A Newtonian Development of the Mean-Axis Dynamics with Example and Simulation

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Dedication

To my parents, Jimmy and Cindy, for their endless support.

Abstract

Mean-axis models of flight dynamics for flexible aircraft are being utilized more frequently in recent dynamics and controls research. The equations of motion resulting from the mean-axis formulation are frequently developed with Lagrangian mechanics. In addition, the models are typically simplified using assumptions regarding the effects of the elastic deformation. Although widely accepted in the literature, the formulation and assumptions may be confusing to a user outside of the flight dynamics field (such as a controls engineer). In this thesis, the equations of motion are derived from first principles utilizing Newtonian, rather than Lagrangian, mechanics. In this framework, the formulation offers a new set of insights into the equations of motion and explanations for the assumptions. A three-lumped-mass idealization of a rolling flexible aircraft is presented as an example of the mean-axis equations of motion. The example is also used to investigate the effects of common simplifying assumptions. The equations of motion are also developed without any such assumptions, and simulation results allow for a comparison of the exact and simplified dynamics.

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Chapter 1

Introduction

The objective of this work is to provide insight into the formulation and assumptions of the mean-axis equations of motion for flexible aircraft. As engineers seek to design and produce more fuel efficient aircraft, the resulting trends are of reduced structural mass and increased wing aspect ratio. This leads to increasingly flexible aircraft, which present unique control challenges. Typically, aircraft are designed such that flutter and excessive vibrations will not occur within the flight envelope, and controllers are designed assuming a rigid aircraft. As aircraft become more flexible, however, they will require that controllers provide integrated rigid body and vibration control. In turn, these controllers require that models are developed which capture the essential dynamics of the system but provide the simplicity necessary for control design. The simplicity requirement translates into a model with a relatively low number of states, on the order of ten rather than thousands as may be found in high fidelity computational models. The requirement that the model capture the essential dynamics means that the model must accurately describe both the rigid body and the elastic degrees of freedom, as well as critical interactions between them such as body-freedom flutter.

When modeling the flight dynamics of flexible aircraft, the use of the mean-axis formulation of the equations of motion dates back to the early work of Milne in the mid 1960s. [1] The modeling approach has grown more popular with the advent of finite element methods for characterizing the free vibrations of the aircraft structure, and mean- axis models have been used in a wide variety of applications. Such applications include nonlinear real-time piloted simulations, [2] flight dynamics and flutter analyses, [3] and control law synthesis for active flutter suppression. [4] Some of the advantages of using mean-axis models are 1) the state vector is a direct extension of the state vector in rigid aircraft models, 2) the nonlinear dynamics of the rigid body degrees of freedom may be modeled, 3) the model may be parameterized using non-dimensional aerodynamic and aeroelastic coefficients, thus making the model valid over a region of the flight envelope rather than just one flight condition, 4) models of low dynamic order may be obtained that are especially attractive when using multivariable control techniques, and 5) the model structure and format is familiar to flight dynamicists. In addition, the validity of mean-axis models has been demonstrated by comparing flutter predictions to those obtained from computational models [3], and by comparing model-based transient responses with those obtained in flight tests. [5]

Despite their benefits, mean-axis models present several challenges. For new users of the mean-axis modeling technique, it can be difficult to gain intuition for the physical meaning of the floating reference frame. Furthermore, the validity of the standard assumptions under different conditions can be unclear. To gain additional insight into the mean-axis technique, an alternate Newtonian derivation for a system of particles is presented. The derivation is typically carried out using Lagrangian methods for a body with distributed mass. The Newtonian derivation may offer new insight as it approaches the dynamics from a momentum perspective, rather than an energy perspective. Moreover, the floating reference frame introduces additional degrees of freedom, and this derivation attempts to reconcile the treatment of those additional degrees of freedom in a precise and definitive manner. Furthermore, the simplicity of the system of particles, as opposed to a body with distributed mass, is meant to allow for additional insight into the derivation and the final equations of motion.

In addition to the theoretical results, a simple example of a lumped-mass rolling aircraft is presented and analyzed in order to gain understanding of the equations of motion. The example also creates the opportunity to investigate how the dynamics differ under various assumptions and conditions.

Chapter 2

The Mean Axes

2.1 Notation and Problem Formulation

Consider a deformable body consisting of n particles with mass m_i ($i = 1, \ldots, n$) as shown on the left of Figure 2.1. Each particle is free to translate in three directions, and the position of particle i in the inertial frame I is denoted by $\boldsymbol{r}_i$. Each particle is acted upon by internal and external forces. The external force on particle i is denoted by $\boldsymbol{F}_i$. The internal force on particle i due to particle j is denoted by $\boldsymbol{F}_{ij}$. By Newton's Third Law, the internal forces between particles i and j are assumed to be equal and opposite: $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. Moreover, the internal forces are assumed to act along the line between the two particles: $\boldsymbol{F}_{ij} = |\boldsymbol{F}_{ij}|(\boldsymbol{r}_j - \boldsymbol{r}_i)$. The dynamics for this deformable body are specified by Newton's Second Law:

$$
\begin{gathered}
m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i + \sum_{j \neq i} \boldsymbol{F}_{ij} \qquad \text{for } i = 1, \ldots, n \\
\text{Initial Conditions: } \{\boldsymbol{r}_i(0)\}_{i=1}^n \text{ and } \{\dot{\boldsymbol{r}}_i(0)\}_{i=1}^n
\end{gathered}
\tag{2.1}
$$

These equations of motion consist of n vector, second-order differential equations for the inertial positions $\boldsymbol{r}_i$. The initial conditions specify the position and velocity for each particle at $t = 0$. These dynamics can be rewritten as $3n$ scalar, second-order differential equations in terms of the (x, y, z) components of the various vectors. In this framework, internal forces are those that act only between particles. This implies that the body is unre-

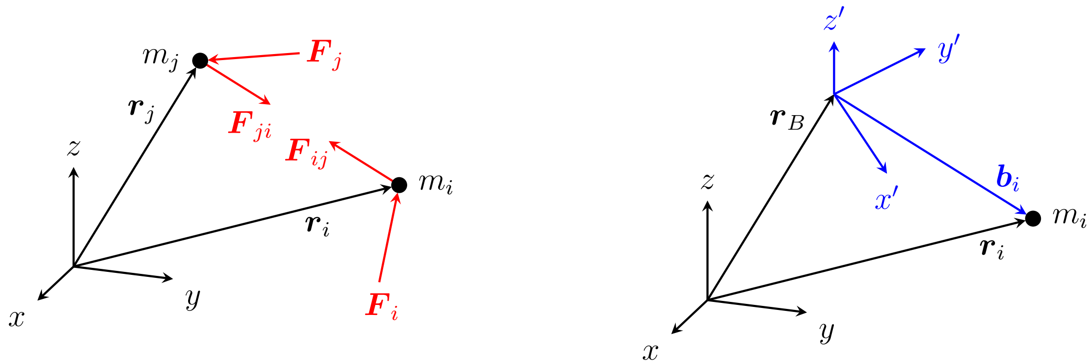

Figure 2.1: Left: Notation for system of particles in an inertial (x, y, z) frame. Right: Notation for system using a body (x', y', z') frame.

strained, meaning that it is not attached to a fixed point outside of the body. [1] The dynamic equations can be rewritten using a *body-reference* frame B. The right side of Figure 2.1 shows the inertial frame I, denoted (x, y, z), and the body frame B, denoted (x', y', z'). The body frame has origin at $\boldsymbol{r}_B$ and orientation given by an arbitrary Euler angle sequence. The body-reference frame moves with the body (in some manner that is not yet specified), but it is not necessarily attached to a particle or material point on the body. Hence, the particles may be located arbitrarily with respect to the origin of the reference frame $\boldsymbol{r}_B$. The vector $\boldsymbol{b}_i$ denotes the position of particle i in the body frame. The particle positions specified in the inertial and body frames are related as follows:

$$\boldsymbol{r}_i = \boldsymbol{r}_B + \boldsymbol{b}_i \qquad \text{for } i = 1, \dots, n \tag{2.2}$$

Expressed using inertial derivatives, the acceleration of the i^{th} particle is simply:

$$\ddot{\boldsymbol{r}}_i = \ddot{\boldsymbol{r}}_B + \ddot{\boldsymbol{b}}_i \qquad \text{for } i = 1, \dots, n \tag{2.3}$$

Assuming that the translation of the reference frame $\boldsymbol{r}_B$ is known, substitute for $\ddot{\boldsymbol{r}}_i$ in Equation 2.1 to obtain an additional form of the dynamics:

$$\begin{gathered} m_i\left(\ddot{\boldsymbol{r}}_B + \ddot{\boldsymbol{b}}_i\right) = \boldsymbol{F}_i + \sum_{j \neq i} \boldsymbol{F}_{ij} \qquad \text{for } i = 1, \dots, n \\ \text{Initial Conditions: } \{\boldsymbol{b}_i(0)\}_{i=1}^n \text{ and } \{\dot{\boldsymbol{b}}_i(0)\}_{i=1}^n \end{gathered} \tag{2.4}$$

[1] Generally, only unrestrained bodies will be considered here. If connections to a fixed point do exist, however, they would be considered external forces in this framework.

Expressing the dynamics with relative derivatives, rather than inertial derivatives, is often more natural when using a reference frame, although some additional work is required. Let $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ denote the angular velocity and acceleration of frame B. The time derivative of a vector $\boldsymbol{b}_i$ in the body frame is given by the Transport Theorem [6]:

$$\dot{\boldsymbol{b}}_i = \overset{\circ}{\boldsymbol{b}}_i + \boldsymbol{\omega} \times \boldsymbol{b}_i \tag{2.5}$$

Here $\dot{\boldsymbol{b}}_i = \frac{d}{dt}\Big|_I \boldsymbol{b}_i$ and $\overset{\circ}{\boldsymbol{b}}_i = \frac{d}{dt}\Big|_B \boldsymbol{b}_i$ denote time derivatives with respect to the inertial and body frames, respectively. Note that the Transport Theorem implies that $\dot{\boldsymbol{\omega}} = \overset{\circ}{\boldsymbol{\omega}}$, i.e. the derivative of $\boldsymbol{\omega}$ is the same in the inertial and body frames since $\boldsymbol{\omega} \times \boldsymbol{\omega} = 0$. It follows from the Transport Theorem that the first and second derivatives of $\boldsymbol{r}_i$ are:

$$\dot{\boldsymbol{r}}_i = \dot{\boldsymbol{r}}_B + \overset{\circ}{\boldsymbol{b}}_i + \boldsymbol{\omega} \times \boldsymbol{b}_i \tag{2.6}$$

$$\ddot{\boldsymbol{r}}_i = \ddot{\boldsymbol{r}}_B + \overset{\circ\circ}{\boldsymbol{b}}_i + \dot{\boldsymbol{\omega}} \times \boldsymbol{b}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b}_i) + 2\boldsymbol{\omega} \times \overset{\circ}{\boldsymbol{b}}_i \tag{2.7}$$

Substitute for $\ddot{\boldsymbol{r}}_i$ in Equation 2.1 to obtain the dynamics expressed using the body frame B:

$$\begin{aligned} & m_i \left(\ddot{\boldsymbol{r}}_B + \overset{\circ\circ}{\boldsymbol{b}}_i + \dot{\boldsymbol{\omega}} \times \boldsymbol{b}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b}_i) + 2\boldsymbol{\omega} \times \overset{\circ}{\boldsymbol{b}}_i \right) = \boldsymbol{F}_i + \sum_{j \neq i} \boldsymbol{F}_{ij} \qquad \text{for } i = 1, \ldots, n \\ & \text{Initial Conditions: } \{\boldsymbol{b}_i(0)\}_{i=1}^n \text{ and } \{\overset{\circ}{\boldsymbol{b}}_i(0)\}_{i=1}^n \end{aligned} \tag{2.8}$$

