Robust LPV Estimator Synthesis Using Integral Quadratic Constraints

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***Abstract*— A method is presented for synthesizing output estimators for a class of continuous time, uncertain, linear parameter-varying (LPV) systems. The uncertain system is described as an interconnection of a nominal LPV system and a block structured perturbation. The nominal LPV system is "gridded" over the space of parameters, with the state matrices being arbitrary functions of the parameters. The input/output behavior of the perturbation is described by integral quadratic constraints. The main contribution of this paper is the derivation of convex conditions for the synthesis of output estimators for uncertain, grid-based LPV plants. Since LPV systems do not have valid frequency response interpretations, the time domain, dissipation inequality approach is followed. Robust performance is measured using the upper-bound on the worst-case induced-L_2 gain of the closed loop. The effectiveness of the proposed method is demonstrated using a numerical example.**

I. INTRODUCTION

Estimation is an important topic of research both in signal processing and feedback control. The well-known Kalman filter is an optimal minimum-covariance estimator for linear systems affected by Gaussian noise [1], [2]. The rise of robust control techniques in the 1980s led to an interest in alternative methods for synthesizing filters/estimators. The most well-known among these are the H_2 and H_∞ filters from robust control theory [3], [4], [5]. These methods assume the signals are generated by a known dynamic model of the plant and robustness to uncertainties is an important consideration. The 1990s saw a rise in methods based on the structured singular value, allowing for the incorporation of linear time-invariant (LTI) model uncertainty in analysis and synthesis problems. Since the 1990s, numerous papers on robust filter design have appeared in the literature [6], [7], [8], [9], [10]. Of these papers, some have presented methods based on the μ-synthesis [6]. Others have cast the robust filter design problem as an infinite dimensional optimization, that can be solved by frequency gridding [11], [12], [13], [14].

This paper considers the problem of synthesizing output estimators for uncertain, linear parameter-varying (LPV) systems. The uncertain system is described as a feedback interconnection of a nominal LPV system and a perturbation. The nominal LPV system is modeled using a grid-based approach, in which the parameter space is gridded and the state-space matrices are defined at each grid point. While a benefit of the grid-based approach is that the state matrices can be arbitrary functions of the scheduling parameters, a disadvantage is that any analysis is conducted over a finite number of grid points [15], [16]. The other major class of LPV systems is based on the linear fractional transform (LFT). While a benefit of the LFT-based approach is that the LPV system can be described completely analytically, a disadvantage is that the state matrices are restricted to depend rationally on the scheduling parameters [17], [18], [19].

The input/output behavior of the perturbation is assumed to satisfy several time-domain integral quadratic constraints (IQCs). IQCs were originally introduced in [20] to analyze the stability of the feedback interconnection between a linear time-invariant (LTI) plant and a perturbation. The stability theorem in [20] was presented using frequency-domain inequalities. A related stability theorem was formulated in the time-domain using dissipativity theory in [21]. This result was extended to analyze the robustness of uncertain LPV systems in [22], [23]. Robust filter design has been considered with static IQC multipliers in [24], [25], [26] and with dynamic IQC multipliers in [26], [27], [28]. In particular, robust filter design using *tall* factorizations of the frequency-domain IQC multipliers [27], [28] lead to additional degrees of freedom in the optimization.

The estimator synthesis problem is a special case of a more general controller synthesis problem. This general controller synthesis problem leads to nonconvex conditions when there is uncertainty in the system. For example, the synthesis of output feedback controllers that are robust to structured LTI uncertainty is a nonconvex problem that is solved using DK-iterations [29]. When the uncertainty is described using IQCs, robust output feedback controllers can be synthesized using similar iterative algorithms. These are commonly referred to in the literature as IQC-synthesis [30], [31], [32]. However, the synthesis conditions can be made convex when there are no uncertainties in the control channel of the closed-loop [33], such as the output estimator problem. A synthesis framework for robust gain-scheduled controllers for uncertain LFT-based LPV systems was presented in [34]. The main contribution of this paper is the derivation of convex conditions for the synthesis of output estimators for grid-based LPV systems.

II. BACKGROUND

A. Integral Quadratic Constraints

Standard notation is used in this paper [35]. The para-Hermitian conjugate of $G \in \mathbb{RL}_\infty^{m\times n}$ is defined as $G^\sim(s) := G(-s)^T$. Symmetric matrix blocks are denoted by $\star$. Figure 1 shows a LFT interconnection of a nominal LPV system G and a perturbation Δ, denoted $\mathcal{F}_u(G,\Delta)$. A state-space representation of G is given in section II-B. Δ has a block-structure that is standard in robust control modeling [35], and can include blocks that are hard nonlinearities (e.g.

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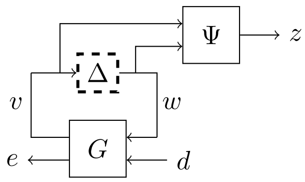

Fig. 1. Interconnection of nominal plant G, perturbation Δ, & IQC Ψ.

