Gain Scheduling for Nonlinear Systems via Integral Quadratic Constraints

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Abstract— **The paper considers a general approach for gain scheduling of Lipschitz continuous nonlinear systems. The approach is based on a linear parameter varying system (LPV) representation of the nonlinear dynamics along with integral quadratic constraints (IQC) to account for the linearization errors. Past results have shown that Jacobian linearization leads to hidden coupling terms in the controlled system. These terms arise due to neglecting the higher order terms of the Taylor series and due to the use of constant (frozen) values of the scheduling parameter. This paper proposes an LPV control synthesis method that accounts for these shortcomings. The higher order terms of the linearization are treated as a memoryless uncertainty whose input/output behavior is described by a parameter varying IQC. It is also shown that if the rate of the scheduling parameter is measurable then it can be treated as a known disturbance in the control synthesis step. A simple numerical example shows that the proposed control design approach leads to improved control performance.**

I. INTRODUCTION

Gain scheduling is a common approach to nonlinear control design [1], [2], [3], [4]. The starting point for gain scheduling design is an LPV model of the nonlinear plant generally obtained by Jacobian linearization about a family of equilibrium (*trim*) points as given in Section II. A linear controller is designed at each trim point of the plant ensuring that the performance criteria are met locally. The nonlinear controller is constructed by interpolating between the linear controllers based on the scheduling parameter. Two main directions exist for LPV system representation, linear fractional transformation (LFT) based LPV systems [4], [5], [6] and "grid-based" LPV systems [7], [8]. The former requires rational dependence on the parameters, but leads to more computationally tractable linear matrix inequality (LMI) conditions while the latter offers arbitrary dependence on the parameter. The paper follows the grid-based approach, but the results may be extended to LFT type LPV systems.

The main advantage of gain scheduling is that it applies well developed linear design tools to nonlinear problems. The induced L_2 control design approach is given in Section III-A. On the other hand, a major limitation of gain scheduling is that the closed-loop system fulfills the stability and performance criteria only in the vicinity of the trim points. It was shown in [2], [9], [10], [11], [12], [13], [14] that hidden coupling terms can appear in the closed loop due to neglecting the higher order terms of the Taylor series in the linearization and due to variation in the scheduling parameter.

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The aim of the paper is to propose a control synthesis method that accounts for these shortcomings of gain scheduling. The paper considers Lipschitz-continuous nonlinear systems. The higher order terms of the linearization of such systems can be treated as a memoryless uncertainty whose input/output signals are described by a parameter varying IQC. IQCs provide a general framework for robustness analysis [15], where the interconnection of a linear system and a perturbation is considered and the input/output behavior of the perturbation is bounded by an IQC in the frequency domain (Section III-B). The IQC framework is extended to the time domain based on the dissipation inequality in [6], [16] and parameter varying IQCs are introduced in [17], [18]. Hidden couplings arise in the linearization process due to the time variation of the scheduling parameter. This variation can be treated as a disturbance in the design (synthesis) model. In addition, the LPV controller can explicitly depend on the parameter rate of variation if it is measurable [7], [8]. This offers guarantees on stability and performance in the case of time-varying scheduling parameter(s). The proposed control design and a numerical example are given in Sections IV–V.

II. PROBLEM FORMULATION

A. Assumptions

Consider the following nonlinear system G:

$$\begin{aligned} \dot{x}(t) &= f(x(t), d(t), u(t), \rho(t)) \\ e(t) &= h_1(x(t), d(t), u(t), \rho(t)) \\ y(t) &= h_2(x(t), d(t), u(t), \rho(t)) \end{aligned} \tag{1}$$

where f, h_1 and h_2 are differentiable functions. The signals are input $u(t) \in \mathbb{R}^{n_u}$, disturbance $d(t) \in \mathbb{R}^{n_d}$, measured output $y(t) \in \mathbb{R}^{n_y}$, performance output $e(t) \in \mathbb{R}^{n_e}$ and state variable $x(t) \in \mathbb{R}^{n_x}$. Finally, $\rho(t) \in \mathbb{R}^{n_\rho}$ is a measurable exogenous parameter vector, called the scheduling parameter. ρ is assumed to be a continuously differentiable function and the admissible trajectories are restricted based on physical considerations to a known compact subset $\mathcal{P} \subset \mathbb{R}^{n_\rho}$. The rates of the parameter variation $\dot{\rho}$ are assumed to be bounded in some applications. The present paper investigates the unbounded rate case for simplicity. The results carry over to the rate bounded case with a more complex notation. The dependence on time t is suppressed to shorten the notation.

Assumption 1: f, h_1 and h_2 are Lipschitz-continuous:

$$\begin{aligned} \|f(\alpha_1) - f(\alpha_2)\| &\leq L_f \|\alpha_1 - \alpha_2\| \quad \forall \alpha_1, \alpha_2 \in \operatorname{dom} f \\ \|h_1(\alpha_1) - h_1(\alpha_2)\| &\leq L_{h_1} \|\alpha_1 - \alpha_2\| \quad \forall \alpha_1, \alpha_2 \in \operatorname{dom} h_1 \\ \|h_2(\alpha_1) - h_2(\alpha_2)\| &\leq L_{h_2} \|\alpha_1 - \alpha_2\| \quad \forall \alpha_1, \alpha_2 \in \operatorname{dom} h_2 \end{aligned} \tag{2}$$

where $L_f, L_{h_1}, L_{h_2} \in \mathbb{R}_0^+$ are the Lipschitz constants for f, h_1 and h_2, respectively.

Assumption 2: There is a family of equilibrium points $(\bar{x}(\rho), \bar{d}(\rho), \bar{u}(\rho))$ such that

$$f(\bar{x}(\rho), \bar{d}(\rho), \bar{u}(\rho), \rho) = 0, \quad \forall \rho \in \mathcal{P} \tag{3}$$

The parameterized trim outputs are defined as

$$\begin{aligned} \bar{e}(\rho) &= h_1(\bar{x}(\rho), \bar{d}(\rho), \bar{u}(\rho), \rho), \quad \forall \rho \in \mathcal{P} \\ \bar{y}(\rho) &= h_2(\bar{x}(\rho), \bar{d}(\rho), \bar{u}(\rho), \rho), \quad \forall \rho \in \mathcal{P} \end{aligned} \tag{4}$$

The general control objective is to ensure that x tracks $\bar{x}(\rho)$. Note that ρ specifies the desired operating point and is effectively a reference command.