The motion of the body frame appears in these dynamics due its translational acceleration $\ddot{\boldsymbol{r}}_B$ as well as its angular velocity $\boldsymbol{\omega}$ and acceleration $\dot{\boldsymbol{\omega}}$. A specific choice for the body frame will be discussed in the subsequent sections. For now, assume the motion of the body frame is given. In this case, Equation 2.8 consists of n vector, second-order differential equations for the positions $\boldsymbol{b}_i$ in the body frame. Again, these dynamics can be rewritten as $3n$ scalar, second-order differential equations in terms of the (x', y', z') components of the various vectors. The initial conditions for Equation 2.8 are specified in the body frame. The following equations relate these body frame initial conditions to those given in the inertial frame:

$$\boldsymbol{b}_i(0) = \boldsymbol{r}_i(0) - \boldsymbol{r}_B(0) \tag{2.9}$$

$$\overset{\circ}{\boldsymbol{b}}_i(0) = \dot{\boldsymbol{r}}_i(0) - \dot{\boldsymbol{r}}_B(0) - \boldsymbol{\omega}(0) \times (\boldsymbol{r}_i(0) - \boldsymbol{r}_B(0)) \tag{2.10}$$

2.2 Mean-Axis Constraints

The dynamics expressed using a body-reference frame B (Equation 2.8) are valid for arbitrary translational and rotational motion of the frame. A particularly useful choice is a frame that satisfies the *mean-axis constraints* $[1, 7, 8]$. Specifically, the mean-axis constraints define a body frame for which there is no internal translational or angular momentum. Internal momentum is defined as momentum due to relative position and velocity with respect to the reference frame. To be precise, the internal translational momentum $\boldsymbol{P}_{int}$ and angular momentum $\boldsymbol{H}_{int}$ in frame B are given by:

$$\boldsymbol{P}_{int} := \sum_{i=1}^{n} m_i \overset{\circ}{\boldsymbol{b}}_i \tag{2.11}$$

$$\boldsymbol{H}_{int} := \sum_{i=1}^{n} m_i \boldsymbol{b}_i \times \overset{\circ}{\boldsymbol{b}}_i \tag{2.12}$$

The mean-axis constraints are $\boldsymbol{P}_{int}(t) = 0$ and $\boldsymbol{H}_{int}(t) = 0$ for all time $t \geq 0$. These constraints implicitly define the motion of the body-reference frame B. However, there is an ambiguity in the mean axes because these constraints are in terms of the internal translational and angular velocity. In particular, the initial position and rotation of the axes are not specified. Hence, the constraints $\boldsymbol{P}_{int}(t) = 0$ and $\boldsymbol{H}_{int}(t) = 0$ only define the mean axes up to constant translational and rotational offsets. The ambiguity in the translational offset is removed by requiring the origin of the mean axes to initially be located at the center of mass. This specific translational offset plays a critical role in simplifying the equations of motion using the body-reference frame. To summarize, the mean axes are formally defined below.

Definition 1 (Mean-Axes)**.** The mean axes[2] are a body-reference frame B that satisfy the following two conditions:

A) *Translational Motion:* Frame B has no internal translational momentum, i.e. $\boldsymbol{P}_{int}(t) = 0$ for all $t \geq 0$. Moreover, the origin of B is located at the center of mass at the initial time, i.e. $\sum_{i=1}^{n} m_i \boldsymbol{b}_i(0) = 0$.

B) *Rotational Motion:* Frame B has no internal angular momentum, i.e. $\boldsymbol{H}_{int}(t) = 0$

[2] Constant offsets in the angular orientation of the mean axes do not play a critical role in simplifying the equations of motion. Hence this definition allows for any angular orientation of the axes at the initial time. As a result there is a set of mean axes all related by constant rotational offsets. "The" mean axes refers to any one of these axes.

for all $t \geq 0$.

As noted above, the mean-axis constraints implicitly define the motion of the mean axes. The next section show that these constraints are equivalent to explicit equations of motion for the translation and rotation of the frame.

2.3 Translation of the Mean Axes

This section focuses on the translational mean-axis constraint in Definition 1.A. First, define the vector $\boldsymbol{p} := \sum_{i=1}^{n} m_i \boldsymbol{b}_i$ and the total mass of the body $m_{tot} := \sum_{i=1}^{n} m_i$. Note that $\frac{1}{m_{tot}}\boldsymbol{p}$ is the center of mass in the body frame. Moreover, the internal translational momentum is $\boldsymbol{P}_{int} = \overset{\circ}{\boldsymbol{p}}$. Sum the n body-referenced differential equations with inertial derivatives only (Equation 2.4) to obtain the following differential equation for $\boldsymbol{p}$:

$$\begin{aligned} & m_{tot}\ddot{\boldsymbol{r}}_B + \ddot{\boldsymbol{p}} = \boldsymbol{F}_{ext} \qquad \text{for } i = 1, \ldots, n \\ & \text{Initial Conditions: } \boldsymbol{p}(0) = \sum_{i=1}^{n} m_i \boldsymbol{b}_i(0) \text{ and } \dot{\boldsymbol{p}}(0) = \sum_{i=1}^{n} m_i \dot{\boldsymbol{b}}_i(0) \end{aligned} \tag{2.13}$$

where $\boldsymbol{F}_{ext} := \sum_{i=1}^{n} \boldsymbol{F}_i$ is the net external force. Note that the internal forces $\boldsymbol{F}_{ij}$ sum to zero since they are assumed to be equal and opposite. Additionally, the Transport Theorem yields a constraint on the solution of equation 2.13. This constraint relates the absolute derivative, $\dot{\boldsymbol{p}}$, and the relative derivative, $\overset{\circ}{\boldsymbol{p}}$:

$$\begin{aligned} & \dot{\boldsymbol{p}} = \overset{\circ}{\boldsymbol{p}} + \boldsymbol{\omega} \times \boldsymbol{p} \qquad \text{for } i = 1, \ldots, n \\ & \text{Initial Conditions: } \boldsymbol{p}(0) = \sum_{i=1}^{n} m_i \boldsymbol{b}_i(0) \text{ and } \overset{\circ}{\boldsymbol{p}}(0) = \sum_{i=1}^{n} m_i \overset{\circ}{\boldsymbol{b}}_i(0) = \boldsymbol{P}_{int}(0) \end{aligned} \tag{2.14}$$

Equations 2.13 and 2.14, which govern $\boldsymbol{p}$, are used in the next theorem to provide an explicit equation of motion corresponding to the translational mean-axis constraint.

Theorem 1. The frame B satisfies the translational mean-axis constraint (Definition 1.A) if and only if $\boldsymbol{r}_B$ satisfies the following equation of motion:

$$\begin{aligned} & m_{tot}\ddot{\boldsymbol{r}}_B = \boldsymbol{F}_{ext} \\ & \text{Initial Conditions: } \boldsymbol{r}_B(0) = \frac{1}{m_{tot}} \sum_{i=1}^{n} m_i \boldsymbol{r}_i(0) \text{ and } \dot{\boldsymbol{r}}_B(0) = \frac{1}{m_{tot}} \sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i(0) \end{aligned} \tag{2.15}$$

Proof. ($\Rightarrow$) Assume the frame B satisfies the translational mean-axis constraint (Definition 1.A). This implies that $\overset{\circ}{\boldsymbol{p}}(t) = \boldsymbol{P}_{int}(t) = 0$ for all $t \geq 0$ and $\boldsymbol{p}(0) = \sum_{i=1}^{n} m_i \boldsymbol{b}_i(0) = 0$. Equation 2.14 reduces to the following constraint:

$$\dot{\boldsymbol{p}}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{p}(t) \tag{2.16}$$

The initial condition is $\boldsymbol{p}(0) = 0$. The unique solution to this differential equation is $\boldsymbol{p}(t) = 0$ for all $t \geq 0$ and for all $\boldsymbol{\omega}$. This further implies $\ddot{\boldsymbol{p}}(t) = 0$ for all $t \geq 0$. In this case Equation 2.13 simplifies to $m_{tot}\ddot{\boldsymbol{r}}_B = \boldsymbol{F}_{ext}$. Moreover, it follows from the relation $\boldsymbol{r}_i = \boldsymbol{r}_B + \boldsymbol{b}_i$ that:

$$\sum_{i=1}^{n} m_i \boldsymbol{r}_i(0) = m_{tot}\boldsymbol{r}_B(0) + \boldsymbol{p}(0) \tag{2.17}$$

Therefore $\boldsymbol{p}(0) = 0$ (assumed by the translational mean-axis constraint) implies $\boldsymbol{r}_B(0) = \frac{1}{m_{tot}} \sum_{i=1}^{n} m_i \boldsymbol{r}_i(0)$. It can similarly be shown that $\dot{\boldsymbol{r}}_B(0) = \frac{1}{m_{tot}} \sum_{i=1}^{n} m_i \dot{\boldsymbol{r}}_i(0)$.

($\Leftarrow$) Assume the frame B satisfies the ODE and initial conditions in Equation 2.15. In this case Equation 2.13 simplifies to the following unforced ODE:

$$\ddot{\boldsymbol{p}}(t) = 0 \tag{2.18}$$

The initial conditions in Equation 2.15 can be rewritten as $\boldsymbol{p}(0) = \dot{\boldsymbol{p}}(0) = 0$. Based on these initial conditions, the unique solution to the unforced ODE in Equation 2.18 is $\boldsymbol{p}(t) = 0$ for all $t \geq 0$. This implies that $\boldsymbol{p}(t) = \boldsymbol{P}_{int}(t) = 0$ for all $t \geq 0$ and $\boldsymbol{p}(0) = \sum_{i=1}^{n} m_i \boldsymbol{b}_i(0) = 0$. Thus the translational mean-axis constraints are satisfied. □

This theorem provides a single vector, second-order differential equation (Equation 2.15) for the translational motion of the mean axes. This corresponds to three scalar, second-order differential equations when expressed in component form. Because $\boldsymbol{p}(t) = 0$ for all $t \geq 0$, $\boldsymbol{r}_B(t) = \frac{1}{m_{tot}} \sum_{i=1}^{n} m_i \boldsymbol{r}_i(t)$ for all $t \geq 0$. Therefore, the differential equation and initial conditions specify that the origin of the mean axes is located at the center of mass of the flexible body. This conclusion is reinforced by the fact that the mean-axis translational equation of motion (Equation 2.15) is identical to the equation of motion for the center of mass of a system of particles given in standard dynamics references. Furthermore, it is also identical to the equation of motion for the center of mass of a rigid body [6].