saturations) as well as model uncertainties. The perturbation $\Delta : L_{2e}^{n_v}[0,\infty) \rightarrow L_{2e}^{n_w}[0,\infty)$ is a bounded, causal operator, relating v and w as $w = \Delta(v)$. Δ is modeled by specifying IQCs on its inputs and outputs. IQCs were introduced in [20] and are defined using frequency-domain multipliers $\Pi : j\mathbb{R} \rightarrow \mathbb{C}^{(n_v+n_w)\times(n_v+n_w)}$ that are measurable Hermitian-valued functions. Any $\Pi \in \mathbb{RL}_{\infty}^{(n_v+n_w)\times(n_v+n_w)}$ can be factorized as $\Pi = \Psi^{\sim} M \Psi$, where $M = M^T \in \mathbb{R}^{n_z \times n_z}$ and $\Psi \in \mathbb{RH}_{\infty}^{n_z\times(n_v+n_w)}$ [21]. Such factorizations are not unique, but can be computed with state-space methods [36]. Let the pair (Ψ, M) denote any factorization of Π. Two signals $v \in L_{2e}^{n_v}[0,\infty)$ and $w \in L_{2e}^{n_w}[0,\infty)$ satisfy the IQC defined by (Ψ, M) if:

$$\int_0^{\infty} z(t)^T M z(t) dt \geq 0. \tag{1}$$

In (1), $z = \Psi\left[v^T, w^T\right]^T \in \mathbb{R}^{n_z}$ is the output of the linear system Ψ, described by the equations,

$$\begin{bmatrix} \dot{x}_\Psi \\ z \end{bmatrix} = \begin{bmatrix} A_\Psi & B_{\Psi v} & B_{\Psi w} \\ C_\Psi & D_{\Psi v} & D_{\Psi w} \end{bmatrix} \begin{bmatrix} x_\Psi \\ v \\ w \end{bmatrix}, \tag{2}$$

where $x_\Psi \in \mathbb{R}^{n_\Psi}$ and $x_\Psi(0) = 0$. Notions of hard and soft factorizations are obtained based on the time horizon in the integral [20]. In particular, a hard factorization is one for which inequality (1) holds for all finite time horizons. Δ satisfies the IQC defined by (Ψ, M) if and only if the time domain constraint (1) holds $\forall v \in L_{2e}^{n_v}[0,\infty)$ and $w = \Delta(v)$. This is denoted as $\Delta \in \text{IQC}(\Psi, M)$. The set of all Δ that satisfy the IQC is defined as $\boldsymbol{\Delta} := \{\Delta : \Delta \in \text{IQC}(\Psi, M)\}$. If the uncertainty Δ satisfies a collection of IQCs $\{\Pi_i\}_{i=1}^N$ with corresponding factorizations $\{(\Psi_i, M_i)\}_{i=1}^N$, then the filters Ψ_i can be stacked as $\Psi := \left[\Psi_1^T, \Psi_2^T, \ldots, \Psi_N^T\right]^T$. Finally, a single IQC multiplier can be obtained that is parameterized as $\Pi(\lambda) = \Psi^{\sim} M(\lambda) \Psi$. Section 4.2 in [32] provides more details about parameterizing IQC multipliers.

B. Robustness Analysis of LPV Systems

In Figure 1, G is an LPV system whose state-space matrices depend on a time-varying parameter $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_\rho}$:

$$\begin{bmatrix} \dot{x}_G \\ v \\ e \end{bmatrix} = \begin{bmatrix} A_G(\rho) & B_{1G}(\rho) & B_{2G}(\rho) \\ C_{1G}(\rho) & D_{11G}(\rho) & D_{12G}(\rho) \\ C_{2G}(\rho) & D_{21G}(\rho) & D_{22G}(\rho) \end{bmatrix} \begin{bmatrix} x_G \\ w \\ d \end{bmatrix}, \tag{3}$$

where $x_G \in \mathbb{R}^{n_G}$ is the state, $w \in \mathbb{R}^{n_w}$ and $d \in \mathbb{R}^{n_d}$ are the inputs, and $v \in \mathbb{R}^{n_v}$ and $e \in \mathbb{R}^{n_e}$ are the outputs. The state-space matrices of G have dimensions compatible with these signals and are assumed to be continuous functions of $\rho(t)$. The parameter $\rho(t)$ is assumed to be a continuously differentiable function of time and admissible trajectories are restricted to a known compact set $\mathcal{P} \subset \mathbb{R}^{n_\rho}$. In addition, bounds on the rate of variation $\dot{\rho}$ can be specified using a hyper-rectangle. The mathematical definition of the set of admissible trajectories is given in [15]. It is assumed that the nominal LPV plant G is *parametrically-dependent stable* [15]. In the remainder of the paper, the explicit dependence of the state-space matrices on ρ is suppressed for brevity.

The robust performance of $\mathcal{F}_u(G,\Delta)$ is measured using the metric of the induced L_2 gain. For a given $\Delta \in \boldsymbol{\Delta}$, the induced L_2 gain from d to e is defined as,

$$\|\mathcal{F}_u(G,\Delta)\| := \sup_{\substack{0 \neq d \in L_2^{n_d}[0,\infty) \\ x_G(0)=0}} \frac{\|e\|}{\|d\|}. \tag{4}$$

The worst-case induced L_2 gain from d to e over the set of uncertainties $\boldsymbol{\Delta}$ is defined as $\sup_{\Delta \in \boldsymbol{\Delta}} \|\mathcal{F}_u(G,\Delta)\|$. The system has robust asymptotic stability if $\lim_{t\rightarrow\infty} x_G(t) \rightarrow 0$ for all $x_G(0) \in \mathbb{R}^{n_G}$, disturbances $d \in L_2$, and uncertainties $\Delta \in \boldsymbol{\Delta}$. In order to assess the robust performance of $\mathcal{F}_u(G,\Delta)$, the filter Ψ is appended to the v and w channels of G, as shown in Figure 1. The extended system, formed by the interconnection of G and Ψ, is:

$$\begin{bmatrix} \dot{x}_e \\ z \\ e \end{bmatrix} = \begin{bmatrix} A_e & B_{1e} & B_{2e} \\ C_{1e} & D_{11e} & D_{12e} \\ C_{2e} & D_{21e} & D_{22e} \end{bmatrix} \begin{bmatrix} x_e \\ w \\ d \end{bmatrix}, \tag{5}$$

where $x_e = \left[x_G^T, x_\Psi^T\right]^T \in \mathbb{R}^{n_G+n_\Psi}$. The robust performance and stability of $\mathcal{F}_u(G,\Delta)$ can be assessed using dissipation inequality conditions, as given in the next Theorem.