B. Jacobian Linearization of Nonlinear Systems

The nonlinear system G given by (1) can be linearized about the equilibrium points via Jacobian linearization based on Taylor series expansion. Define the deviation variables as

$$\begin{aligned} &\delta_x := x - \bar{x}(\rho),\ \delta_u := u - \bar{u}(\rho),\ \delta_e := e - \bar{e}(\rho) \\ &\delta_y := y - \bar{y}(\rho),\ \delta_d := d - \bar{d}(\rho) \end{aligned} \tag{5}$$

Differentiating the δ_x term of (5) results in

$$\dot{\delta}_x = \dot{x} - \dot{\bar{x}}(\rho) = f(x, d, u, \rho) - \dot{\bar{x}}(\rho) \tag{6}$$

The Taylor series expansion of f, h_1 and h_2 about the equilibrium point yields

$$\begin{aligned} \dot{\delta}_x &= \nabla_x f|_0\, \delta_x + \nabla_d f|_0\, \delta_d + \nabla_u f|_0\, \delta_u + \epsilon_f(\delta_x, \delta_d, \delta_u, \rho) - \dot{\bar{x}}(\rho) \\ \delta_e &= \nabla_x h_1|_0\, \delta_x + \nabla_d h_1|_0\, \delta_d + \nabla_u h_1|_0\, \delta_u + \epsilon_{h_1}(\delta_x, \delta_d, \delta_u, \rho) \\ \delta_y &= \nabla_x h_2|_0\, \delta_x + \nabla_d h_2|_0\, \delta_d + \nabla_u h_2|_0\, \delta_u + \epsilon_{h_2}(\delta_x, \delta_d, \delta_u, \rho) \end{aligned} \tag{7}$$

where the $|_0$ denotes evaluation at the trim point $(\bar{x}(\rho), \bar{d}(\rho), \bar{u}(\rho), \rho)$. Terms ϵ_f, ϵ_{h_1} and ϵ_{h_2} represent the higher order terms of the Taylor series expansion. The term $\dot{\bar{x}}(\rho)$ arises due to the time variation in ρ. The linearization is performed with respect to (x, d, u) but the nonlinear dependence on ρ is retained. Define $L(\rho) := -\nabla \bar{x}(\rho)$. The linearization about the family of trim points becomes

$$\begin{aligned} \dot{\delta}_x =& A(\rho)\delta_x + B_d(\rho)\delta_d + B_u(\rho)\delta_u + L(\rho)\dot{\rho} + \epsilon_f(\delta_x, \delta_d, \delta_u, \rho) \\ \delta_e =& C_e(\rho)\delta_x + D_{ed}(\rho)\delta_d + D_{eu}(\rho)\delta_u + \epsilon_h(\delta_x, \delta_d, \delta_u, \rho) \\ \delta_y =& C_y(\rho)\delta_x + D_{yd}(\rho)\delta_d + D_{yu}(\rho)\delta_u + \epsilon_h(\delta_x, \delta_d, \delta_u, \rho) \end{aligned} \tag{8}$$

where the parameter-dependent state matrices are given by the gradients appearing in (7), e.g. $A(\rho) := \nabla_x f|_0$. The LPV system is commonly obtained by assuming that $\epsilon_f, \epsilon_{h_1}, \epsilon_{h_2} \approx 0$. In addition, it is typically assumed that the parameter variation is sufficiently slow, thus $\dot{\rho} \approx 0$. Under these assumptions, the LPV system G_ρ is given by

$$\begin{aligned} \dot{\delta}_x =& A(\rho)\delta_x + B_d(\rho)\delta_d + B_u(\rho)\delta_u \\ \delta_e =& C_e(\rho)\delta_x + D_{ed}(\rho)\delta_d + D_{eu}(\rho)\delta_u \\ \delta_y =& C_y(\rho)\delta_x + D_{yd}(\rho)\delta_d + D_{yu}(\rho)\delta_u \end{aligned} \tag{9}$$

The goal of the paper is to propose an LPV control synthesis method, which addresses these shortcomings of the Jacobian linearization. The terms ϵ_f, ϵ_{h_1} and ϵ_{h_2} are treated as a memoryless uncertainty satisfying a parameter varying IQC. The term $L(\rho)\dot{\rho}$ is treated as a disturbance in the design (synthesis) model.

III. BACKGROUND

This section reviews existing material on LPV systems and IQCs.

A. Induced L_2 Control of LPV Systems

Consider an LPV system G_ρ, obtained via Jacobian linearization of the nonlinear system G,

$$\begin{bmatrix} \dot{x} \\ e \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_d(\rho) & B_u(\rho) \\ C_e(\rho) & D_{ed}(\rho) & D_{eu}(\rho) \\ C_y(\rho) & D_{yd}(\rho) & D_{yu}(\rho) \end{bmatrix} \begin{bmatrix} x \\ d \\ u \end{bmatrix} \tag{10}$$

The δ notation that appears in (9) for the (linearized) deviation variables is dropped here in order to simplify the notation. Let K_ρ be an LPV controller of the form:

$$\begin{bmatrix} \dot{x}_K \\ u \end{bmatrix} = \begin{bmatrix} A_K(\rho) & B_K(\rho) \\ C_K(\rho) & D_K(\rho) \end{bmatrix} \begin{bmatrix} x_K \\ y \end{bmatrix} \tag{11}$$

The controller K_ρ generates the control input u with a linear dependence on the measurement y but an arbitrary dependence on the scheduling parameter ρ. A lower LFT $\mathcal{F}_l(G_\rho, K_\rho)$ defines the closed-loop interconnection of G_ρ and K_ρ (see Fig. 1.a). The performance of $\mathcal{F}_l(G_\rho, K_\rho)$ can be specified in terms of the induced L_2 gain from d to e over all allowable parameter trajectories as

$$\|\mathcal{F}_l(G_\rho, K_\rho)\| = \sup_{d \neq 0, d \in L_2, \rho \in \mathcal{P}, x_{cl}(0)=0} \frac{\|e\|}{\|d\|} \tag{12}$$

where x_{cl} denotes the closed loop state variables. The objective is to synthesize a controller K_ρ to minimize the closed-loop induced L_2 gain from d to e. The following theorem gives the sufficient condition to upper bound the induced L_2 gain of $\mathcal{F}_l(G_\rho, K_\rho)$.