2.4 Rotation of the Mean Axes

This section focuses on the rotational mean-axis constraint in Definition 1.B. The total angular momentum about the origin of frame B is:

$$\boldsymbol{H}_{tot} = \sum_{i=1}^{n} \boldsymbol{b}_i \times m_i \dot{\boldsymbol{r}}_i \tag{2.19}$$

The inertial derivative of $\boldsymbol{H}_{tot}$ is given by:

$$\dot{\boldsymbol{H}}_{tot} = \sum_{i=1}^{n} \left(\dot{\boldsymbol{b}}_i \times m_i \dot{\boldsymbol{r}}_i + \boldsymbol{b}_i \times m_i \ddot{\boldsymbol{r}}_i \right) \tag{2.20}$$

To simplify this expression, use the equations of motion for the particles in the inertial frame (Equation 2.1) to substitute for $m_i \ddot{\boldsymbol{r}}_i$. In addition, substitute $\dot{\boldsymbol{r}}_i = \dot{\boldsymbol{r}}_B + \dot{\boldsymbol{b}}_i$. This yields the following form for $\dot{\boldsymbol{H}}_{tot}$ after some re-arrangement:

$$\dot{\boldsymbol{H}}_{tot} = \left(\sum_{i=1}^{n} m_i \dot{\boldsymbol{b}}_i \right) \times \dot{\boldsymbol{r}}_B + \sum_{i=1}^{n} m_i \left(\dot{\boldsymbol{b}}_i \times \dot{\boldsymbol{b}}_i \right) + \sum_{i=1}^{n} \boldsymbol{b}_i \times \left(\boldsymbol{F}_i + \sum_{j \neq i} \boldsymbol{F}_{ij} \right) \tag{2.21}$$

The second term is equal to zero because $\dot{\boldsymbol{b}}_i \times \dot{\boldsymbol{b}}_i = 0$. Moreover, the third term simplifies to $\boldsymbol{M}_{ext} := \sum_{i=1}^{n} \boldsymbol{b}_i \times \boldsymbol{F}_i$ because the internal forces are assumed to be equal, opposite, and acting along the line between the particles. The vector $\boldsymbol{M}_{ext}$ is the net moment about the origin of frame B due to the external forces. Finally, the translational mean-axis condition implies the first term is zero. Specifically, $\sum_{i=1}^{n} m_i \dot{\boldsymbol{b}}_i$ is equal to $(\sum_{i=1}^{n} m_i \mathring{\boldsymbol{b}}_i) + \boldsymbol{\omega} \times (\sum_{i=1}^{n} m_i \boldsymbol{b}_i)$ by the Transport theorem. The translational mean-axis condition implies $\sum_{i=1}^{n} m_i \boldsymbol{b}_i = 0$ and $\sum_{i=1}^{n} m_i \mathring{\boldsymbol{b}}_i = 0$ as shown in Theorem 1. Hence the first term is zero. As a result of these simplifications, the inertial derivative of $\boldsymbol{H}_{tot}$ simplifies to

$$\dot{\boldsymbol{H}}_{tot} = \boldsymbol{M}_{ext} \tag{2.22}$$

In other words, the rate of change of the total angular momentum about the center of mass is equal to the total moment. This result is consistent with equations from standard dynamics references [6].

Before stating the rotational mean-axis result, it is useful to rewrite the total angular momentum in an alternative form that involves the internal angular momentum $\boldsymbol{H}_{int}$. First substitute for $\dot{\boldsymbol{r}}_i$ in the definition of $\boldsymbol{H}_{tot}$ (Equation 2.19) using the expression derived from the Transport Theorem (Equation 2.6):

$$\boldsymbol{H}_{tot} = \sum_{i=1}^{n} \boldsymbol{b}_i \times m_i \left(\dot{\boldsymbol{r}}_B + \overset{\circ}{\boldsymbol{b}}_i + (\boldsymbol{\omega} \times \boldsymbol{b}_i) \right) \tag{2.23}$$

$$= \left(\sum_{i=1}^{n} m_i \boldsymbol{b}_i \right) \times \dot{\boldsymbol{r}}_B + \sum_{i=1}^{n} m_i \boldsymbol{b}_i \times \overset{\circ}{\boldsymbol{b}}_i + \sum_{i=1}^{n} m_i \left(\boldsymbol{b}_i \times (\boldsymbol{\omega} \times \boldsymbol{b}_i) \right) \tag{2.24}$$

The translational mean-axis condition implies $\sum_{i=1}^{n} m_i \boldsymbol{b}_i = 0$ as shown in Theorem 1. Hence the first term involving $\dot{\boldsymbol{r}}_B$ drops out of the expression. The second term is simply the internal angular momentum $\boldsymbol{H}_{int}$ as defined in Equation 2.12. The third term can be rewritten using the vector triple product identity:

$$\sum_{i=1}^{n} m_i \left(\boldsymbol{b}_i \times (\boldsymbol{\omega} \times \boldsymbol{b}_i) \right) = \sum_{i=1}^{n} m_i \left(|\boldsymbol{b}_i|^2 \boldsymbol{\omega} - (\boldsymbol{b}_i \cdot \boldsymbol{\omega}) \boldsymbol{b}_i \right) \tag{2.25}$$

This term is simply $\boldsymbol{J}\boldsymbol{\omega}$ where $\boldsymbol{J}$ is the instantaneous moment of inertia tensor.[3] This represents the angular momentum associated with the rotation of the frame itself. The moment of inertia tensor $\boldsymbol{J}$ depends on the particle locations $\boldsymbol{b}_i$ expressed in the body frame B. The vectors $\boldsymbol{b}_i$ can vary in time due to deformations and hence $\boldsymbol{J}$ can also vary in time. To summarize, if the translational mean-axis condition holds then the total angular momentum is:

$$\boldsymbol{H}_{tot} = \boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{H}_{int} \tag{2.27}$$

Combining Equations 2.22 and 2.27 yields the following dynamic equation:

$$\left. \frac{d}{dt} \right|_I (\boldsymbol{J}\boldsymbol{\omega} + \boldsymbol{H}_{int}) = \boldsymbol{M}_{ext} \tag{2.28}$$

This differential equation is used in the next theorem to provide an explicit equation of motion corresponding to the rotational mean-axis constraint.

[3]Let $[\omega_x, \omega_y, \omega_z]^T$ and $[x_i, y_i, z_i]^T$ be the components of the vectors $\boldsymbol{\omega}$ and $\boldsymbol{b}_i$ expressed in the body frame. Then the components of the vector $\sum_{i=1}^{n} m_i \left(|\boldsymbol{b}_i|^2 \boldsymbol{\omega} - (\boldsymbol{b}_i \cdot \boldsymbol{\omega}) \boldsymbol{b}_i \right)$ are given by:

$$\begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{2.26}$$

where $J_{xx} = \sum_{i=1}^{n} m_i(y_i^2 + z_i^2)$, $J_{xy} = -\sum_{i=1}^{n} m_i(x_i y_i)$, etc.

Theorem 2. Assume the frame B satisfies the translational mean-axis constraint (Definition 1.A). Then B also satisfies the rotational mean-axis constraint (Definition 1.B) if and only if $\boldsymbol{\omega}$ satisfies the following equation of motion:

$$\begin{aligned}&\left.\frac{d}{dt}\right|_I (\boldsymbol{J}\boldsymbol{\omega}) = \boldsymbol{M}_{ext}\\&\text{Initial Condition: } \boldsymbol{\omega}(0) = \boldsymbol{J}^{-1}(0)\boldsymbol{H}_{tot}(0)\end{aligned} \tag{2.29}$$

Proof. ($\Rightarrow$) Assume the frame B satisfies the rotational mean-axis constraint (Definition 1.B), i.e. $\boldsymbol{H}_{int}(t) = 0$ for all $t \geq 0$. Hence $\boldsymbol{H}_{tot} = \boldsymbol{J}\boldsymbol{\omega}$ by Equation 2.27. Thus $\boldsymbol{H}_{tot}(0) = \boldsymbol{J}(0)\boldsymbol{\omega}(0)$ and the dynamics in Equation 2.29 follow by simplifying Equation 2.28.

($\Leftarrow$) Assume the frame B satisfies the ODE and initial conditions in Equation 2.29. Then Equation 2.22 simplifies to the following unforced ODE:

$$\left.\frac{d}{dt}\right|_I (\boldsymbol{H}_{int}) = 0 \tag{2.30}$$

Moreover, $\boldsymbol{H}_{int}(0) = \boldsymbol{H}_{tot}(0) - \boldsymbol{J}(0)\boldsymbol{\omega}(0) = 0$ by the assumed initial condition for $\boldsymbol{\omega}$. Based on these initial conditions, the solution to the unforced ODE in Equation 2.30 is $\boldsymbol{H}_{int}(t) = 0$ for all $t \geq 0$. $\square$

This theorem provides a single vector, second-order differential equation (Equation 2.29) for the rotational motion of the mean axes. This corresponds to three scalar, second-order differential equations when expressed in components. The differential equation for $\boldsymbol{\omega}$ can be expanded using the Transport Theorem:

$$\boldsymbol{J}\dot{\boldsymbol{\omega}} + \mathring{\boldsymbol{J}}\,\boldsymbol{\omega} + \boldsymbol{\omega} \times (\boldsymbol{J}\boldsymbol{\omega}) = \boldsymbol{M}_{ext} \tag{2.31}$$

This is similar to the standard rotational equations of motion for a rigid body, except that $\boldsymbol{J}$ can vary in time due to deformations of the body. These time variations introduce the term $\mathring{\boldsymbol{J}}\boldsymbol{\omega}$ where $\mathring{\boldsymbol{J}}$ denotes the rate of change of the moment of inertia tensor measured in the body frame. For small deformations, the changes to the inertia tensor $\boldsymbol{J}$ may become negligibly small. In this case, it may be assumed that $\mathring{\boldsymbol{J}}\,\boldsymbol{\omega}$ is zero and Equation 2.31 reduces to $\boldsymbol{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\boldsymbol{J}\boldsymbol{\omega}) = \boldsymbol{M}_{ext}$. This is identical in form to the standard Newton-Euler equations for the rotational motion of a rigid body [6]. The ODE in Theorem 2 specifies an initial condition on the rate $\boldsymbol{\omega}$ but not on the initial orientation of the frame. As a result, the mean axes are only unique up to constant offsets in the orientation based on the specified initial conditions.

Several observations may be made about the similarities and differences of the mean-axis frame and rigid body equations of motion. The distinction between a rigid body reference frame and the mean-axis frame is important for a deformable body. Not only is the body deformable, the reference frame B is a body-reference frame but not a *body-fixed* frame. The reference frame for a rigid body is typically body-fixed, meaning that it is attached to a material point on the body. As previouly mentioned, the reference frame B is more abstractly related to the body using momentum constraints, which do not necessarily constrain the frame to a fixed material point. The mean-axis constraints result in equations of motion for the mean axes which are similar in form to a body-fixed frame for a rigid body, but a body-fixed frame for a deformable body may have significantly different equations of motion. However, the mean-axis frame becomes indistinguishable from a body-fixed frame if the stiffness increases without bound (i.e. the stiffness of the deformable body increases until it becomes a rigid body). Equivalently stated, the mean-axis reference frame for a rigid body is in fact a body-fixed frame.

Chapter 3

Linear Deformation and the Mean Axes

This chapter again considers a deformable body consisting of n particles, but with the additional assumption that the internal forces arise due to linear stiffness between the particles. The equations of motion are first derived in an inertial frame. This is mainly to introduce notation including the modal form for the dynamics, as well as important properties of the modal dynamics. This is followed by a derivation of the dynamics in a body-reference frame undergoing arbitrary motion. Finally, the dynamics in the mean-axis frame are calculated using the modal coordinates.