Theorem 1: Let G be a parametrically stable LPV system defined by (3). In addition, let $\Delta : L_{2e}^{n_v}[0,\infty) \rightarrow L_{2e}^{n_w}[0,\infty)$ be a bounded, causal operator such that $\mathcal{F}_u(G,\Delta)$ is well-posed $\forall \Delta \in \boldsymbol{\Delta}$. Let the IQC multipliers be parameterized by λ. The interconnection of G and Ψ has a state-space representation as given in (5). If

1) the combined multiplier, partitioned as $\Pi(\lambda) = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^{\sim} & \Pi_{22} \end{bmatrix}$, satisfies $\Pi_{11}(j\omega) \in \mathbb{C}^{n_v \times n_v} > 0$ and $\Pi_{22}(j\omega) \in \mathbb{C}^{n_w \times n_w} < 0 \ \forall \omega \in \mathbb{R} \cup \{\infty\}$, and
2) there exists a continuously differentiable function $\bar{P} : \mathcal{P} \rightarrow \mathbb{S}^{n_G+n_\Psi}$, a scalar $\gamma > 0$, and parameters λ such that condition (6) holds for all admissible parameter trajectories,

$$\begin{bmatrix} A_e^T \bar{P} + \bar{P} A_e & \star & \star \\ B_{1e}^T \bar{P} & 0 & \star \\ B_{2e}^T \bar{P} & 0 & -\gamma I_{n_d} \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C_{2e}^T \\ D_{21e}^T \\ D_{22e}^T \end{bmatrix} (\star) + \begin{bmatrix} C_{1e}^T \\ D_{11e}^T \\ D_{12e}^T \end{bmatrix} M(\lambda)(\star) < 0 \tag{6}$$

then,

a) $\lim_{T\rightarrow\infty} x_e(T) = 0 \ \forall x_e(0) \in \mathbb{R}^{n_G+n_\Psi}$, $\forall d \in L_2$, and $\forall \Delta \in \boldsymbol{\Delta}$, and
b) $\sup_{\Delta \in \boldsymbol{\Delta}} \|\mathcal{F}_u(G,\Delta)\| \leq \gamma$.

Proof: The full proof involves arguments from game theory and can be found as Theorem 2 in [23]. However, in order to give some intuition to the reader, a version of the proof is given under the technical assumptions that $\bar{P} \geq 0$ and $(\Psi, M(\lambda))$ is a hard factorization of $\Pi(\lambda)$. To show b), define a parameter-dependent storage function $V : \mathbb{R}^{n_G+n_\Psi} \times \mathbb{R}^{n_\rho} \rightarrow \mathbb{R}^+$ by $V(x_e, \rho) = x_e^T \bar{P}(\rho) x_e$ and let $d \in L_2^{n_d}[0, \infty)$ be any input signal and ρ be any allowable parameter trajectory. From well-posedness, the interconnection of $\mathcal{F}_u(G, \Delta)$ has a solution that satisfies the dynamics in (5). Left and right multiply (6) by $\left[x_e^T, w^T, d^T\right]$ and $\left[x_e^T, w^T, d^T\right]^T$ to show that V satisfies,

$$\dot{V} + z(t)^T M(\lambda) z(t) \leq \gamma d(t)^T d(t) - \frac{1}{\gamma} e(t)^T e(t). \quad (7)$$

The dissipation inequality (7) is integrated from $t = 0$ to $t = T$ with the initial condition $x_e(0) = 0$. Then, the hard IQC condition is applied, along with $V \geq 0$, to yield $\frac{1}{\gamma} \int_0^T e(t)^T e(t)\, dt \leq \gamma \int_0^T d(t)^T d(t)\, dt$.

The proof for a) is more subtle but follows arguments similar to those given in [37]. First, note that (6) still holds if the term $\epsilon \cdot \text{diag}(I_{n_G+n_\Psi}, 0_{n_w+n_d})$ is added to the left hand side with $\epsilon > 0$ sufficiently small. Left and right multiply the modified inequality (6) by $\left[x_e^T, w^T, d^T\right]$ and $\left[x_e^T, w^T, d^T\right]^T$ to yield,

$$\dot{V}(t) + z(t)^T M(\lambda) z(t) + \epsilon \cdot x_e(t)^T x_e(t) \leq \gamma d(t)^T d(t) - \frac{1}{\gamma} e(t)^T e(t). \quad (8)$$