Theorem 1: ([7], [8]): The interconnection $\mathcal{F}_l(G_\rho, K_\rho)$ is exponentially stable and $\|\mathcal{F}_l(G_\rho, K_\rho)\| \leq \gamma$ if there exists a matrix $P = P^T \in \mathbb{R}^{n_{x_{cl}} \times n_{x_{cl}}}$ such that $P \geq 0$ and $\forall \rho \in \mathcal{P}$

$$\begin{bmatrix} PA_{cl} + A_{cl}^T P & PB_{cl} \\ B_{cl}^T P & -I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C_{cl}^T \\ D_{cl}^T \end{bmatrix} \begin{bmatrix} C_{cl} & D_{cl} \end{bmatrix} < 0 \tag{13}$$

where subscript cl stands for closed loop. The dependence of the state matrices on ρ has been omitted in (13).

Proof: The proof is based on a dissipation inequality satisfied by the storage function $V : \mathbb{R}^{n_{x_{cl}} \times n_{x_{cl}}} \to \mathbb{R}^+$ given as $V(x_{cl}) := {x_{cl}}^T P x_{cl}$. Multiplying (13) on the left/right by $[{x_{cl}}^T, d^T]$ and $[{x_{cl}}^T, d^T]^T$ gives

$$\dot{V} \leq d^T d - \gamma^{-2} e^T e \tag{14}$$

The dissipation inequality (14) can be integrated with the initial condition $x_{cl}(0) = 0$, which yields $\|e\| \leq \gamma \|d\|$. ■

This analysis theorem forms the basis for the induced L_2 norm controller synthesis of [7], [8], achieved by solving bounded-real type LMI conditions that are sufficient to upper bound the gain of an LPV system. The LMI conditions and the controller reconstruction steps are given in [7], [8].

B. Robustness Analysis of LPV Systems via Integral Quadratic Constraints

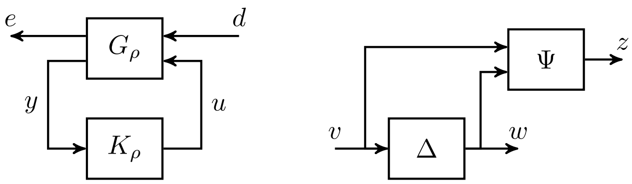
IQCs provide a framework for robustness analysis [15]. The IQC specifies constraint on the input/outputs signals of the perturbation.

Definition 1: Let M be a symmetric matrix, i.e. $M = M^T \in \mathbb{R}^{n_z \times n_z}$ and Ψ a stable linear system, i.e. $\Psi \in \mathbb{RH}^{n_z \times (n_v + n_w)}$. Operator $\Delta : L_{2e}^{n_v} \to L_{2e}^{n_w}$ satisfies IQC defined by (M, Ψ) if the following inequality holds for all $v \in L_{2e}^{n_v}[0, \infty)$, $w = \Delta(v)$ and $T \geq 0$:

$$\int_0^T z^T M z dt \geq 0 \tag{15}$$

where z is the output of the linear system Ψ with inputs (v, w) and zero initial conditions.

The notation $\Delta \in IQC(\Psi, M)$ is applied if Δ satisfies IQC defined by (Ψ, M). Fig. 1.b shows a graphic interpretation of the IQC, where the input and output signals of Δ are filtered through Ψ. There is a wide class of IQCs available for



(a) LFT for LPV synthesis (b) Graphic interpretation of IQCs

Fig. 1. Graphic interconnection for LPV synthesis and IQCs

various uncertainties or nonlinearities. The remainder of the section focuses on IQCs for a memoryless operator Δ based on time varying sector bounds. The input/output behavior of Δ is described by $w = \Delta(v, \rho)$ where signals v and w are assumed to satisfy the following condition:

$$v^T S^T(\rho) S(\rho) v - w^T w \geq 0, \forall v \in \mathbb{R}^{n_v},\ w \in \mathbb{R}^{n_w},\ \rho \in \mathcal{P} \quad (16)$$

where $S(\rho)$ is a parameter dependent diagonal matrix that scales signal v. The uncertainty Δ therefore satisfies the quadratic constraint (QC)

$$\begin{bmatrix} v \\ w \end{bmatrix}^T M(\rho) \begin{bmatrix} v \\ w \end{bmatrix} \geq 0, \quad \forall v \in \mathbb{R}^{n_v}, w \in \mathbb{R}^{n_w},\ \rho \in \mathcal{P} \qquad (17)$$

where $M(\rho)$ is defined as

$$M(\rho) := \begin{bmatrix} S(\rho)^T S(\rho) I_{n_v} & 0 \\ 0 & -I_{n_w} \end{bmatrix} \qquad (18)$$

Selecting $\Psi = I_{n_v + n_w}$, therefore $z = \begin{bmatrix} v^T & w^T \end{bmatrix}^T$, and integrating (17) implies $\Delta \in IQC(I, M(\rho))$.

The uncertain LPV system denoted by upper LFT as $\mathcal{F}_u(H_\rho, \Delta)$ is defined by the interconnection of an LPV system H_ρ and uncertainty Δ. H_ρ is defined as

$$\begin{bmatrix} \dot{x} \\ v \\ e \end{bmatrix} = \begin{bmatrix} A(\rho) & B_w(\rho) & B_d(\rho) \\ C_v(\rho) & D_{vw}(\rho) & D_{vd}(\rho) \\ C_e(\rho) & D_{ew}(\rho) & D_{ed}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ d \end{bmatrix} \qquad (19)$$

The worst-case L_2 gain of $\mathcal{F}_u(H_\rho, \Delta)$ can be defined as

$$\gamma := \sup_{\Delta \in \mathrm{IQC}(I, M(\rho)), \rho \in \mathcal{P}} \|\mathcal{F}_u(H_\rho, \Delta)\| \qquad (20)$$

An upper bound to the worst-case L_2 gain γ can be defined as a dissipation inequality based on equations (17) and (19) in the form of an LMI [17], [18].