3.1 Equations of Motion in an Inertial Frame

The notation used to define the internal forces in an inertial frame is shown in Figure 3.1. The undeformed position of particle i is denoted by the vector $\boldsymbol{s}_i$. The vectors $\boldsymbol{s}_i$ are assumed to be constant and to satisfy $\sum_i m_i \boldsymbol{s}_i = 0$, i.e. the origin of the inertial frame is the center of mass when the particles are in their undeformed positions. The deformation of particle i from its undeformed position is denoted as $\boldsymbol{\delta}_i$. It is assumed that the internal forces are proportional to the deformation. Specifically, the force on particle i due to deformation $\boldsymbol{\delta}_j$ is given by $-\boldsymbol{K}_{ij}\boldsymbol{\delta}_j$. In addition, the body is assumed to be unrestrained so that the linear stiffness produces zero force due to a rigid body translation or (small) rotation. The assumption of an unrestrained body implies certain conditions on the stiffness matrices $\boldsymbol{K}_{ij}$ as discussed further below. The inertial position of particle i is given by $\boldsymbol{r}_i = \boldsymbol{s}_i + \boldsymbol{\delta}_i$. Thus $\dot{\boldsymbol{r}}_i = \dot{\boldsymbol{\delta}}_i$ and $\ddot{\boldsymbol{r}}_i = \ddot{\boldsymbol{\delta}}_i$ because $\boldsymbol{s}_i$ is assumed to be constant. Hence the dynamics for this

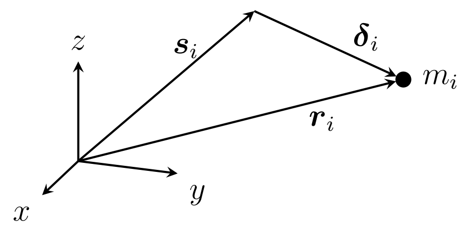

Figure 3.1: Position of particle i with $\boldsymbol{s}_i$ and $\boldsymbol{\delta}_i$ denoting the undeformed position and deformation in an inertial frame.

deformable body are specified by Newton's Second Law as:

$$\begin{aligned} m_i \ddot{\boldsymbol{\delta}}_i &= \boldsymbol{F}_i - \sum_{j=1}^{n} \boldsymbol{K}_{ij} \boldsymbol{\delta}_j \qquad \text{for } i = 1, \ldots, n \\ &\text{Initial Conditions: } \{\boldsymbol{\delta}_i(0)\}_{i=1}^{n} \text{ and } \{\dot{\boldsymbol{\delta}}_i(0)\}_{i=1}^{n} \end{aligned} \tag{3.1}$$

These equations of motion consist of n vector, second-order differential equations for the deformations $\boldsymbol{\delta}_i$. These dynamics can be rewritten as $3n$ scalar, second-order differential equations in terms of the (x, y, z) components of the various vectors. The component form is now given as it leads to the *modal form* for the dynamics. Let $\delta_i := [\delta_{i,x}, \delta_{i,y}, \delta_{i,z}]^T \in \mathbb{R}^3$ denote the components of the vector $\boldsymbol{\delta}_i$ $(i = 1, \ldots, n)$ expressed in the inertial frame.[1] Stack these components into a single vector: $\delta := [\delta_1^T, \delta_2^T, \ldots, \delta_n^T]^T \in \mathbb{R}^{3n}$. Moreover, define the block diagonal mass matrix as $M := \text{diag}(m_1 I_3, m_2 I_3, \ldots, m_n I_3) \in \mathbb{R}^{3n \times 3n}$. The external force vector $F \in \mathbb{R}^{3n}$ and stiffness matrix $K \in \mathbb{R}^{3n \times 3n}$ can be defined similarly. The equations of motion, expressed in these stacked inertial components, are given by:

$$\begin{aligned} &M\ddot{\delta} + K\delta = F \\ &\text{Initial Conditions: } \delta(0) \text{ and } \dot{\delta}(0) \end{aligned} \tag{3.2}$$

The remainder of this section summarizes known results related to the modal form of the system dynamics [9, 10]. The mass matrix M is symmetric (in fact, diagonal) and positive definite. The stiffness matrix K is assumed to be symmetric and positive semidefinite. Thus there exists a set of non-negative generalized eigenvalues $\lambda_i \in \mathbb{R}$ $(i = 1, \ldots, 3n)$ and corresponding eigenvectors $\Phi_i \in \mathbb{R}^{3n}$, also called *mode shapes*, such that $K\Phi_i = \lambda_i M \Phi_i$. The physical significance of these mode shapes is that any deformation may be expressed as a linear combination of the mode shapes (given that the deformation lies within a linear

[1] Bold font is reserved for true vectors and tensors while unbold font is used for variables with components (vectors and matrices) expressed in a particular frame.

regime). This means that the deformation may be written as $\delta(t) = \sum_{i=1}^{3n} \Phi_i \eta_i(t)$ for all t, where $\eta_i(t)$ are modal coordinates that dynamically scale the mode shapes. Note that the actual mode shapes Φ_i do not vary with time. Under the conditions described here (specifically a lack of damping), the mode shapes may be excited individually, and hence represent independent ways in which the body may move or deform. The assumption of an unrestrained body implies that there are six rigid body mode shapes and $3n - 6$ elastic mode shapes. Three of the rigid body mode shapes correspond to translation and can be concretely expressed as follows:

$$\Phi_{T,x} := \begin{bmatrix} \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right] \\ \vdots \\ \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right] \end{bmatrix}, \quad \Phi_{T,y} := \begin{bmatrix} \left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right] \\ \vdots \\ \left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right] \end{bmatrix}, \quad \Phi_{T,z} := \begin{bmatrix} \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right] \\ \vdots \\ \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right] \end{bmatrix} \tag{3.3}$$

The mode shapes $\Phi_{T,x}$, $\Phi_{T,y}$, and $\Phi_{T,z} \in \mathbb{R}^{3n}$ correspond to a translational deformation of each particle along the x, y, and z directions, respectively. For notational simplicity, these translational mode shapes are stacked together in the matrix $\Phi_T = [\Phi_{T,x}, \Phi_{T,y}, \Phi_{T,z}] \in \mathbb{R}^{3n\times 3}$. The other three rigid body mode shapes correspond to (small) rotations about the coordinate axes. Consider, for example, a small rotation of angle θ_x about the x axis. This will shift particle i from the undeformed position s_i to the deformed position $s_i + \left(\theta_x \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]\right) \times s_i$. This corresponds to the deformation $\delta_i = -s_i \times \left(\theta_x \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]\right)$. This deformation can be rewritten as a matrix / vector multiplication:

$$\delta_i = -s_i^{\times} \left(\theta_x \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]\right) \quad \text{where} \quad s_i^{\times} := \begin{bmatrix} 0 & -s_{i,z} & s_{i,y} \\ s_{i,z} & 0 & -s_{i,x} \\ -s_{i,y} & s_{i,x} & 0 \end{bmatrix} \tag{3.4}$$

Here, the superscript in $s_i^{\times}$ denotes the skew-symmetric cross-product matrix formed from the vector s_i. Thus the three rotational mode shapes can be expressed as follows (normalizing the rotational angle to $\theta_\cdot = 1$):

$$\Phi_{R,x} = \begin{bmatrix} -s_1^{\times} \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right] \\ \vdots \\ -s_n^{\times} \left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right] \end{bmatrix}, \Phi_{R,y} = \begin{bmatrix} -s_1^{\times} \left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right] \\ \vdots \\ -s_n^{\times} \left[\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right] \end{bmatrix}, \Phi_{R,z} = \begin{bmatrix} -s_1^{\times} \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right] \\ \vdots \\ -s_n^{\times} \left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right] \end{bmatrix} \tag{3.5}$$

These rotational mode shapes are also combined as $\Phi_R = [\Phi_{R,x}, \Phi_{R,y}, \Phi_{R,z}] \in \mathbb{R}^{3n\times 3}$. The remaining $3n-6$ elastic mode shapes are combined and denoted as $\Phi_E \in \mathbb{R}^{3n\times(3n-6)}$. These are assumed, by convention, to be normalized as $\|\Phi_{E,i}\| = 1$ $(i = 1,\ldots,3n-6)$. All modes $3n$ shapes are stacked together as $\Phi := [\Phi_T, \Phi_R, \Phi_E] \in \mathbb{R}^{3n\times 3n}$.

The mode shapes diagonalize the mass and stiffness matrices. Specifically, the generalized mass matrix $\mathcal{M} := \Phi^T M \Phi$ and generalized stiffness matrix $\mathcal{K} := \Phi^T K \Phi$ are both diagonal. Moreover, no restoring forces arise due to rigid body motions of an unrestrained body, i.e. $K\Phi_T = K\Phi_R = 0$. These properties can be used to express the dynamics in modal form. Define a change of coordinates $\delta(t) := \Phi\eta(t)$ where $\eta(t) \in \mathbb{R}^{3n}$ is a vector of the modal coordinates. Substituting $\Phi\eta$ for δ in Equation 3.2 and left multiplying by Φ^T yields the following modal form for the dynamics in an inertial frame:

$$\begin{aligned}&\mathcal{M}\ddot{\eta} + \mathcal{K}\eta = \mathcal{F}\\&\text{Initial Conditions: } \eta(0) = \Phi^{-1}\delta(0) \text{ and } \dot{\eta}(0) = \Phi^{-1}\dot{\delta}(0)\end{aligned} \tag{3.6}$$

where $\mathcal{F} := \Phi^T F$ is the modal forcing. These dynamics consist of $3n$ scalar, second-order differential equations in modal coordinates η. The left side of the equations is decoupled because both $\mathcal{M}$ and $\mathcal{K}$ are diagonal matrices, although coupling may appear on the right side if the external forces depend on η or $\dot{\eta}$.

3.2 Equations of Motion in a Body-Reference Frame

This section derives the equations of motion in a body-reference frame using the notation shown in Figure 3.2. The undeformed position of particle i, relative to the reference frame B, is denoted by the vector $\boldsymbol{s}_i$. This vector is assumed to be constant in the body-referenced frame, i.e. $\overset{\circ}{\boldsymbol{s}}_i = 0$. The deformation of particle i from its undeformed position in the reference frame is denoted as $\boldsymbol{\delta}_i$. These vectors can also be written in component form. In this case, all vectors will be expressed in terms of their body-referenced (x', y', z') components. Let $\delta_i := [\delta_{i,x}, \delta_{i,y}, \delta_{i,z}]^T \in \mathbb{R}^3$ now denote the components of $\boldsymbol{\delta}_i$ $(i = 1,\ldots,n)$ expressed in the body frame B. Again stack these components into a single vector: $\delta := [\delta_1^T, \delta_2^T, \ldots, \delta_n^T]^T \in \mathbb{R}^{3n}$. The vectors $\boldsymbol{s}_i$, $\boldsymbol{b}_i$ and $\boldsymbol{F}_i$ are similarly expressed in terms of body frame components and stacked as s, b, and F. As in the previous subsection, it is assumed that the body is unrestrained with internal forces given by $-\boldsymbol{K}_{ij}\boldsymbol{\delta}_j$. Note that any translations or small rotations of the frame correspond to rigid body translations and rotations, and therefore will not generate any internal forces. Deformation in this frame will simply appear superimposed on some combination of rigid body modes. The inertial

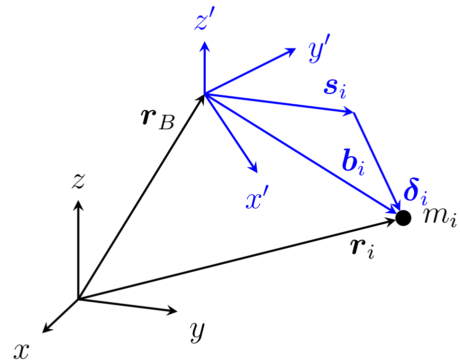

Figure 3.2: Position of particle i with $\boldsymbol{s}_i$ and $\boldsymbol{\delta}_i$ denoting the undeformed position and deformation in an the body-reference frame.