Consider now the response for any initial condition $x_e(0)$, input $d \in L_2$, and allowable trajectory ρ. Integrate (8) from $t = 0$ to $t = T$, apply the hard IQC conditions, and $V \geq 0$ to show that, as $T \rightarrow \infty$, we obtain $\epsilon \|x_e\|_2^2 \leq \gamma \|d\|_2^2 - \frac{1}{\gamma} \|e\|_2^2 + V(x_e(0), \rho(0)) < \infty$. It follows that $x_e \in L_2$. A similar perturbation argument can be used to show that $v \in L_2$ and hence $w = \Delta(v) \in L_2$ by the assumed boundedness of Δ. The time derivative of x_e is $\dot{x}_e = A_e x_e + B_{1e} w + B_{2e} d$. Therefore $\dot{x}_e \in L_2$ since $(x_e, w, d) \in L_2$ and A_e, B_{1e}, and B_{2e} are bounded on $\mathcal{P}$. Finally, $(x_e, \dot{x}_e) \in L_2$ implies that $\lim_{T \rightarrow \infty} x_e(T) = 0$ (see Appendix B of [38]). ∎

In Theorem 1, a) indicates robust asymptotic stability of x_e and b) indicates bounded worst-case gain. Theorem 1 is a sufficient condition for the existence of an upper bound on $\sup_{\Delta \in \boldsymbol{\Delta}} \|\mathcal{F}_u(G, \Delta)\|$. Theorem 1 is used to derive convex synthesis conditions for the robust LPV estimator.

III. ROBUST LPV ESTIMATOR SYNTHESIS

A. Problem Formulation

The robust estimator synthesis problem is formulated using the interconnection shown in Figure 2. The nominal plant P is an LPV system whose state-space matrices depend on ρ. Let $x_P \in \mathbb{R}^{n_P}$ denote the states, $y \in \mathbb{R}^{n_y}$ denote the measurable outputs, and $q \in \mathbb{R}^{n_q}$ denote the outputs to be estimated. The problem is to synthesize an estimator F that uses the measurements y to generate an estimate of q. F has

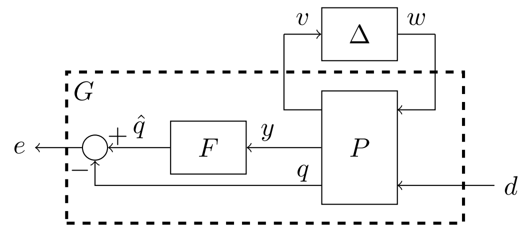

Fig. 2. Interconnection of the nominal LPV system P, perturbation Δ, and output estimator F.

the state-space representation,

$$\begin{bmatrix} \dot{x}_F \\ \hat{q} \end{bmatrix} = \begin{bmatrix} A_F(\rho) & B_F(\rho) \\ C_F(\rho) & D_F(\rho) \end{bmatrix} \begin{bmatrix} x_F \\ y \end{bmatrix}, \quad (9)$$

where $x_F \in \mathbb{R}^{n_F}$ are the filter states, $\hat{q} \in \mathbb{R}^{n_q}$ are the estimated outputs, and $e = \hat{q} - q$ are the estimation errors. As shown by the dashed box in Figure 2, the interconnection of P and F can be condensed into G, with states $x_G = \left[x_P^T, x_F^T\right]^T$. This fits into the notation used in section II-B. Before presenting the main result, some interconnections need to be defined. First, the interconnection of P and Ψ is described by the system of equations,

$$\begin{bmatrix} \dot{x} \\ z \\ y \\ q \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \\ C_3 & D_{31} & D_{32} \end{bmatrix} \begin{bmatrix} x \\ w \\ d \end{bmatrix}, \quad (10)$$

where $x = \left[x_P^T, x_\Psi^T\right]^T \in \mathbb{R}^n$ and $n = n_P + n_\Psi$. Next, the interconnection of P, Ψ, and F is described by the system of equations (5), where $x_e = \left[x_P^T, x_\Psi^T, x_F^T\right]^T \in \mathbb{R}^{n+n_F}$ and the state-space matrices are decomposed as,

$$A_e = \begin{bmatrix} A & 0 \\ B_F C_2 & A_F \end{bmatrix}, \; B_{1e} = \begin{bmatrix} B_1 \\ B_F D_{21} \end{bmatrix}, \quad (11)$$

$$B_{2e} = \begin{bmatrix} B_2 \\ B_F D_{22} \end{bmatrix}, \; C_{1e} = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad (12)$$

$$C_{2e} = \begin{bmatrix} (D_F C_2 - C_3) & C_F \end{bmatrix}, \quad (13)$$

$$D_{11e} = D_{11}, \; D_{12e} = D_{12}, \quad (14)$$

$$D_{21e} = D_F D_{21} - D_{31}, \; D_{22e} = D_F D_{22} - D_{32}. \quad (15)$$

B. Main Result

The general robust synthesis problem has two main sources of conservatism. First, the analysis result in Theorem 1 is only a sufficient condition for the existence of an upper bound on the worst-case gain. Second, when Theorem 1 is applied to the general robust synthesis problem, the resulting synthesis conditions are nonconvex and require the use of IQC-synthesis to solve for the controller [30], [31], [32]. However, the synthesis conditions can be made convex for the output estimator synthesis problem [33]. The main contribution of this paper is the derivation of convex conditions for the synthesis of output estimators for grid-based LPV plants. Hence, the second source of conservatism is removed. Given that the starting analysis condition in Theorem 1 is only sufficient, the main result provides a sufficient condition for the existence of an LPV estimator F such that the uncertain closed-loop system $\mathcal{F}_u(G, \Delta)$ achieves robust asymptotic stability and bounded worst-case gain. Hence, the main result introduces no additional conservatism with respect to Theorem 1. In general, the state dimension of the estimator F will be at most as large as $n_P + n_\Psi$. However, full-order estimators ($n_P + n_\Psi = n_F = n$) are synthesized in this paper.