Theorem 2: Assume $\mathcal{F}_u(H_\rho, \Delta)$ is well posed for all $\Delta \in$ IQC(I,$M(\rho)$). Then $\|\mathcal{F}_u(H_\rho, \Delta)\| \leq \gamma$ if there exists matrix $P = P^T \in \mathbb{R}^{n_x \times n_x}$ and a scalar $\lambda \geq 0$ such that $P \geq 0$ and $\forall \rho \in \mathcal{P}$

$$\begin{bmatrix} PA + A^T P & PB_w & PB_d \\ B_w^T P & 0 & 0 \\ B_d^T P & 0 & -I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C_e^T \\ D_{ew}^T \\ D_{ed}^T \end{bmatrix} \begin{bmatrix} C_e & D_{ew} & D_{ed} \end{bmatrix}$$
$$+ \lambda \begin{bmatrix} C_v^T & 0 \\ D_{vw}^T & I \\ D_{vd}^T & 0 \end{bmatrix} M \begin{bmatrix} C_v & D_{vw} & D_{vd} \\ 0 & I & 0 \end{bmatrix} < 0 \qquad (21)$$

The dependence on ρ has been omitted in (21).

Proof: The proof is based on defining the storage function $V : \mathbb{R}^{n_x \times n_x} \to \mathbb{R}^+$ by $V(x) := x^T P x$. Left and right multiply (21) by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ to show that V satisfies the dissipation inequality:

$$\lambda \begin{bmatrix} v \\ w \end{bmatrix}^T M \begin{bmatrix} v \\ w \end{bmatrix} + \dot{V} \leq \gamma^2 d^T d - e^T e \qquad (22)$$

The dissipation inequality (22) can be integrated from $t = 0$ to $t = T$ with the initial condition $x(0) = 0$. The QC condition (17) along with $\lambda \geq 0$ and $P \geq 0$ imply $\|e\| \leq \gamma \|d\|$. Details of the proof are given in [16], [18]. ■

The results of the section can be considered as a specific case of parameter varying IQCs, where uncertainty Δ satisfies a more strict QC. The theory of IQC is more general in principle [6], [15], [16], [17], which can contain dynamic, parameter varying filters and integral constraints.

IV. THE PROPOSED CONTROL DESIGN

Consider the LPV system G_ρ given by (9), obtained by Jacobian linearization of G in (1). G_ρ is an approximation of G since terms ϵ_f, ϵ_{h_1}, ϵ_{h_2} and $L(\rho)\dot{\rho}$ are considered negligible in the linearization step. The aim of this section is to propose an LPV control design method for G_ρ based on [7], [8] that accounts for these neglected terms. These terms can be formulated as perturbations to system G_ρ and can be sorted into two groups.

The goal is to treat the higher order terms ϵ_f, ϵ_{h_1} and ϵ_{h_2} as a memoryless uncertainty Δ whose input/output signals satisfy a QC. Uncertainty Δ can be derived based on Assumption 1. Interconnection $\mathcal{F}_u(G_\rho, \Delta)$ allows IQC-based robustness analysis. Additionally, the aim is apply scalings to G_ρ and Δ such that LMI (21) becomes equivalent to LMI (13) for the resulting interconnection. Therefore, the LPV control synthesis of [7], [8] accounts for terms ϵ_f, ϵ_{h_1} and ϵ_{h_2} of the interconnection. The term $L(\rho)\dot{\rho}$ can be treated as an additional disturbance or it can be incorporated as an input to the controller in the LPV design in case $\dot{\rho}$ is measurable. By accounting for these terms, the proposed control synthesis method gives an upper bound for the induced L_2 gain from input d to output e for interconnection of the resulting LPV controller and the *original nonlinear system G*.

A. Quadratic Constraints for the Higher Order Terms of Taylor Series Expansion

The first goal of this section is to derive uncertainty Δ that satisfies a QC and captures the terms ϵ_f, ϵ_{h_1} and ϵ_{h_2}. The second aim is to apply scalings to G_ρ and Δ in order to bring LMI (21) to the form of LMI (13) for the resulting interconnection. This can be achieved by two scalings. The first scaling accounts for the parameter dependent $M(\rho)$ of (17). $M(\rho)$ is transformed to an identity matrix via parameter dependent scalings. The second scaling accounts for optimizing over λ by keeping $\lambda = 1$ in LMI (21). The optimal value of λ can be found by evaluating analysis over a gridded domain of λ.

Lemma 1: Let the nonlinear system G of (1) fulfill Assumptions 1 and 2. Then terms ϵ_f, ϵ_{h_1} and ϵ_{h_2} of (7) are also Lipschitz-continuous. The behavior of these terms can be captured by a memoryless uncertainty Δ. The input/output signals of uncertainty Δ satisfy a QC.