position of particle i is given by $\boldsymbol{r}_i = \boldsymbol{r}_B + \boldsymbol{b}_i$ where $\boldsymbol{b}_i = \boldsymbol{s}_i + \boldsymbol{\delta}_i$ is the position of the particle in the body-referenced frame. Note that $\overset{\circ}{\boldsymbol{b}}_i = \overset{\circ}{\boldsymbol{\delta}}_i$ and $\overset{\circ\circ}{\boldsymbol{b}}_i = \overset{\circ\circ}{\boldsymbol{\delta}}_i$ because $\boldsymbol{s}_i$ is assumed to be constant in the body frame. Hence the dynamics expressed using the body frame B (simplifying the equations previously given in Equation 2.8 for particle dynamics in a relative frame) are given below. To simplify notation, some terms are expressed using $\boldsymbol{b}_i$ rather than $\boldsymbol{s}_i + \boldsymbol{\delta}_i$.

$$m_i \left(\ddot{\boldsymbol{r}}_B + \overset{\circ\circ}{\boldsymbol{\delta}}_i + \dot{\boldsymbol{\omega}} \times \boldsymbol{b}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b}_i) + 2\boldsymbol{\omega} \times \overset{\circ}{\boldsymbol{\delta}}_i \right) = \boldsymbol{F}_i - \sum_{j=1}^{n} \boldsymbol{K}_{ij} \boldsymbol{\delta}_j \quad \text{for } i = 1, \ldots, n$$
$$\text{Initial Conditions: } \{\boldsymbol{\delta}_i(0)\}_{i=1}^n \text{ and } \{\overset{\circ}{\boldsymbol{\delta}}_i(0)\}_{i=1}^n \tag{3.7}$$

Moreover, let $\ddot{r}_B := [r_{B,x}, r_{B,y}, r_{B,z}]^T$ and $\omega := [\omega_x, \omega_y, \omega_z]^T$ denote the components of $\ddot{\boldsymbol{r}}_B$ and $\boldsymbol{\omega}$ expressed in frame B.[2] The mass matrix is defined, as before, as $M := \text{diag}(m_1 I_3, m_2 I_3, \ldots, m_n I_3) \in \mathbb{R}^{3n \times 3n}$. Finally, the matrices $\Omega^\times := \text{diag}(\omega^\times, \ldots, \omega^\times) \in \mathbb{R}^{3n \times 3n}$ and $\dot{\Omega}^\times := \text{diag}(\dot{\omega}^\times, \ldots, \dot{\omega}^\times) \in \mathbb{R}^{3n \times 3n}$ are used for cross-product terms. With this notation the equations of motion can be expressed in these body-referenced components as follows:

$$M \left(\Phi_T \ddot{r}_B + \ddot{\delta} + \dot{\Omega}^\times (s + \delta) + \Omega^\times \Omega^\times (s + \delta) + 2\Omega^\times \dot{\delta} \right) + K\delta = F$$
$$\text{Initial Conditions: } \delta(0) \text{ and } \dot{\delta}(0) \tag{3.8}$$

[2]Note that the components of $\ddot{r}_B$ must be rotated from frame B to frame I before integrating to obtain the velocity and position in terms of the inertial frame components.

In this equation $\frac{d}{dt}$ is simply the time derivative of the components and there is no need to distinguish between $\frac{d}{dt}|_I$ and $\frac{d}{dt}|_B$. Hence the simple overdot, e.g. $\dot{\delta}$, is used to represent all time derivatives. Also note that $\Phi_T := [I_3, \ldots, I_3]^T$ by definition of the translational mode shapes (Equation 3.3). Hence $\Phi_T \ddot{r}_B$ is simply $\ddot{r}_B$ stacked on itself: $[\ddot{r}_B^T, \ldots \ddot{r}_B^T]^T \in \mathbb{R}^{3n}$.

A related modal form can be derived for these body-referenced equations of motion. The modal form will first be derived for a body-referenced frame undergoing arbitrary motion (not necessarily the mean axes). Recall that the rotational mode shapes defined in Equation 3.5 are given by:

$$\Phi_R(s) := \begin{bmatrix} -s_1^\times \\ \vdots \\ -s_n^\times \end{bmatrix} \tag{3.9}$$

Here the dependence on the undeformed positions s_i is made explicit in the notation $\Phi_R(s)$. The modal form previously derived in the inertial frame (Equation 3.6) involved the change of coordinates $\delta = \Phi(s)\eta$ and left multiplication of the dynamic equations by $\Phi(s)^T$. The matrix $\Phi(s) := [\Phi_T, \Phi_R(s), \Phi_E]$ is used in both steps. The derivation in the body-referenced frame relies on one minor but important distinction. The change of coordinates $\delta = \Phi(s)\eta$ will again be used. However, Equation 3.8 will instead be left multiplied by $\Phi(b)^T := [\Phi_T, \Phi_R(b), \Phi_E]^T$. In other words, the deformed positions b will be used rather than the undeformed positions s in this left multiplication. This yields an equivalent and valid set of dynamic equations as long as $\Phi(b) \in \mathbb{R}^{3n \times 3n}$ is non-singular at each point in time (which will be assumed). This leads to the following modal form for the equations of motion:

$$\Phi_T^T \left(Ma + K\Phi(s)\eta\right) = \Phi_T^T F \tag{3.10a}$$

$$\Phi_R(b)^T \left(Ma + K\Phi(s)\eta\right) = \Phi_R(b)^T F \tag{3.10b}$$

$$\Phi_E^T \left(Ma + K\Phi(s)\eta\right) = \Phi_E^T F \tag{3.10c}$$

$$\text{Initial Conditions: } \eta(0) = \Phi(s)^{-1}\delta(0) \text{ and } \dot{\eta}(0) = \Phi(s)^{-1}\dot{\delta}(0) \tag{3.10d}$$

where the vector of particle accelerations $a \in \mathbb{R}^{3n}$ is defined to simplify the notation:

$$a := \Phi_T \, \ddot{r}_B + \Phi(s)\ddot{\eta} + \dot{\Omega}^\times(s + \Phi(s)\eta) + \Omega^\times\Omega^\times(s + \Phi(s)\eta) + 2\Omega^\times\Phi(s)\dot{\eta} \tag{3.11}$$

The three block equations in Equations 3.10a, 3.10b, and 3.10c will be referred to as the modal translational, rotational, and elastic dynamics, respectively. These equations are significantly more complicated than the ones derived in an inertial frame. However, the equations simplify when expressed in the mean-axis frame as shown in the next section.

3.3 Equations of Motion in the Mean-Axis Frame

Equation 3.10 describes the modal dynamics for the system of n particles in a body-reference frame. The system has $3n$ degrees of freedom and there are exactly $3n$ differential equations to describe their motion. The $3n$ degrees of freedom can be expanded in modal form as follows:

$$\delta = \Phi\eta = \Phi_T\eta_T + \Phi_R(s)\eta_R + \Phi_E\eta_E \tag{3.12}$$

The vectors $\eta_T \in \mathbb{R}^3$, $\eta_R \in \mathbb{R}^3$, and $\eta_E \in \mathbb{R}^{3n-6}$ represent the modal coordinates for the translational, (small) rotational, and elastic motion. The motion of the body frame adds another 6 degrees of freedom. These additional translational and rotational degrees of freedom are redundant, and hence the motion of the body frame can be chosen arbitrarily. This section considers the specific case of the mean-axis body frame. It is assumed that the vectors $\boldsymbol{s}_i$ describe particle positions relative to the center of mass of the undeformed shape, implying that $\sum m_i \boldsymbol{s}_i = 0$ for all time.[3] By Theorems 1 and 2, the mean-axis constraints are equivalent to the following dynamics for the translation and rotation of the body frame expressed in component form:

$$m_{tot}\ddot{r}_B = F_{ext} \tag{3.13a}$$

$$J\dot{\omega} + \dot{J}\omega + \omega \times (J\omega) = M_{ext} \tag{3.13b}$$

where $m_{tot} := \sum_{i=1}^{n} m_i$ is the total mass and $J := \sum_{i=1}^{n} m_i(b_i^T b_i I_3 - b_i b_i^T)$ is the instantaneous moment of inertia. Moreover, $F_{ext} := \sum_{i=1}^{n} F_i$ and $M_{ext} := \sum_{i=1}^{n} b_i \times F_i$ are the total external force and moment, respectively.

It is typical when using the mean axes to discard the modal rigid body degrees of freedom η_T and η_R. The corresponding modal translational and rotational dynamics (Equations 3.10a and 3.10b) are then replaced by the equations of motion for the mean-axis frame B (Equation 3.13a and 3.13b). However, discarding the rigid body degrees of freedom η_T

[3] This assumption is necessary for the mean-axis reference frame because the origin must always coincide with the center of mass, including conditions under which the body is not deformed. Specifically, $\sum_i m_i \boldsymbol{b}_i(t) = \sum_i m_i \boldsymbol{s}_i + \sum_i m_i \boldsymbol{\delta}_i(t) = 0$ and hence $\sum_i m_i \boldsymbol{s}_i = 0$ since $\sum_i m_i \boldsymbol{\delta}_i(t) = 0$ when $\boldsymbol{\delta}_i = 0$.

and η_R must be done with some care. In particular, it is not possible to both arbitrarily assign the motion of the body frame **and** set $\eta_T = \eta_R = 0$. This would only leave the remaining $3n - 6$ elastic degrees of freedom η_E to satisfy the $3n$ modal equations of motion in Equation 3.10, i.e. it would overconstrain the solution. The remainder of the section works through the details of this derivation.

First consider the modal translational dynamics in Equation 3.10a. These dynamics simplify to the following form after some straightforward but lengthy algebra (given in Appendix A.2):

$$m_{tot}\left(\ddot{r}_B + \ddot{\eta}_T + \dot{\omega}^\times \eta_T + \omega^\times \omega^\times \eta_T + 2\omega^\times \dot{\eta}_T\right) = F_{ext} \tag{3.14}$$

This is the component form of the analogous vector dynamics given in Equation 2.13 expressed with components in a relative frame. This differential equation is used in the next theorem.

Theorem 3. If the frame B satisfies the translational mean-axis constraint (Definition 1.A) then the modal translational dynamics in Equation 3.10a simplify to the following unforced ODE:

$$\begin{aligned} &m_{tot}\left(\ddot{\eta}_T + \dot{\omega}^\times \eta_T + \omega^\times \omega^\times \eta_T + 2\omega^\times \dot{\eta}_T\right) = 0 \\ &\text{Initial Conditions: } \eta_T(0) = 0 \text{ and } \dot{\eta}_T(0) = 0 \end{aligned} \tag{3.15}$$

Moreover, the solution of this unforced ODE is $\eta_T(t) = 0$ for all $t \geq 0$.