Theorem 2: Let P be a parametrically stable LPV system. In addition, let $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ be a bounded, causal operator such that $\mathcal{F}_u(P, \Delta)$ is well-posed $\forall \Delta \in \mathbf{\Delta}$. Let the IQC multipliers be parameterized by λ. The interconnection of P and Ψ has a state-space representation as given in (10). If

1) the combined multiplier, partitioned as $\Pi(\lambda) = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^{\sim} & \Pi_{22} \end{bmatrix}$, satisfies $\Pi_{11}(j\omega) \in \mathbb{C}^{n_v \times n_v} > 0$ and $\Pi_{22}(j\omega) \in \mathbb{C}^{n_w \times n_w} < 0 \; \forall \omega \in \mathbb{R} \cup \{\infty\}$, and
2) there exist continuously differentiable functions $X : \mathcal{P} \to \mathbb{S}^n$, $Y : \mathcal{P} \to \mathbb{S}^n$, continuous functions $\bar{A} : \mathcal{P} \to \mathbb{R}^{n \times n}$, $\bar{B} : \mathcal{P} \to \mathbb{R}^{n \times n_y}$, $\bar{C} : \mathcal{P} \to \mathbb{R}^{n_q \times n}$, $\bar{D} : \mathcal{P} \to \mathbb{R}^{n_q \times n_y}$, a scalar $\gamma > 0$, and parameters λ such that conditions (16) and (17) hold for all admissible parameter trajectories,

then, there exists an estimator F such that the interconnection of P, F, and Δ, shown in Figure 2, satisfies,

a) $\lim_{T\to\infty} x_e(T) = 0 \; \forall x_e(0) \in \mathbb{R}^{2n}$, $\forall d \in L_2$, and $\forall \Delta \in \mathbf{\Delta}$, and
b) $\sup_{\Delta \in \mathbf{\Delta}} \|\mathcal{F}_u(G, \Delta)\| \leq \gamma$.

Proof: First, the closed-loop Lyapunov matrix is partitioned compatibly with $x_e = \left[x^T, x_F^T\right]^T \in \mathbb{R}^{n+n}$ as $\bar{P} := \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}$. Then, the following change of variables is applied to (16),

$$Y := X - X_2 X_3^{-1} X_2^T, \tag{18}$$

$$\bar{A} := X_2 \left(B_F C_2 - A_F X_3^{-1} X_2^T\right), \tag{19}$$

$$\bar{B} := X_2 B_F, \tag{20}$$

$$\bar{C} := -C_F X_3^{-1} X_2^T, \tag{21}$$

$$\bar{D} := D_F, \tag{22}$$

where A_F, B_F, C_F, and D_F are the state-space matrices of the filter to be synthesized. Following the change of variables, a congruence transformation is applied by multiplying on the right by T^{-1} and the left by T^{-T}, where $T = \operatorname{diag}\left(\tilde{T}, I\right)$ and $\tilde{T} := \begin{bmatrix} I & I \\ -X_3^{-1} X_2^T & 0 \end{bmatrix}$. Next, using the expressions in (11) to (15), the following inequality is obtained.

$$\begin{bmatrix} A_e^T \bar{P} + \star & \star & \star & \star \\ B_{1e}^T \bar{P} & 0 & \star & \star \\ B_{2e}^T \bar{P} & 0 & -\gamma I & \star \\ C_{2e} & D_{21e} & D_{22e} & -\gamma I \end{bmatrix} + \begin{bmatrix} C_{1e}^T \\ D_{11e}^T \\ D_{12e}^T \\ 0 \end{bmatrix} M(\lambda)(\star) < 0 \tag{23}$$

Applying the Schur complement lemma to (23), the condition in (6) is recovered. Finally, Theorem 1 is applied to conclude that statements a) and b) are true. ■

The condition $X - Y > 0$ ensures that a stable estimator is synthesized. To show this, consider the (1,1) block of (23),

$$A_e^T \bar{P} + \bar{P} A_e + C_{1e}^T M(\lambda) C_{1e} < 0. \tag{24}$$

Expressing A_e and C_{1e} using the decompositions listed in (11) to (15), (24) can be rewritten as,

$$\begin{bmatrix} A & 0 \\ B_F C_2 & A_F \end{bmatrix}^T \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} + \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} \begin{bmatrix} A & 0 \\ B_F C_2 & A_F \end{bmatrix} + \begin{bmatrix} C_1^T M C_1 & 0 \\ 0 & 0 \end{bmatrix} < 0. \tag{25}$$

X is the storage matrix for $x \in \mathbb{R}^n$ and X_3 is the storage matrix for $x_F \in \mathbb{R}^n$. The (2,2) block of (25) is $A_F^T X_3 + X_3 A_F < 0$. Hence, $X_3 > 0$ (equivalent to $X - Y > 0$) is an LMI condition to ensure that F is a parametrically stable LPV system. The state-space matrices of F are obtained using the following transformation,

$$\begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} = \begin{bmatrix} X_2^{-1}\left(\bar{B} C_2 - \bar{A}\right) X_2^{-T} X_3 & X_2^{-1} \bar{B} \\ -\bar{C} X_2^{-T} X_3 & \bar{D} \end{bmatrix} \tag{26}$$

A suitable reconstruction for the estimator is obtained using $X_3 = I$ and $X_2 X_2^T = X - Y$, where X_2 is the lower-triangular Cholesky factor of $X - Y$. Inequalities (16) and (17) are LMIs in X, Y, $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, λ, and γ. Theorem 2 can be formulated as a semidefinite program with γ as the cost function that is to be minimized, subject to LMI constraints.