Proof: Consider the first element of ϵ_f, denoted by ϵ_{f_1}, which can be expressed based on (8) as

$$\begin{aligned}&\epsilon_{f_1}(\delta_x,\delta_d,\delta_u,\rho) = f_1(x,d,u,\rho) - f_1(\bar{x},\bar{d},\bar{u},\rho) - A_1(\rho)\delta_x \\ &- B_{d_1}(\rho)\delta_d - B_{u_1}(\rho)\delta_u = f_1(\bar{x}+\delta_x,\bar{d}+\delta_d,\bar{u}+\delta_u,\rho) \\ &- f_1(\bar{x},\bar{d},\bar{u},\rho) - A_1(\rho)\delta_x - B_{d_1}(\rho)\delta_d - B_{u_1}(\rho)\delta_u\end{aligned} \tag{23}$$

where $A_1(\rho)$, $B_{d_1}(\rho)$ and $B_{u_1}(\rho)$ denote the first rows of matrices $A(\rho)$, $B_d(\rho)$ and $B_u(\rho)$ respectively. The norm of (23) satisfies the following inequality

$$\begin{aligned}&\|\epsilon_{f_1}(\delta_x,\delta_d,\delta_u,\rho)\| \leq \\ &\left\|f_1(\bar{x}+\delta_x,\bar{d}+\delta_d,\bar{u}+\delta_u,\rho) - f_1(\bar{x},\bar{d},\bar{u},\rho)\right\| \\ &+ \|A_1(\rho)\delta_x\| \, \|B_{d_1}(\rho)\delta_d\| + \|B_{u_1}(\rho)\delta_u\| \\ &\leq \left\|f_1(\bar{x}+\delta_x,\bar{d}+\delta_d,\bar{u}+\delta_u,\rho) - f_1(\bar{x},\bar{d},\bar{u},\rho)\right\| \\ &+ \|A_1(\rho)\| \, \|\delta_x\| + \|B_{d_1}(\rho)\| \, \|\delta_d\| + \|B_{u_1}(\rho)\| \, \|\delta_u\|\end{aligned} \tag{24}$$

Substituting Lipschitz condition (2) into (24) results in

$$\begin{aligned}&\|\epsilon_{f_1}(\delta_x,\delta_d,\delta_u,\rho)\| \leq L_{f_1}(\rho)\left\|\begin{bmatrix}\delta_x^T & \delta_d^T & \delta_u^T\end{bmatrix}^T\right\| \\ &+ \|A_1(\rho)\| \, \|\delta_x\| + \|B_{d_1}(\rho)\| \, \|\delta_d\| + \|B_{u_1}(\rho)\| \, \|\delta_u\|\end{aligned} \tag{25}$$

The following inequality holds based on the Euclidean norm

$$\begin{aligned}&\|\epsilon_{f_1}(\delta_x,\delta_d,\delta_u,\rho)\|^2 \leq ((L_{f_1}(\rho) + \|A_1(\rho)\|) \, \|\delta_x\| \\ &(L_{f_1}(\rho) + \|B_{d_1}(\rho)\|) \, \|\delta_d\| + (L_{f_1}(\rho) + \|B_{u_1}(\rho)\|) \, \|\delta_u\|)^2\end{aligned} \tag{26}$$

Applying Jensen's inequality to (26) leads to

$$\begin{aligned}&\|\epsilon_{f_1}(\delta_x,\delta_d,\delta_u,\rho)\|^2 \leq 3((L_{f_1}(\rho) + \|A_1(\rho)\|) \, \|\delta_x\|)^2 \\ &+ 3((L_{f_1}(\rho) + \|B_{d_1}(\rho)\|) \, \|\delta_d\|)^2 \\ &+ 3((L_{f_1}(\rho) + \|B_{u_1}(\rho)\|) \, \|\delta_u\|)^2 \\ &= a^2_{x\epsilon_{f_1}}(\rho) \, \|\delta_x\|^2 + a^2_{d\epsilon_{f_1}}(\rho) \, \|\delta_d\|^2 + a^2_{u\epsilon_{f_1}}(\rho) \, \|\delta_u\|^2\end{aligned} \tag{27}$$

Finally, the following inequality can be obtained for ϵ_{f_1}

$$\begin{aligned}&\epsilon_{f_1}(\delta_x,\delta_d,\delta_u,\rho)^2 \\ &\leq a^2_{x\epsilon_{f_1}}(\rho)\delta_x^T\delta_x + a^2_{d\epsilon_{f_1}}(\rho)\delta_d^T\delta_d + a^2_{u\epsilon_{f_1}}(\rho)\delta_u^T\delta_u\end{aligned} \tag{28}$$

Let signals $w_{\epsilon_{f_1}}$ and $v_{\epsilon_{f_1}}$ be defined as $w_{\epsilon_{f_1}} := \epsilon_{f_1}$ and $v_{\epsilon_{f_1}} := \begin{bmatrix}\delta_x^T & \delta_d^T & \delta_u^T\end{bmatrix}^T$. Then the following QC holds

$$\begin{gathered}\begin{bmatrix}v_{\epsilon_{f_1}} \\ w_{\epsilon_{f_1}}\end{bmatrix}^T M_{\epsilon_{f_1}}(\rho)\begin{bmatrix}v_{\epsilon_{f_1}} \\ w_{\epsilon_{f_1}}\end{bmatrix} \geq 0 \\ \forall v_{\epsilon_{f_1}} \in \mathbb{R}^{n_{v\epsilon_{f_1}}}, \, w_{\epsilon_{f_1}} \in \mathbb{R}^{n_{w\epsilon_{f_1}}}, \, \rho \in \mathcal{P}\end{gathered} \tag{29}$$

where $M_{\epsilon_{f_1}}(\rho)$ is given as

$$M_{\epsilon_{f_1}}(\rho) := \begin{bmatrix}a^2_{x\epsilon_{f_1}}(\rho)I_{n_x} & 0 & 0 & 0 \\ 0 & a^2_{d\epsilon_{f_1}}(\rho)I_{n_d} & 0 & 0 \\ 0 & 0 & a^2_{u\epsilon_{f_1}}(\rho)I_{n_u} & 0 \\ 0 & 0 & 0 & -1\end{bmatrix} \tag{30}$$

A memoryless uncertainty $\Delta_{\epsilon_{f_1}}$ can be constructed whose input/output behavior is given by $w_{\epsilon_{f_1}} = \Delta_{\epsilon_{f_1}}(v_{\epsilon_{f_1}},\rho)$ and $\Delta_{\epsilon_{f_1}} \in QC(I, M_{\epsilon_{f_1}}(\rho))$. QCs can be constructed for each term of ϵ_f, ϵ_{h_1} and ϵ_{h_2} in the same manner. A block diagonal Δ can then be constructed as

$$\Delta = \begin{bmatrix}\Delta_{\epsilon_{f_1}} & & \\ & \ddots & \\ & & \Delta_{\epsilon_{h_{ny}}}\end{bmatrix} \tag{31}$$

where input/output behavior of $\Delta \in QC(I, M(\rho))$ is given by $w = \Delta(v,\rho)$ where $v = \begin{bmatrix}v^T_{\epsilon_{f_1}} \dots v^T_{\epsilon_{h_{2ny}}}\end{bmatrix}^T$ and $w = \begin{bmatrix}w_{\epsilon_{f_1}} \dots w_{\epsilon_{h_{2ny}}}\end{bmatrix}^T$. ■

Note that using a single IQC to cover all the Taylor series linearization errors can be very conservative. In practice it is possible to exploit the structure of the problem and bound each term individually. This would give (possibly many) IQCs each with their own scaling variable. Efficient methods to implement/solve this could be explored as future work.