Proof. By Theorem 1, if frame B satsfies the translational mean-axis constraint then $m_{tot}\ddot{r}_B = F_{ext}$. Thus the simplified modal translational dynamics (Equation 3.14) reduce to those given in Equation 3.15. The initial conditions $\eta_T(0) = 0$ and $\dot{\eta}_T(0) = 0$ also follow from Theorem 1. In particular, the translational mean-axis constraint implies $\sum_{i=1}^{n} m_i \boldsymbol{b}_i(0) = 0$ and $\sum_{i=1}^{m} m_i \mathring{\boldsymbol{b}}_i(0) = 0$. Note that these initial conditions can be expressed in component form as $\Phi_T^T M b(0) = 0$ and $\Phi_T^T M \dot{b}(0) = 0$. Furthermore, the assumption that $\sum_{i=1}^{n} m_i \boldsymbol{s}_i = 0$ (or in component form $\Phi_T^T M s = 0$) reduces these initial conditions to $\Phi_T^T M \Phi \eta(0) = 0$ and $\Phi_T^T M \Phi \dot{\eta}(0) = 0$. Thus $m_{tot}\eta_T(0) = 0$ and $m_{tot}\dot{\eta}_T(0) = 0$ by orthogonality of the mode shapes and the form of Φ_T. Thus, $\eta_T(t) = 0$ follows from equation 3.15 and the corresponding initial conditions.

□

Theorem 3 justifies setting $\eta_T \equiv 0$ and replacing the modal translational dynamics in Equation 3.10a by the mean-axis translational dynamics in Equation 3.13a. In particular, the theorem shows that the modal translational dynamics are trivially satisfied by $\eta_T(t) = 0$ when the body frame satisfies the translational mean-axis conditions. Hence the modal coordinate η_T and its associated dynamics can be discarded. This result is not surprising, given that Theorem 1 implies that the origin of the mean-axis frame is coincident with the center of mass for all time. This may be stated as $\Phi_T^T M \Phi \eta(t) = 0$ and $\Phi_T^T M \Phi \dot{\eta}(t) = 0$, which imply that $\eta(t) = 0$ and $\dot{\eta}(t) = 0$ as shown in the proof of Theorem 3.

Next consider the rotational dynamics in Equation 3.10b. Assume that frame B satisfies the translational mean-axis constraint and hence $\eta_T \equiv 0$ by Theorem 3. Then the rotational dynamics simplify to the following form after some straightforward but lengthy algebra (given in Appendix A.3):

$$\begin{aligned} &\Big(J\dot{\omega} + \dot{J}\omega + \omega \times (J\omega)\Big) + \left(J_{rig}\ddot{\eta}_R + \omega \times (J_{rig}\dot{\eta}_R)\right) \\ +&\left(\sum_{i=1}^{n} \delta_i\left(\eta_R, \eta_E\right) \times m_i \ddot{\delta}_i\left(\eta_R, \eta_E\right) + \omega \times \sum_{i=1}^{n}\Big(\delta_i\left(\eta_R, \eta_E\right) \times m_i \dot{\delta}_i\left(\eta_R, \eta_E\right)\Big)\right) \\ &= M_{ext} + M_{int} \end{aligned} \tag{3.16}$$

where J is the instantaneous moment of inertia (as defined earlier), $J_{rig} := \sum_{i=1}^{n} m_i(s_i^T s_i I_3 - s_i s_i^T)$ is the moment of inertia in the undeformed (rigid body) position, and $\delta_i\left(\eta_R, \eta_E\right)$ is displacement in the reference frame due to (small) modal rotations and elastic deformation. Equation 3.16 is the component form of the analogous vector dynamics given in Equation 2.22.

The first three terms of the first line of Equation 3.16, grouped in parentheses, represent the change in angular momentum associated with the rotation of the body frame and the instantaneous inertia tensor, J, which varies with deformation. The remaining terms on the first and second line are exactly the rate of change of internal angular momentum, H_{int}, as shown in Appendix A.4 (recall that the total angular momentum is $H_{tot} = J\omega + H_{int}$). The second grouping of terms of the first line, specifically $(J_{rig}\ddot{\eta}_R + \omega \times (J_{rig}\dot{\eta}_R))$, represents the rate of change of the internal angular momentum from first-order effects. These effects are associated with (small) modal rotations only, not elastic deformation.

The remaining terms, grouped together on the second line, represent the rate of change of the internal angular momentum due to second-order effects. The terms affected by both elastic and rotational displacement are expressed in terms of δ_i and its derivatives to highlight that this is changing internal angular momentum due to a nonlinear, second-order effect. Specifically, Theorem 3 says that $\eta_T = 0$, and hence translational motion will not affect these deformation terms but they may be the result of both small rotations and small elastic deformation. Also note that the moment on the right hand side of this equation is the summation of the total external moment M_{ext} as well as the total internal moment M_{int}. If the model is perfect, there will be no internal moment since all forces are equal and opposite. In general, however, there may be modeling errors, e.g. due to the use of (linearized) stiffness matrices, which lead to non-zero internal moments. Thus, this term is retained here for clarity.

The second-order terms are a source of difficulty in the derivation and simplification of the equations in the mean-axis frame. In some cases, it is appropriate to assume that the elastic deformation occurs primarily in one direction within the body frame. This implies that the elastic deflection is collinear. If the deflection is collinear, it follows that its derivatives are also collinear to the deflection. Hence $\delta_{E,i} \times m_i \dot{\delta}_{E,i} = \delta_{E,i} \times m_i \ddot{\delta}_{E,i} = 0$ where the subscript E indicates that the deformation is due to elastic motion only. This assumption of collinearity is typically valid for beam and plate-like structures, e.g. aircraft [3, 8], and it will be used to simplify the equations of motion that follow.

Theorem 4. Assume the following: (1) frame B satisfies the translational mean-axis constraint (Definition 1.A), (2) the net internal moment M_{int} is zero, (3) the elastic deformation is collinear, and (4) the initial condition $\eta_R(0) = 0$ holds. If the frame B also satisfies the rotational mean-axis constraint (Definition 1.B), then the modal rotational dynamics in Equation 3.10b simplify to the following ODE:

$$\begin{aligned} J_{rig}\ddot{\eta}_R + \omega \times J_{rig}\dot{\eta}_R + \sum_{i=1}^{n} \Big(\delta_i\left(\eta_R, \eta_E\right) \times m_i \ddot{\delta}_i\left(\eta_R, \eta_E\right)\Big) + \\ \omega \times \sum_{i=1}^{n} \Big(\delta_i\left(\eta_R, \eta_E\right) \times m_i \dot{\delta}_i\left(\eta_R, \eta_E\right)\Big) = 0 \\ \text{Initial Conditions: } \eta_R(0) = 0 \text{ and } \dot{\eta}_R(0) = 0 \end{aligned} \tag{3.17}$$

Moreover, the solution of this ODE is $\eta_R(t) = 0$ for all $t \geq 0$.

Proof. By Theorem 2, if frame B satisfies the rotational mean-axis constraint then $J\dot{\omega} + \dot{J}\omega + \omega \times (J\omega) = M_{ext}$. In this case, Equation 3.16 simplifies to Equation 3.17 (recall that $M_{int} = 0$ by assumption). The initial condition $\dot{\eta}_R(0) = 0$ also follows from Theorem 2.

In particular, the rotational mean-axis constraint implies $\sum_{i=1}^{n} \boldsymbol{b}_i(0) \times m_i \overset{\circ}{\boldsymbol{b}}_i(0) = 0$. Note that this initial condition can be expressed in component form as $\sum_{i=1}^{n} b_i(0) \times m_i \dot{\delta}_i(0) = 0$, which simplifies to $\sum_{i=1}^{n} s_i \times m_i \dot{\delta}_i(0) = 0$ under the assumption of collinearity. This can be expressed in terms of mode shapes as $\Phi_R(s)^T M \Phi \dot{\eta}(0) = 0$. By orthogonality of mode shapes, this equation reduces to $\Phi_R(s)^T M \Phi_R(s) \dot{\eta}_R(0) = J_{rig} \dot{\eta}_R(0) = 0$.

Equation 3.17 can be written in an alternate form for simplification. The displacement $\delta_i\left(\eta_R, \eta_E\right)$ is a function of both elastic deformation and displacement due to (small) modal rotations. Specifically, the total displacement is a linear combination of these contributions: $\delta_i\left(\eta_R, \eta_E\right) = \delta_{E,i} + \delta_{R,i}$ (where $\delta_{E,i}$ is displacement due to elastic deformation and $\delta_{R,i}$ is displacement due to modal rotations). By the collinearity assumption, any cross products between elastic deformation and elastic deformation rates ($\delta_{E,i} \times \dot{\delta}_{E,i}$ and $\delta_{E,i} \times \ddot{\delta}_{E,i}$) are zero. Using these facts, Equation 3.17 can be expanded and written as follows:

$$\begin{aligned} J_{rig}\ddot{\eta}_R + \omega \times J_{rig}\dot{\eta}_R + \sum_{i=1}^{n} \left(\delta_{E,i} \times m_i \ddot{\delta}_{R,i} + \delta_{R,i} \times m_i \ddot{\delta}_{R,i} + \delta_{R,i} \times m_i \ddot{\delta}_{E,i}\right) + \\ \omega \times \sum_{i=1}^{n} \left(\delta_{E,i} \times m_i \dot{\delta}_{R,i} + \delta_{R,i} \times m_i \dot{\delta}_{R,i} + \delta_{R,i} \times m_i \dot{\delta}_{E,i}\right) = 0 \\ \text{Initial Conditions: } \eta_R(0) = 0 \text{ and } \dot{\eta}_R(0) = 0 \end{aligned} \tag{3.18}$$

The displacement has been decomposed into contributions from elastic motion $\delta_{E,i}$ and (small) modal rotations $\delta_{R,i}$. All cross products between elastic deformation terms become zero, leaving a set of cross products between terms depending on the rotational motion of the body. Given that $\delta_R = \Phi_R(s)\eta_R$ and $\dot{\delta}_R = \Phi_R(s)\dot{\eta}_R$, the initial conditions imply that $\delta_{R,i}(0) = \dot{\delta}_{R,i}(0) = 0$ in addition to $\eta_R(0) = \dot{\eta}_R(0) = 0$. With these initial conditions, the solution to the ODE in Equation 3.18 is $\eta_R(t) = 0$, which implies also that $\delta_{R,i}(t) = 0$.

□

This theorem justifies setting $\eta_R \equiv 0$ and replacing the modal rotational dynamics in Equation 3.10b by the mean-axis rotational dynamics in Equation 3.13b. As discussed previously, the mean axes (Definition 1) specify a constraint on rotational angular velocities. This leads to constant offsets in the angular orientation. However, to eliminate rotational motion within the mean-axis frame and therefore replace the modal rotational dynamics by the mean-axis rotational dynamics, Theorem 4 states that the rotational modal coordinates must always be equal to zero. Thus, if the initial orientation satisfies $\eta_R(0) = 0$ then

$\eta_R(t) = 0$ for all $t \geq 0$ and the ambiguity in rotational orientation is eliminated.

As previously mentioned, the rotational equation for the mean-axis reference frame B (Theorem 2) is $J\dot{\omega} + \dot{J}\omega + \omega \times (J\omega) = M_{ext}$. By Theorems 3 and 4, $\eta_T = \eta_R \equiv 0$, which implies that any change in the inertia tensor J is due to elastic motion η_E. Therefore, any terms in the rotational equation involving J couple the rotational motion of the mean-axis frame with the elastic deformation of the body. If it is assumed that changes in the inertia tensor are negligible, the rotational motion becomes decoupled from the elastic motion as J becomes a constant that only depends on the undeformed positions of the particles $\boldsymbol{s}_i$.