Theorem 2 provides guarantees on the worst-case gain for all admissible parameter trajectories, wherein the parameters are restricted to a known compact set and their rates of variation are restricted to a known hyper-rectangle. In implementation, the parameter space is discretized into a finite number of grid points and the LMI constraints are enforced at each grid point. The LMIs at each grid point share a common parameter-dependent closed-loop Lyapunov matrix, making this approach significantly different from a pointwise design. Theorem 2 is different from the existing results because it allows for grid-based LPV plants whose state matrices are arbitrary functions of the scheduling parameter. In contrast, the results in [34] are applicable to LFT-LPV plants.

IV. NUMERICAL EXAMPLE

Figure 3 shows a spring-mass-damper system, consisting of two masses, two springs, and two dampers. The masses are $m_1 = 1\text{kg}$ and $m_2 = 0.5\text{kg}$. Both springs have the same spring constant $k = 1\,\text{N}\,\text{m}^{-1}$. The damping coefficient c_1 is certain, but depends on a time-varying scheduling parameter $\rho(t)$ as $c_1 = |\sin(\rho(t))|$. Admissible trajectories are restricted to the interval $\rho \in \mathcal{P} := \left[0, \frac{\pi}{3}\right]$, with infinite bounds on the rate of variation $\dot{\rho}$. Since c_1 is a transcendental function of ρ, this problem is not directly solvable by the LFT-LPV approach [34]. Following the grid-based LPV approach, the parameter space is gridded into three points $\left\{0, \frac{\pi}{6}, \frac{\pi}{3}\right\}$. These three points are chosen for demonstration purposes and the grid may be made as dense as needed [15].

The damping coefficient c_2 is time-invariant, but uncertain within the interval $[0.5, 3.5]\,\text{N}\,\text{s}\,\text{m}^{-1}$. The uncertainty in the real parameter c_2 is normalized to unity and represented as

$$\begin{bmatrix} A^TY+YA & \star & \star & \star & \star \\ A^TY+XA+\bar{A} & A^TX+\bar{B}C_2+\star & \star & \star & \star \\ B_1^TY & B_1^TX+D_{21}^T\bar{B}^T & 0 & \star & \star \\ B_2^TY & B_2^TX+D_{22}^T\bar{B}^T & 0 & -\gamma I_{n_d} & \star \\ -C_3+\bar{D}C_2+\bar{C} & -C_3+\bar{D}C_2 & -D_{31}+\bar{D}D_{21} & -D_{32}+\bar{D}D_{22} & -\gamma I_{n_q} \end{bmatrix} + \begin{bmatrix} C_1^T \\ C_1^T \\ D_{11}^T \\ D_{12}^T \\ 0 \end{bmatrix} M(\lambda)(\star) < 0 \qquad (16)$$

$$X - Y > 0 \qquad (17)$$

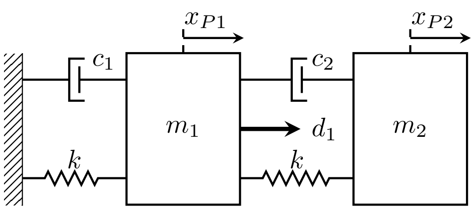

Fig. 3. Spring-mass-damper system, where c_1 is parameter-varying and c_2 is uncertain.

$\Delta = \delta_c I_2$, where $|\delta_c| \leq 1$. Mass m_1 is disturbed by an external force d_1. The positions of m_1 and m_2 relative to their respective equilibrium positions are denoted by x_{P1} and x_{P2}. The objective is to estimate x_{P2} using a measurement of x_{P1} that is corrupted by measurement noise d_2. The LPV plant model P can be expressed in state-space as:

$$\begin{bmatrix} \dot{x}_{P1} \\ \dot{x}_{P2} \\ \dot{x}_{P3} \\ \dot{x}_{P4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-2k}{m_1} & \frac{k}{m_1} & \frac{-(c_2+c_1(\rho))}{m_1} & \frac{c_2}{m_1} \\ \frac{k}{m_2} & \frac{-k}{m_2} & \frac{c_2}{m_2} & \frac{-c_2}{m_2} \end{bmatrix} \begin{bmatrix} x_{P1} \\ x_{P2} \\ x_{P3} \\ x_{P4} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1.5 & 1.5 & \frac{1}{m_1} & 0 \\ 3 & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ d_1 \\ d_2 \end{bmatrix}. \qquad (27)$$

The output equations are: $v = \left[x_{P3}^T, x_{P4}^T\right]^T$, $y = x_{P1} + d_2$, and $q = x_{P2}$.

Since δ_c is a constant real parametric uncertainty, Δ satisfies all IQCs defined by multipliers of the form,

$$\Pi(j\omega) = \begin{bmatrix} \mathcal{X}(j\omega) & \mathcal{Y}(j\omega) \\ \mathcal{Y}(j\omega)^\sim & -\mathcal{X}(j\omega) \end{bmatrix}, \qquad (28)$$

where $\mathcal{X}(j\omega) = \mathcal{X}(j\omega)^\sim \geq 0$ and $\mathcal{Y}(j\omega) = -\mathcal{Y}(j\omega)^\sim$ are bounded and measurable matrix functions [20]. Π can be factorized as $\Pi(j\omega) = \Psi(j\omega)^\sim M \Psi(j\omega)$ with $\Psi(j\omega) = \text{diag}(\psi(j\omega), \psi(j\omega))$ and $M = \begin{bmatrix} P_M & R_M \\ R_M^T & -P_M \end{bmatrix}$. Moreover, $\psi^\sim P \psi > 0$ on $\mathbb{C}^0$, $R = -R^T$ and ψ is taken as:

$$\psi(j\omega) = \left[I_2, \left(\frac{j\omega-\alpha}{j\omega+\alpha}\right) I_2, \ldots, \left(\frac{j\omega-\alpha}{j\omega+\alpha}\right)^{n_\psi} I_2\right]^T.$$