The LPV system $\check{G}_\rho$ can be obtained by extending G_ρ with signals v and w as

$$\begin{bmatrix}\dot{x} \\ v \\ e \\ y\end{bmatrix} = \begin{bmatrix}A(\rho) & B_w(\rho) & B_d(\rho) & B_u(\rho) \\ C_v(\rho) & D_{vw}(\rho) & D_{vd}(\rho) & D_{vu}(\rho) \\ C_e(\rho) & D_{ew}(\rho) & D_{ed}(\rho) & D_{eu}(\rho) \\ C_y(\rho) & D_{yw}(\rho) & D_{yd}(\rho) & D_{yu}(\rho)\end{bmatrix}\begin{bmatrix}x \\ w \\ d \\ u\end{bmatrix} \tag{32}$$

Given LPV controller K_ρ, IQC-based robust stability analysis can be done for the interconnection $\mathcal{F}_u(\mathcal{F}_l(\check{G}_\rho, K_\rho), \Delta)$. The aim is however, to propose an LPV control design method that accounts for the effect of Δ already in the design step. For this the following two scalings are applied to $\check{G}_\rho$ and Δ as depicted in Fig. 2.

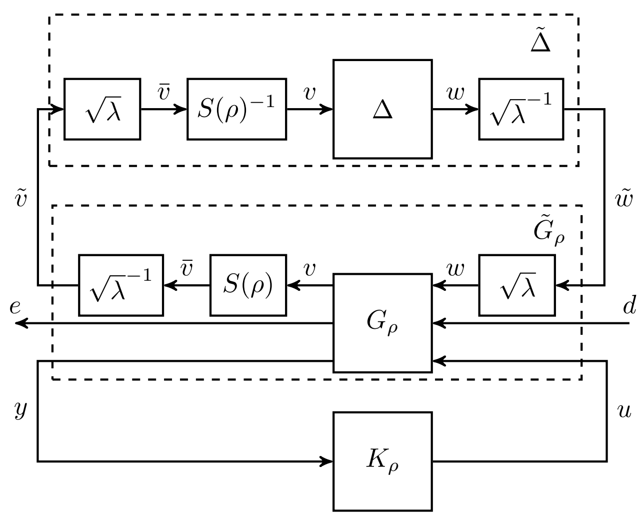


Fig. 2. Scaled system

The first scaling accounts for the term $M(\rho)$ of LMI (21). Signal v can be scaled in the following way

$$\bar{v} = S(\rho)v \tag{33}$$

where $S(\rho)$ is defined as

$$S(\rho) := \begin{bmatrix}a_{x\epsilon_{f_1}}(\rho)I_{n_x} & & \\ & \ddots & \\ & & a_{\epsilon_{h_{2ny}}}(\rho)I_{n_y}\end{bmatrix} \tag{34}$$

$\bar{\Delta}$ can be constructed as $w = \bar{\Delta}(\check{v},\rho)$. The input/output behavior of $\bar{\Delta}$ satisfies

$$\bar{v}^T\bar{v} - w^Tw \geq 0, \quad \forall \bar{v} \in \mathbb{R}^{n_{\bar{v}}}, \, w \in \mathbb{R}^{n_w} \tag{35}$$

The uncertainty $\bar{\Delta}$ therefore satisfies the QC

$$\begin{bmatrix}\bar{v} \\ w\end{bmatrix}^T \bar{M}\begin{bmatrix}\bar{v} \\ w\end{bmatrix} \geq 0, \quad \forall \bar{v} \in \mathbb{R}^{n_{\bar{v}}}, w \in \mathbb{R}^{n_w} \tag{36}$$

where $\bar{M}$ is defined as

$$\bar{M} := \begin{bmatrix} I_{n_{\bar{v}}} & 0 \\ 0 & -I_{n_w} \end{bmatrix} \tag{37}$$

The second scaling accounts for the optimization over λ. The aim is to pull λ out from LMI (21) and treat it as scalings to $\bar{G}_\rho$ and $\bar{\Delta}$. This can be achieved by scaling $\bar{G}_\rho$ and $\bar{\Delta}$ as

$$\begin{aligned} \tilde{G}_\rho &= \begin{bmatrix} \sqrt{\lambda}^{-1} I_{n_{\bar{v}}} & 0 \\ 0 & I_{n_e+n_y} \end{bmatrix} \bar{G}_\rho \begin{bmatrix} \sqrt{\lambda} I_{n_{\bar{w}}} & 0 \\ 0 & I_{n_d+n_u} \end{bmatrix} \\ \tilde{\Delta} &= \sqrt{\lambda}\bar{\Delta}\sqrt{\lambda}^{-1} \end{aligned} \tag{38}$$

B. Rate of Variation of ρ as an External Disturbance

The aim of this section is to include the effect of $L(\rho)\dot{\rho}$ into $\tilde{G}_\rho$. $\dot{\rho}$ can be treated as an additional disturbance signal. Therefore, signal d can be extended as $\hat{d} = \begin{bmatrix} d^T & \dot{\rho}^T \end{bmatrix}^T$. $\hat{G}_\rho$ can be derived from $\tilde{G}_\rho$ by extending the input matrices

$$\begin{aligned} \hat{B}_d(\rho) &= \begin{bmatrix} B_d(\rho) & L(\rho) \end{bmatrix}, \hat{D}_{vd}(\rho) = \begin{bmatrix} D_{vd}(\rho) & 0 \end{bmatrix}, \\ \hat{D}_{ed}(\rho) &= \begin{bmatrix} D_{ed}(\rho) & 0 \end{bmatrix}, \hat{D}_{yd}(\rho) = \begin{bmatrix} D_{yd}(\rho) & 0 \end{bmatrix} \end{aligned} \tag{39}$$

$\hat{G}_\rho$ and $\tilde{\Delta}$ form the basis of the following theorem.

