2}\|\mathcal{D}_{\bar{\tau}}\| + \frac{1}{2}$. The varying delay is bounded as $\|\mathcal{D}_{\bar{\tau},r}\| \leq \frac{1}{\sqrt{1-r}}$ Gu et al. (2002); Kao and Rantzer (2007) while the constant delay is bounded by $\|\mathcal{D}_{\bar{\tau}}\| \leq 1$. □

Lemma 2. $\|(\mathcal{S}_{\bar{\tau},r} - \frac{1}{2}\mathcal{S}_{\bar{\tau}}) \circ \frac{1}{s}\| \leq \frac{1}{2}\bar{\tau}$.

Proof. The proof is only sketched as it is similar to that given for Lemma 1 in Kao and Lincoln (2004). To simplify notation define $\Delta := (\mathcal{S}_{\bar{\tau},r} - \frac{1}{2}\mathcal{S}_{\bar{\tau}}) \circ \frac{1}{s}$. Consider $w = \Delta v$

for some $v \in L_2[0,\infty)$ and define $y(t) := \int_0^t v(\alpha)d\alpha$. Thus $w = (\mathcal{S}_{\bar{\tau},r} - \frac{1}{2}\mathcal{S}_{\bar{\tau}})(y)$ which, after some algebra, gives $w(t) = \int_{t-\bar{\tau}}^t s(\alpha)v(\alpha)d\alpha$, where $s(\alpha) = +\frac{1}{2}$ for $\alpha \in [t-\bar{\tau}, t-\tau(t)]$ and $s(\alpha) = -\frac{1}{2}$ for $\alpha \in [t-\tau(t), t]$. The Cauchy-Schwartz inequality can then be used to show $w(t)^2 \le \frac{\bar{\tau}}{4}\int_{t-\bar{t}}^t v^2(\alpha)d\alpha$. Integrate this inequality from $t = 0$ to $t = \infty$ and perform a change of variables to obtain $\|w\|^2 \le \frac{\bar{\tau}^2}{4}\|v\|^2$. □

B IQC Factorizations

This appendix provides numerical procedures to factorize $\Pi = \Pi^\sim \in \mathbb{RL}_\infty^{m\times m}$ as $\Psi^\sim M\Psi$. Such factorizations are not unique and this appendix provides two specific factorizations. The second of these factorizations (Lemma 3) is particularly useful. First, let $(A_\pi, B_\pi, C_\pi, D_\pi)$ be a minimal state-space realization for Π. Separate Π into its stable and unstable parts $\Pi = G_S + G_U$. Let (A, B, C, D_π) denote a state space realization for the stable part G_S so that A is Hurwitz. The assumptions on Π imply that G_U has a state space realization of the form $(-A^T, -C^T, B^T, 0)$ (Section 7.3 of Francis (1987)). Thus $\Pi = G_S + G_U$ can be written as $\Pi = \Psi^\sim M\Psi$ where $\Psi(s) := \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix}$ and $M := \begin{bmatrix} 0 & C^T \\ C & D_\pi \end{bmatrix}$. This provides a factorization $\Pi = \Psi^\sim M\Psi$ where $M = M^T \in \mathbb{R}^{n_z\times n_z}$ and $\Psi \in \mathbb{RH}_\infty^{n_z\times m}$. For this factorization Ψ is, in general, non-square ($n_z \ne m$) and it may have right-half plane zeros.

The stability theorems in this paper require a special factorization such that Ψ is square ($n_z = m$), stable, and minimum phase. More precisely, given non-negative integers p and q, let $J_{p,q}$ denote the signature matrix $\begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$. Ψ is called a $J_{p,q}$-spectral factor of Π if $\Pi = \Psi^\sim J_{p,q}\Psi$ and $\Psi, \Psi^{-1} \in \mathbb{RH}_\infty^{m\times m}$. The term J-spectral factor will be used if the values of p and q are not important. Lemma 3 provides a necessary and sufficient condition for constructing a J-spectral factorization of Π. Finally, Lemma 4 below gives a simple frequency domain condition that is sufficient for the existence of a J-spectral factor.

Lemma 3. Let $\Pi(s) := \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix}^\sim \begin{bmatrix} 0 & C^T \\ C & D_\pi \end{bmatrix} \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix}$ with A Hurwitz and $D_\pi = D_\pi^T$. Then the following statements are equivalent:

(1) D_π is nonsingular and there exists a unique real solution $X = X^T$ to the Algebraic Riccati Equation

$$A^T X + XA - (XB + C^T)D_\pi^{-1}(B^T X + C) = 0 \quad \text{(B.1)}$$

such that $A - BD_\pi^{-1}\left(B^T X + C\right)$ is Hurwitz.

(2) Π has a $J_{p,q}$ spectral factorization where p and q are the number of positive and negative eigenvalues of D_π, respectively. Moreover, Ψ is a $J_{p,q}$-spectral factor of Π if and only if it has a state-space realization $\left(A, B, J_{p,q}W^{-T}\left(B^T X + C\right), W\right)$ where W is a solution of $D_\pi = W^T J_{p,q} W$.

Proof. This lemma is based on the canonical factorization in Bart et al. (1979) and summarized in Chapter 7 of Francis (1987). The precise wording of this lemma is a special case of Theorem 2.4 in Meinsma (1995). □

Lemma 4. Let $\Pi = \Pi^\sim \in \mathbb{RL}_\infty^{(m_1+m_2)\times(m_1+m_2)}$ be partitioned as $\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\sim & \Pi_{22} \end{bmatrix}$ where $\Pi_{11} \in \mathbb{RL}_\infty^{m_1\times m_1}$ and $\Pi_{22} \in \mathbb{RL}_\infty^{m_2\times m_2}$. Assume $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$. Then Π has a J_{m_1,m_2}-spectral factorization.

Proof. The sign definite conditions on Π_{11} and Π_{22} can be used to show that Π has no equalizing vectors as defined in Meinsma (1995). Thus the Riccati Equation (B.1) has a unique stabilizing solution (Theorem 2.4 in Meinsma (1995)). Details are given in Seiler (2014). □