Robust estimator synthesis is performed for two cases: the LTI plant defined at the frozen parameter value $\rho = 0$ and the LPV plant defined on the grid $\left\{0, \frac{\pi}{6}, \frac{\pi}{3}\right\}$. The upper bounds on $\sup_{\Delta \in \mathbf{\Delta}} \|\mathcal{F}_u(G, \Delta)\|$ for various values of α and n_ψ are listed in Table I. For the LTI case, the results presented in section 5 of [28] are recovered. Conservatism due to the choice of the multiplier can be reduced by increasing n_ψ. However, in this particular example, the upper bounds obtained for $n_\psi > 2$ are no lower than those obtained for $n_\psi = 2$. In addition, selecting faster poles for $\psi(j\omega)$ increases the conservatism of the upper bounds.

TABLE I

UPPER BOUND ON THE WORST-CASE GAIN OF $\|\mathcal{F}_u(G, \Delta)\|$.

Plant	LTI			LPV		
n_ψ	0	1	2	0	1	2
$\alpha = 0.1$	4.53	3.07	2.64	4.56	3.25	2.97
$\alpha = 1$	4.53	3.37	2.64	4.56	3.48	2.97
$\alpha = 10$	4.53	3.63	2.64	4.56	3.73	2.98

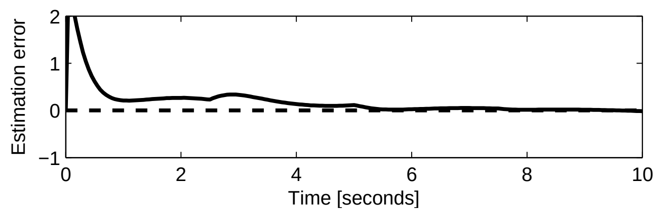

Fig. 4. LPV closed-loop simulation with $\alpha = 1$ and $n_\psi = 1$.

For the LPV case, the infinite rate bounds require the use of a common, parameter-independent, Lyapunov matrix $\bar{P}$ across all grid points. The result is a parameter-varying estimator. In order to ensure that the estimators across the grid points share the same state coordinates, $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ are expressed using common parameter-dependent basis functions. Second, the upper bounds obtained for the LPV case are greater than those for the LTI case because of the consideration of all admissible parameter trajectories. As an example, consider the trajectory $\rho(t) = |\sin(0.4\pi t)|$. In order to illustrate the performance of the estimator, a closed-loop simulation is performed with this trajectory and the initial condition $x_{P1}(0) = 1$. Figure 4 shows the estimation error asymptotically converging to zero.

In addition, the above example has two copies of the real parametric uncertainty δ_c. This can be reduced to a single copy, since the first two columns of the B matrix in (27) are not linearly independent. However, two copies of δ_c are retained in order to provide more degrees of freedom in $M(\lambda)$, resulting in lesser conservatism. This also allows for comparison with the results presented in [28].

Finally, the results presented in this paper can be generalized to include parameter-dependent IQCs, wherein Ψ and/or M depend on the scheduling parameter ρ. The parameter dependence of Ψ and M get captured through the LMI conditions that are enforced at each grid point. More details on parameter-varying IQCs can be found in section III.B of [22]. Robust synthesis for the LTI case of the example presented above can be solved by several existing methods [7], [14], [28]. The main purpose of this example is to demonstrate the usefulness of the dissipation inequality approach in solving synthesis problems that involve grid-based LPV plants.

V. CONCLUSION

This paper derived convex conditions for the synthesis of output estimators for uncertain, grid-based linear parameter-varying (LPV) plants. In particular, the general robust synthesis problem yields nonconvex conditions, and is addressed using ad-hoc procedures. However, the main result presented a convex solution for the robust output estimation problem using a suitable congruence transformation and change of variables. The proof of the main result used dissipativity theory and time-domain integral quadratic constraints (IQCs). The effectiveness of the proposed method was demonstrated on an uncertain, grid-based LPV plant whose state matrices were transcendental functions of the scheduling parameter.

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REFERENCES

[1] T. Kailath, A. Sayed, and B. Hassibi, *Linear Estimation*. Prentice Hall, 2000.

[2] R. Kalman, “A new approach to linear filtering and prediction problems,” *Journal of Fluids Engineering*, vol. 82, no. 1, pp. 35–45, 1960.

[3] B. Hassibi, T. Kailath, and A. Sayed, *Indefinite-quadratic estimation and control: a unified approach to H_2 and H_∞ theories*. SIAM, 1999.

[4] U. Shaked and Y. Theodor, “H_∞-optimal estimation: A tutorial,” in *IEEE Conference on Decision and Control*, 1992, pp. 2278–2286.

[5] T. Iwasaki and R. Skelton, “All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas,” *Automatica*, vol. 30, no. 8, pp. 1307–1317, 1994.

[6] B. D. Appleby, J. R. Dowdle, and W. VanderVelde, “Robust estimator design using μ-synthesis,” in *IEEE Conference on Decision and Control*, vol. 1, 1991, pp. 640–645.