Theorem 3: Let the interconnection of controller K_ρ in the form of (11) and the nonlinear system G of (1) be denoted by T. Assume that $\mathcal{F}_u(\hat{G}_\rho, \tilde{\Delta})$ is well posed for all $\tilde{\Delta} \in$ QC(I,$\bar{M}$). Then controller K_ρ can be designed such that $\|T\| \leq \gamma$ if there exists $\gamma \leq 1$, matrix $P = P^T \in \mathbb{R}^{n_{x_{cl}} \times n_{x_{cl}}}$ such that $P \geq 0$ and $\forall \rho \in \mathcal{P}$

$$\begin{bmatrix} P\hat{A}_{cl} + \hat{A}_{cl}^T P & P\hat{B}_{cl} \\ \hat{B}_{cl}^T P & -I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} \hat{C}_{cl}^T \\ \hat{D}_{cl}^T \end{bmatrix} \begin{bmatrix} \hat{C}_{cl} & \hat{D}_{cl} \end{bmatrix} < 0 \tag{40}$$

where $\hat{A}_{cl}$, $\hat{B}_{cl} = \begin{bmatrix} \hat{B}_{w_{cl}} & \hat{B}_{\hat{d}_{cl}} \end{bmatrix}$, $\hat{C}_{cl} = \begin{bmatrix} \hat{C}_{v_{cl}}^T & \hat{C}_{e_{cl}}^T \end{bmatrix}^T$ and $\hat{D}_{cl} = \begin{bmatrix} \hat{D}_{vw_{cl}} & \hat{D}_{v\hat{d}_{cl}} \\ \hat{D}_{ew_{cl}} & \hat{D}_{e\hat{d}_{cl}} \end{bmatrix}$ are the state matrices for the closed loop lower LFT $\mathcal{F}_l(\hat{G}_\rho, K_\rho)$. The dependence of the state matrices on ρ has been omitted in (40).

Proof: Note that LMI (40) can be derived from LMI (21). This comes from substituting (37) into (21) and applying $\lambda = 1$ based on (38). The third term of (21) can be multiplied by γ^{-2} without loss of generality. The proof is based on the dissipation inequality satisfied by the storage function $V : \mathbb{R}^{n_{x_{cl}} \times n_{x_{cl}}} \to \mathbb{R}^+$ as $V(x_{cl}) := {x_{cl}}^T P x_{cl}$. LMI (40) can be multiplied on the left/right by $\begin{bmatrix} {x_{cl}}^T & \tilde{w}^T & \hat{d}^T \end{bmatrix}$ and $\begin{bmatrix} {x_{cl}}^T & \tilde{w}^T & \hat{d}^T \end{bmatrix}^T$ to show that V satisfies the dissipation inequality:

$$\begin{aligned} &\dot{V} - \begin{bmatrix} \tilde{w}^T & \hat{d}^T \end{bmatrix} \begin{bmatrix} \tilde{w} \\ \hat{d} \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} \tilde{v}^T & e^T \end{bmatrix} \begin{bmatrix} \tilde{v} \\ e \end{bmatrix} = \\ &\dot{V} + \frac{1}{\gamma^2}\tilde{v}^T\tilde{v} - \tilde{w}^T\tilde{w} + \frac{1}{\gamma^2}e^T e - \hat{d}^T\hat{d} < 0 \end{aligned} \tag{41}$$

Integrating the dissipation inequality and applying condition (35) and (38) results in $\|e\| \leq \gamma \left\|\hat{d}\right\|$ for $\mathcal{F}_u(\mathcal{F}_l(\hat{G}_\rho, K_\rho), \tilde{\Delta})$ in case $\gamma \leq 1$. The condition $\gamma \leq 1$ can be always achieved by scaling signal $\hat{d}$. $\|e\| \leq \gamma \left\|\hat{d}\right\|$ implies $\|e\| \leq \gamma \|d\|$ based on

$$\begin{aligned} &\left\|\mathcal{F}_u(\mathcal{F}_l(\hat{G}_\rho, K_\rho), \tilde{\Delta})\right\| = \sup_{\hat{d}\neq 0, \hat{d}\in L_2, \rho\in\mathcal{P}, x(0)=0} \frac{\|e\|}{\left\|\hat{d}\right\|} \geq \\ &\sup_{\hat{d}\neq 0, \dot{\rho}=0, \hat{d}\in L_2, \rho\in\mathcal{P}, x(0)=0} \frac{\|e\|}{\left\|\hat{d}\right\|} = \sup_{d\neq 0, d\in L_2, \rho\in\mathcal{P}, x(0)=0} \frac{\|e\|}{\|d\|} \end{aligned} \tag{42}$$

The inequality follows because $\sup_{\hat{d}\neq 0, \hat{d}\in L_2, \rho\in\mathcal{P}, x(0)=0} \frac{\|e\|}{\|\hat{d}\|}$ can only decrease if the constraint $\dot{\rho} = 0$ is added. $\hat{G}_\rho$ and $\tilde{\Delta}$ capture all the terms that are neglected in the Jacobian linearization of the nonlinear system G. Therefore, $\left\|\mathcal{F}_u(\mathcal{F}_l(\hat{G}_\rho, K_\rho), \tilde{\Delta})\right\| \leq \gamma$ implies $\|e\| \leq \gamma \|d\|$ for K_ρ interconnected with the nonlinear system G. ■

The form of dissipation inequality (41) implies a connection to nominal induced L_2 gain performance. Therefore, controller K_ρ can be designed based on [8], [7]. The optimal value of λ can be found by constructing $\hat{G}_\rho$ and $\tilde{\Delta}$ and evaluating LMI (40) over a gridded domain of λ. The conditions of Theorem 3 can be relaxed by applying Zames-Falb multipliers and/or using multiple IQCs for Δ and solving the synthesis problem as presented in [19].