Theorem 4 requires the additional collinearity assumption and this limits its applicability to certain types of flexible structures. If the structure does not satisfy the collinearity assumption, then it is possible that an internal moment is present from linearization errors and the modal rotational dynamics in Equation 3.16 simplify to (assuming the body frame satisfies the mean-axis constraints):

$$\begin{aligned} J_{rig}\ddot{\eta}_R + \omega \times J_{rig}\dot{\eta}_R + \sum_{i=1}^{n} \delta_i\left(\eta_R, \eta_E\right) \times m_i\ddot{\delta}_i\left(\eta_R, \eta_E\right) \\ +\omega \times \sum_{i=1}^{n}\Big(\delta_i\left(\eta_R, \eta_E\right) \times m_i\dot{\delta}_i\left(\eta_R, \eta_E\right)\Big) = M_{int} \end{aligned} \tag{3.19}$$

This equation is simply a statement that the rate of change of internal angular momentum in the body frame B is equal to the net internal moment when the frame satisfies the equations of motion from Theorem 2 (which was derived under the assumption that the net internal moment M_{int} is zero). The nonlinear relationship given in Equation 3.19 specifies the dynamics of the modal rotational degrees of freedom. If the modal rotations and elastic deformations are small, then terms that are second-order in displacement $\delta_i\left(\eta_R, \eta_E\right)$ may be neglected. Furthermore, if the net internal moment is negligible, as would be expected for a linear approximation, then Equation 3.19 simplifies to $J_{rig}\ddot{\eta}_R + \omega \times J_{rig}\dot{\eta}_R \approx 0$. In a similar statement to Theorem 4, $\eta_R(t) \approx 0$ if $\eta_R(0) = \dot{\eta}_R(0) = 0$. This approximation is called the *practical mean-axis* or *linearized mean-axis* condition because the internal angular momentum constraint is only approximately satisfied [3, 7–9]. In other words, neglecting the second order elastic deformation terms is equivalent to:

$$H_{int} := \sum_{i=1}^{n} m_i b_i \times \dot{b}_i \approx \sum_{i-1}^{n} m_i s_i \times \dot{b}_i \tag{3.20}$$

The practical mean-axis constraint refers to the approximation $H_{int} \approx \sum_{i-1}^{n} m_i s_i \times \dot{b}_i = 0$ rather than the nonlinear constraint $H_{int} = 0$. For linear deformation in a mean-axis frame, it is shown in Appendix A.4 that $\sum_{i-1}^{n} m_i s_i \times \dot{b}_i = J_{rig}\dot{\eta}_R$, which agrees with the simplification of Equation 3.19. The effect of this approximation is a topic of ongoing consideration, though it is widely accepted as a reasonable approximation throughout the mean-axis literature. [2,3,8,9,11,12]

Finally, consider the elastic dynamics in Equation 3.10c. Continue to assume that frame B satisfies the translational and rotational mean-axis constraints. In addition, assume the structure satisfies the collinearity assumption. Hence $\eta_T = \eta_R \equiv 0$ by Theorems 3 and 4. Then the elastic dynamics simplify to the following form after some straightforward but lengthy algebra (given in Appendix A.5):

$$\mathcal{M}_E \ddot{\eta}_E + \mathcal{K}_E \eta_E + \Phi_E^T M \Omega^\times \Omega^\times (s + \Phi_E \eta_E) = \mathcal{F}_E \tag{3.21}$$

where $\mathcal{M}_E := \Phi_E^T M \Phi_E$ and $\mathcal{K}_E := \Phi_E^T K \Phi_E$, are the elastic modal mass and stiffness matrices. Recal that the modal mass and stiffness matrices are diagonal. Moreover, $\mathcal{F}_E := \Phi_E^T F$ is the elastic modal forcing due to external forces. This equation is similar to the standard vibrational equations of motion with the exception of the last term on the left side, which couples the deformation with the rotational rate of the body frame.

3.4 Final Equations of Motion

The final $3n$ equations of motion may be collected to fully specify the dynamics of the deformable body. The mean-axis translation, rotation, and modal coordinate displacement (Equations 3.13a, 3.13b, and 3.21 respectively) are now described in one system of equations:

$$\begin{aligned} m_{tot}\ddot{r}_B &= F_{ext} \\ J\dot{\omega} + \dot{J}\omega + \omega \times (J\omega) &= M_{ext} \\ \mathcal{M}_E \ddot{\eta}_E + \mathcal{K}_E \eta_E + \Phi_E^T M \Omega^\times \Omega^\times (s + \Phi_E \eta_E) &= \mathcal{F}_E \end{aligned} \tag{3.22}$$

Comparing Equation 3.22 to the modal dynamics for an arbitrary reference frame (Equations 3.10), it is clear that using the mean-axis frame has greatly simplified the equations of motion. Additionally, various assumptions can be made to further simplify the system. These assumptions revolve around additional decoupling of the rotational motion of the mean axes and the elastic deformation of the body. One such assumption is that the instantaneous inertia tensor is equivalent to the undeformed, or rigid body, inertia tensor

$J \approx J_{rig}$. In the context of aircraft, where the inertia tensor is typically large and the deformation is such that the change in the inertia tensor is usually small, it is commonly assumed that the change in the inertia tensor is negligible in comparison to the inertia tensor of the undeformed body.

There are several arguments about how the assumption that $J \approx J_{rig}$ affects the remaining coupling terms. The most common argument in the mean-axis literature suggests that if the inertia tensor is assumed constant, it may be assumed that all derivatives of the inertia tensor are zero [3]. This assumption will decouple the left hand side of the rotational equation of motion (second line) from Equation 3.22 by eliminating the term $\dot{J}\omega$. Moreover, it will also decouple the equations of motion for the elastic modal coordinates (third line). It may be shown that $\Phi_E^T M \Omega^\times \Omega^\times (s + \Phi_E \eta_E)$ is related to the change in the inertia tensor. Specifically, this term is related to the partial derivatives of $\frac{1}{2}\omega^T (J - J_{rig})\omega$ with respect to the elastic modal coordinates. The term $\frac{1}{2}\omega^T (J - J_{rig})\omega$ may be recognized as the kinetic energy associated with rotation of the body reference frame and a change in the inertia tensor. This alternate form of $\Phi_E^T M \Omega^\times \Omega^\times (s + \Phi_E \eta_E)$ is used in [3], and it is neglected accordingly. Hence, if all derivatives of the inertia tensor are assumed to be zero, then the equations of motion are fully *inertially decoupled*. Specifically, coupling arising from the acceleration terms has been removed and the only coupling that remains is through the aerodynamic forces and moments (which depend on translation, rotation, and elastic deformation).

Alternatively, it may be assumed that although the change in the inertia tensor is negligibly small, it is not constant and the rates of change may be significant. Therefore, J would be replaced by J_{rig}, but $\dot{J}$ would remain in the equations. In this case, the elastic equation would be unchanged and the rotational equation would become $J_{rig}\dot{\omega} + \dot{J}\omega + \omega \times (J_{rig}\omega) = M_{ext}$. It can be observed that the rotational and elastic coupling terms (specifically $\dot{J}\omega$ and $\Phi_E^T M \Omega^\times \Omega^\times (s + \Phi_E \eta_E)$) depend explicitly on the rotation rate ω. Therefore, in straight and level flight, or flight with only gentle maneuvers, these coupling terms will be very small due to the small value of ω. When the rotation rate is not negligibly small, however, it is unclear when the coupling terms may be neglected. If the coupling terms are considered as apparent moments and modal forces, it can be hypothesized that they will be very small compared to the aerodynamic moments and forces. Even as the coupling terms grow larger, the maneuvers will be more aggressive and the deflection more pronounced, which will in turn increase the aerodynamic forces and moments. When the instantaneous inertia tensor is replaced by the undeformed inertia tensor and the coupling terms are neglected, the form of the equations is greatly simplified.

$$\begin{aligned} m_{tot}\ddot{r}_B &= F_{ext} \\ J_{rig}\dot{\omega} + \omega \times (J_{rig}\omega) &= M_{ext} \\ \mathcal{M}_E\ddot{\eta}_E + \mathcal{K}_E\eta_E &= \mathcal{F}_E \end{aligned} \tag{3.23}$$

As previously mentioned, this simplified version of the equations of motion is inertially decoupled, though coupling will enter the system through the aerodynamic forces and moments, which depend on the motion of the mean axes as well as the deflection of the structure. In this simplified form, the reference frame translational and rotational equations of motion appear identical to those of a rigid body. Additionally, the equations of motion for the elastic coordinates appear identical to classic vibrational equations [10]. Due to the diagonal form of $\mathcal{M}_E$ and $\mathcal{K}_E$, the decoupling extends to the elastic modal coordinates. It should be noted that the aerodynamic force calculation is critical to the equations of motion. Aerodynamics are the dominant source of coupling, determining how the elastic states will interact with the mean axes. Although this topic is important, it is beyond the scope of this work and will not be discussed outside of the context of the example presented in Chapter 4.

3.5 Newtonian versus Lagrangian Derivation

The equations of motion presented here are fully derived with Newton's Laws, as opposed to the more commonly used Lagrangian method for the mean-axis equations of motion. [2, 3, 8, 12] It should be noted that the final equations of motion are consistent across both methods. Although the final equations are identical, the Newtonian derivation provides an approach that may be more natural for some readers. Furthermore, the mean-axis constraints are momentum-based, and appear more explicitly in the Newtonian derivation than in the energy-based Lagrangian derivation. In this way, it may be more obvious how the constraints enter in a simplifying manner.

It should also be noted that the rotational equations are completely derived without specifying a particular Euler angle rotation sequence. Although this is possible with the Lagrangian method, it requires an advanced technique using quasi-coordinates that is described in [13]. Thus, the Newtonian approach allows for a high level of abstraction throughout the derivation while only relying on basic first-principles. This level of abstraction is preferred so that the equations and assumptions can be stated in the most general form.

In the derivation presented, assumptions are made about the nature of the deformation (linear elastic deformation that satisfies collinearity), but no further assumptions are made until the equations are fully developed. This allows individual coupling terms to be included or neglected as desired. If the assumptions are made early on in the derivation, however, the Lagrangian approach has advantages over the Newtonian approach . If simplifying assumptions are made at the energy level, the kinetic energy decouples and the remainder of the derivation is straightforward and relatively simple [8]. By contrast, simplifications at the momentum level offer no clear advantage in the Newtonian approach to determining the equations of motion for the elastic coordinates, which is particularly laborious.

Chapter 4

Three Mass Example

4.1 Introduction to the Three Mass Example

As an illustrative example, a highly simplified aircraft is modeled as 3 lumped masses, or particles, and is shown in Figure 4.1. Particle b represents the fuselage with a mass of m_f, while particles a and c represent the wings with masses m_w. The masses are connected by rigid and massless elements of lengths l. Bending of the structure through an angle θ is resisted by a torsional spring with linear stiffness k. The aircraft is modeled to be

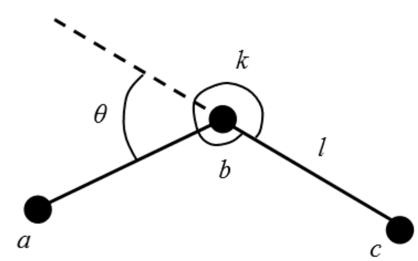

Figure 4.1: Lumped 3 Mass Structure.

representative of the Mini-MUTT, which is an unmanned testbed aircraft utilized by the Performance Adaptive Aeroelastic Wing (PAAW) project. The Mini-MUTT has a 10 ft. wing span and weighs 14.7 lbs. It is described in greater detail in [14]. The spring stiffness of the three-mass example is chosen such that the natural frequency of the linearized vibration is approximately 35 rad/s, which is near that of the first bending mode of the Mini-MUTT. [3] Specific numerical values are given in Appendix A.6.1.