[7] J. Geromel, “Optimal linear filtering under parameter uncertainty,” *IEEE Transactions on Signal Processing*, vol. 47, no. 1, pp. 168–175, 1999.

[8] J. Geromel and M. de Oliveira, “H_2 and H_∞ robust filtering for convex bounded uncertain systems,” *IEEE Transactions on Automatic Control*, vol. 46, no. 1, pp. 100–107, 2001.

[9] K. Sun, “Robust linear filter design via LMIs and controller design with actuator saturation via SOS programming,” Ph.D. dissertation, University of California Berkeley, 2004.

[10] K. Sun and A. Packard, “Robust H_2 and H_∞ filters for uncertain LFT systems,” *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 715–720, 2005.

[11] A. Giusto and F. Paganini, “Robust synthesis of feedforward compensators,” *IEEE Transactions on Automatic Control*, vol. 44, no. 8, pp. 1578–1582, 1999.

[12] G. Ferreres and C. Roos, “Efficient convex design of robust feedforward controllers,” in *IEEE Conference on Decision and Control*, 2005, pp. 6460 – 6465.

[13] ——, “Robust feedforward design in the presence of LTI/LTV uncertainties,” *International Journal of Robust and Nonlinear Control*, vol. 17, no. 14, pp. 1278–1293, 2007.

[14] P. Seiler, B. Vanek, J. Bokor, and G. Balas, “Robust H_∞ filter design using frequency gridding,” in *American Control Conference*, 2011, pp. 1801–1806.

[15] F. Wu, “Control of linear parameter varying systems,” Ph.D. dissertation, University of California, Berkeley, 1995.

[16] F. Wu, X. H. Yang, A. Packard, and G. Becker, “Induced $\mathcal{L}_2$ norm control for LPV systems with bounded parameter variation rates,” *Int. Journal of Robust and Nonlinear Control*, vol. 6, pp. 983–998, 1996.

[17] P. Apkarian and P. Gahinet, “A convex characterization of gain-scheduled H_∞ controllers,” *IEEE Transactions on Automatic Control*, vol. 40, pp. 853–864, 1995.

[18] A. Packard, “Gain scheduling via linear fractional transformations,” *Systems and Control Letters*, vol. 22, pp. 79–92, 1994.

[19] C. Scherer, *Advances in linear matrix inequality methods in control*. SIAM, 2000, ch. Robust mixed control and linear parameter-varying control with full-block scalings, pp. 187–207.

[20] A. Megretski and A. Rantzer, “System analysis via integral quadratic constraints,” *IEEE Transactions on Automatic Control*, vol. 42, pp. 819–830, 1997.

[21] P. Seiler, “Stability analysis with dissipation inequalities and integral quadratic constraints,” *IEEE Transactions on Automatic Control*, vol. 60, no. 6, pp. 1704–1709, 2015.

[22] H. Pfifer and P. Seiler, “Robustness analysis of linear parameter varying systems using integral quadratic constraints,” *International Journal of Robust and Nonlinear Control*, vol. 25, pp. 2843–2864, 2015.

[23] ——, “Less conservative robustness analysis of linear parameter varying systems using integral quadratic constraints,” *accepted to International Journal of Robust and Nonlinear Control*, 2016.

[24] H. Li and M. Fu, “A linear matrix inequality approach to robust H_∞ filtering,” *IEEE Transactions on Signal Processing*, vol. 45, no. 9, pp. 2338–2350, 1997.

[25] M. Ravuri and H. Asada, “Robust state estimation for uncertain nonlinear systems described using integral quadratic constraints: An LMI formulation,” in *American Control Conference*, vol. 2, 2000, pp. 1293–1297.

[26] G. Scorletti and V. Fromion, “Further results on the design of robust H_∞ feedforward controllers and filters,” in *IEEE Conference on Decision and Control*, 2006, pp. 3560–3565.

[27] C. Scherer and I. Kose, “Robust H_2 estimation with dynamic IQCs: A convex solution,” in *IEEE Conference on Decision and Control*, 2006, pp. 4746–4751.

[28] ——, “Robustness with dynamic IQCs: An exact state-space characterization of nominal stability with applications to robust estimation,” *Automatica*, vol. 44, no. 7, pp. 1666–1675, 2008.

[29] J. Doyle, “Structured uncertainty in control system design,” in *IEEE Conference on Decision and Control*, 1985, pp. 260 – 265.

[30] A. Helmerssonn, “IQC synthesis based on inertia constraints,” 1998.

[31] S. Wang, H. Pfifer, and P. Seiler, “Robust synthesis for linear parameter varying systems using integral quadratic constraints,” *accepted to Automatica*, 2016.

[32] J. Veenman and C. Scherer, “IQC-synthesis with general dynamic multipliers,” *International Journal of Robust and Nonlinear Control*, vol. 24, no. 17, pp. 3027–3056, 2014.

[33] C. Scherer, *Robust controller synthesis is convex for systems without control channel uncertainties*. Springer, 2009.

[34] J. Veenman and C. Scherer, “A synthesis framework for robust gain-scheduling controllers,” *Automatica*, vol. 50, no. 11, pp. 2799–2812, 2014.

[35] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*. New Jersey: Prentice-Hall, 1996.

[36] C. Scherer and S. Wieland, “Linear matrix inequalities in control,” Delft University of Technology,” Lecture Notes for a course of the Dutch Institute of Systems and Control, 2004.

[37] U. Jönsson, “Robustness analysis of uncertain and nonlinear systems,” Ph.D. dissertation, Lund Institute of Technology, 1996.

[38] C. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*. Academic Press, 1975.