V. EXAMPLE

A simple numerical example is presented to show the benefits of the proposed control design method. Consider the nonlinear system (similar to the example in [13]) given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ \dfrac{2}{1+e^{-2x_2}} + 1 \end{bmatrix}, \; y = x_2 \tag{43}$$

The aim is to design an output-feedback controller K_ρ, which ensures step response settling time of less than 2 seconds with zero steady state error. The scheduling parameter $\rho := x_2$ is restricted to the interval [-10,10] with a grid of 5 equidistant points. The trim points $(\bar{x}_1(\rho), \rho, \bar{u}(\rho))$ are

$$\bar{x}_1(\rho) = \bar{u}(\rho), \quad \bar{x}_1(\rho) = 1 - \frac{2}{1+e^{-2\rho}} \tag{44}$$

The LPV system G_ρ is obtained by Jacobian linearization of (43) about the trim points. It is assumed that $\dot{\rho}$ is measurable and can be incorporated as an input to the controller in the LPV design. Four control design cases are examined as given in Table I. The linearization error terms that are accounted for in the control design are denoted by ✓. The synthesis interconnection is shown in Fig. 3. Fig. 3.a gives the interconnection for **Cases 1–2** (term $\tilde{\Delta}$ is omitted in **Case 1**). Fig. 3.b depicts the interconnection for **Cases 3–4**. The

TABLE I

CONTROL DESIGN CASES

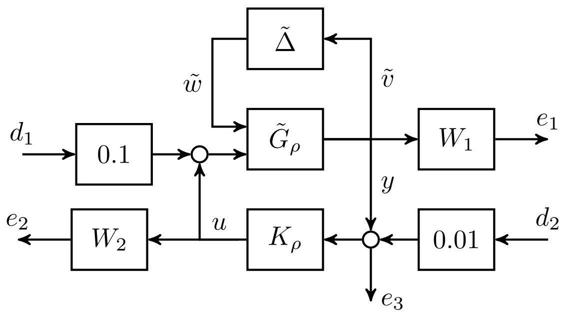
	Taylor series error	Time variation of ρ
Case 1	–	–
Case 2	✓	–
Case 3	–	✓
Case 4	✓	✓

tracking error e_1 is specified by weighting function W_1 and the control signal is penalized by the weighting function W_2, both with a bandwidth of $25\,rad/s$ as

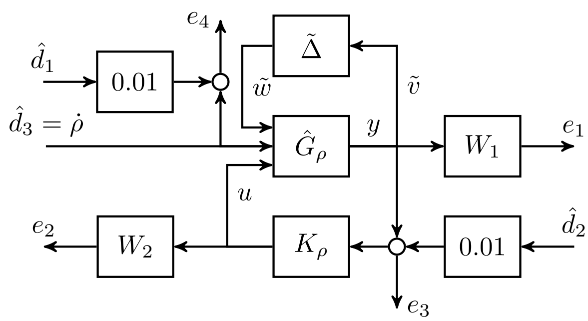
$$W_1(s) = \frac{0.33s + 23.69}{s + 2.369}, \; W_2(s) = \frac{0.0004s + 8.66^{-5}}{s + 43.3} \tag{45}$$

A robust LPV controller is designed for each case using the proposed method given in Section IV. All controllers achieve similar worst case gain of $\gamma \approx 0.12$. The effect of parameter λ on the worst case gain γ is shown in Fig. 4.a.

The responses of the four controllers interconnected with the nonlinear system are depicted in Fig. 4.b, which shows that **Cases 2–4** clearly outperform the nominal control design.

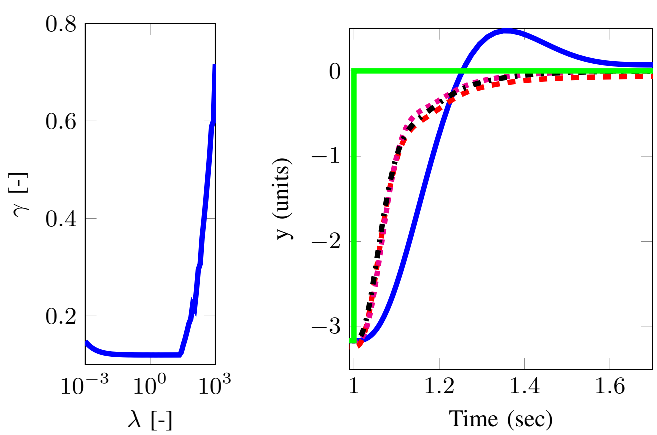


(a) **Case 1** and **Case 2**



(b) **Case 3** and **Case 4**

Fig. 3. Synthesis interconnections



(a) Results of parameter λ (b) Reference tracking performance

Fig. 4. Results of parameter λ and reference tracking performance: Case 1 ( ); Case 2 ( ); Case 3 ( ); Case 4 ( ); reference( )

VI. CONCLUSIONS

The paper proposed an approach for gain scheduling of Lipschitz continuous nonlinear systems based on LPV system representation along with parameter varying IQC with the aim to account for the Jacobian linearization errors. These errors lead to hidden coupling terms in the controlled system. The higher order terms of the Taylor series expansion are treated as a memoryless uncertainty whose input/output behavior is described by a parameter varying IQC. The effect of the time variation of the scheduling parameter is captured by an additional disturbance. The resulting control synthesis gives guarantees for the interconnection of the nonlinear system and the resulting LPV controller. The benefits of the proposed method are shown by a simple numerical example. Future work will consider extending the results, with some restrictions, to non Lipschitz continuous nonlinear systems.

ACKNOWLEDGMENT

This work was supported by the National Science Foundation under Grant No. NSF-CMMI-1254129 entitled ”CAREER: Probabilistic Tools for High Reliability Monitoring and Control of Wind Farms.” This work was also supported by the University of Minnesota Institute on the Environment, IREE Grants No. RS-0039-09 and RL-0011-13.

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