In this example, the aircraft is restricted to planar motion in the inertial y-z plane, while maintaining a constant velocity in the forward x-direction. Figure 4.2 describes the variables used to specify the system. The position of the center of mass, given by the two-dimensional vector $\boldsymbol{r}_B$ (expressed in component form as r_B), denotes the origin of the reference frame, the angle ϕ denotes the orientation of the reference frame, and θ denotes the total angle of elastic deformation (as seen in Figure 4.1).

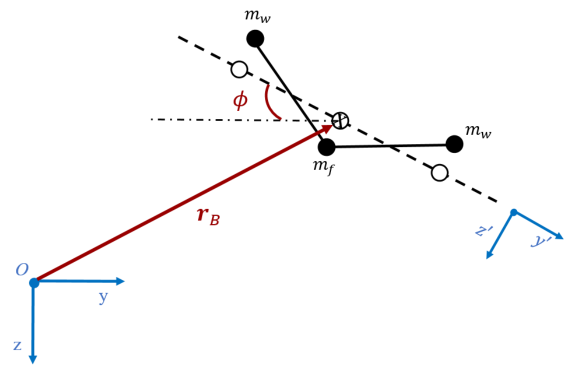

Figure 4.2: Variables describing the 3-Body Problem.

Because the body has planar motion and only three particles, the deformation consists of only one symmetric bending motion. The origin of the reference frame is located at the center of mass, which satisfies the translational mean-axis constraint (Definition 1.A) as shown by Theorem 1. The orientation is then chosen such that the y'-axis is aligned symmetrically between the wing masses a and c. In this way, the relative motion of the wing masses and the corresponding moment arms with respect to the center of mass are symmetric, preventing the generation of any net internal angular momentum and satisfying the nonlinear rotational mean-axis constraint (Definition 1.B).

4.2 The General Equations of Motion

For comparison with simplified equations, the general equations of motion for the three mass example were found (including the nonlinear effects of bending). Because the coordinates for the example were chosen to be consistent with the mean axes, they are more abstract than actual particle locations. For this reason, Lagrange's equations of motion for generalized coordinates were used, rather than Newtonian mechanics. The generalized coordinates are the previously defined r_B, ϕ, and θ. It can be shown that the general equations of motion for the three mass example are:

$$\begin{aligned}
m_{\text{tot}}\ddot{r}_B &= Q_B \\
\left[J_{\text{rig}}\cos^2\left(\frac{\theta}{2}\right) + \mathcal{M}_{\text{vib}}\sin^2\left(\frac{\theta}{2}\right)\right]\ddot{\phi} + \left(\mathcal{M}_{\text{vib}} - J_{\text{rig}}\right)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\dot{\theta}\dot{\phi} &= Q_\phi \\
\frac{1}{4}\left[J_{\text{rig}}\sin^2\left(\frac{\theta}{2}\right) + \mathcal{M}_{\text{vib}}\cos^2\left(\frac{\theta}{2}\right)\right]\ddot{\theta} + \left(J_{\text{rig}} - \mathcal{M}_{\text{vib}}\right)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{8}\dot{\theta}^2\right) + k\theta & \\
&= Q_\theta
\end{aligned} \tag{4.1}$$

where the Q terms are the generalized forces for the coordinates indicated.[1] Additionally, $m_{\text{tot}} = 2m_w + m_f$, $J_{\text{rig}} = 2m_w l^2$, and $\mathcal{M}_{\text{vib}} = (2m_w m_f l^2)/(2m_w + m_f)$. Q_B can be shown to be the sum of all forces acting on the body. The internal forces cancel and therefore $Q_B = F_{ext}$. Additionally, Q_ϕ can be shown to be the sum of all moments on the body about the center of mass ($M_{ext} + M_{int}$). In this example, all internal moments cancel and therefore $Q_\phi = M_{ext}$. All quantities in equation 4.1 are scalar, with the exception of the vector translational equation of motion. Note that F_{ext} is two-dimensional (it only account for forces in the y and z directions) and M_{ext} is scalar (it only accounts for the moment about the x axis). The translational equation may be decomposed and written as two scalar equations, but it is left in vector form here for compactness.

The form of the generalized force Q_θ corresponding to the generalized coordinate θ is more complex. Following the method of virtual work to find the generalized forces, as is often done with Lagrange's equations of motion [6], it can be shown that:

$$Q_\theta = \frac{l}{2}\left[\sin\left(\tfrac{\theta}{2}\right) \quad \tfrac{m_f}{2m_w+m_f}\cos\left(\tfrac{\theta}{2}\right) \quad 0 \quad -\tfrac{2m_w}{m_f+2m_w}\cos\left(\tfrac{\theta}{2}\right) \quad -\sin\left(\tfrac{\theta}{2}\right) \quad \tfrac{m_f}{2m_w+m_f}\cos\left(\tfrac{\theta}{2}\right)\right] F \tag{4.2}$$

where F is the vector comprised of the force components acting on the particles in the body frame. Specifically, $F = \left[F_{y',a} \quad F_{z',a} \quad \ldots \quad F_{y',c} \quad F_{z',c}\right]^T$.

If the elastic deformation is restricted to be small, then the small angle approximation for

[1] The nonlinear equations of motion were calculated with Lagrange's Equations, as outlined in standard dynamics references [6].

θ can be applied ($\sin\theta \approx \theta$ and $\cos\theta \approx 1$) and the equations become:

$$\begin{aligned} m_{\text{tot}}\ddot{r}_B &= F_{ext} \\ \left(J_{\text{rig}} + \frac{1}{4}\mathcal{M}_{vib}\theta^2\right)\ddot{\phi} + \frac{1}{2}\left(\mathcal{M}_{\text{vib}} - J_{\text{rig}}\right)\theta\dot{\theta}\dot{\phi} &= M_{ext} \\ \left(\frac{1}{4}\mathcal{M}_{vib} + \frac{1}{16}J_{rig}\theta^2\right)\ddot{\theta} + \frac{1}{2}\left(J_{rig} - \mathcal{M}_{vib}\right)\left(\frac{1}{8}\dot{\theta}^2 + \frac{1}{2}\dot{\phi}^2\right)\theta + k\theta &= Q_\theta \end{aligned} \tag{4.3}$$

In addition to the small angle approximation, the deformation may be assumed sufficiently small that terms quadratic in θ are negligible. Furthermore, if $\dot{\theta}$ and $\dot{\phi}$ are also sufficiently small, products of these terms may be assumed negligible in comparison to M_{ext} and Q_θ. Under these stronger assumptions, the equations become inertially decoupled:

$$\begin{aligned} m_{\text{tot}}\ddot{r}_B &= F_{ext} \\ J_{\text{rig}}\ddot{\phi} &= M_{ext} \\ \frac{1}{4}\mathcal{M}_{vib}\ddot{\theta} + k\theta &= Q_\theta \end{aligned} \tag{4.4}$$

4.3 Equations of Motion in Terms of Vibration Modal Coordinate

In order to analyze the three mass problem in the standard mean-axis framework, the linear vibration problem must first be solved. Considering the notation and mean axes defined in Figures 4.1 and 4.2, a relationship can be derived between the bending angle θ and the z' component of elastic deformation in the mean-axis frame. It can be shown that θ can be written as $\theta = \theta_{ab} + \theta_{bc}$, where θ_{ab} and θ_{bc} are shown in Figure 4.3.

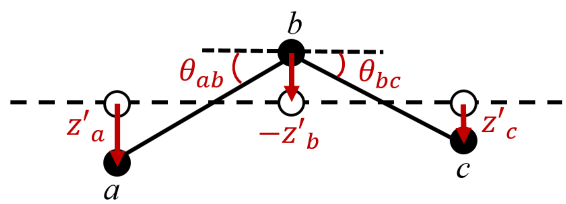

Figure 4.3: Bending and translational deflection.

From Figure 4.3 (and recalling that the wing length is l), it can be shown that $\theta_{ab} = \sin^{-1}\left(\frac{z'_a - z'_b}{l}\right)$ and $\theta_{bc} = \sin^{-1}\left(\frac{z'_c - z'_b}{l}\right)$. From this relationship, the total deflection θ can be written as:

$$\theta = \sin^{-1}\left(\frac{z'_a - z'_b}{l}\right) + \sin^{-1}\left(\frac{z'_c - z'_b}{l}\right) \tag{4.5}$$

The small angle assumption may be made to linearly relate θ to each particle's deflection in the mean-axis z'-direction:

$$\theta \approx \frac{z_a' - 2z_b' + z_c'}{l} \tag{4.6}$$

After some algebraic manipulations, the equations of motion for the linear vibration problem are as shown in Equation 4.7. It is assumed that the deformation occurs primarily in the mean-axis z'-direction and that any deformation in the mean-axis y'-direction is negligible. The linearized stiffness matrix can be found as shown in Appendix A.6.2, and the linear equations of motion for the particles in the local z' direction can be written as:

$$\begin{bmatrix} m_w & & \\ & m_f & \\ & & m_w \end{bmatrix} \begin{bmatrix} \ddot{z}_a' \\ \ddot{z}_b' \\ \ddot{z}_c' \end{bmatrix} + \frac{k}{l^2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_a' \\ z_b' \\ z_c' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4.7}$$

The mode shapes and natural frequencies of the modes are then found by solving the generalized eigenvalue problem. The single vibration mode shape can be expressed as

$$\Phi_{\mathrm{e}} = \begin{bmatrix} m_f \\ -2m_w \\ m_f \end{bmatrix} \alpha \tag{4.8}$$

where α is an arbitrary scale factor.[2]

Because the mode shape assumes all deformation occurs in the z'-direction, the deformation and deformation rates satisfy the collinearity assumption. Hence, it would be expected that the mode shape has the useful properties discussed in Chapter 3. Upon closer examination, it is clear that the elastic mode will not displace the center of mass of the body. Neither will it generate translational or angular momentum with respect to the mean axes, as expected. This implies that the mean-axis constraints will be satisfied if Equation 3.22 is used to write the equations of motion. The specific equations for the three mass example with linear vibrations are:

$$\begin{aligned} m_{\mathrm{tot}}\ddot{r}_B &= F_{ext} \\ \left(J_{\mathrm{rig}} + \mathcal{M}\eta^2\right)\ddot{\phi} + 2\mathcal{M}\,\eta\,\dot{\eta}\,\dot{\phi} &= M_{ext} \\ \mathcal{M}\ddot{\eta} - \mathcal{M}\dot{\phi}^2\eta + \mathcal{K}\eta &= \mathcal{F} \end{aligned} \tag{4.9}$$

[2]To normalize the mode shape with magnitude of 1 (as mentioned in Chapter 3), set α equal to $1/\sqrt{2(m_f^2 + 2m_w^2)}$. For simplicity of the equations, it is left as an arbitrary value here and has no effect on the analysis presented.

where $\mathcal{M} = 2m_w m_f(2m_w + m_f)\alpha^2$ and $\mathcal{K} = 4k(2m_w + m_f)^2\alpha^2/l^2$. In comparison to the general equations of motion in Equation 3.22, the number of equations are reduced due to the planar motion and restricted motion of the particles in the local z'-direction only. The vector r_B is two-dimensional rather than three-dimensional, and the angular orientation is given by a scalar, rather than a three-dimensional vector. Furthermore, there is only one elastic modal coordinate since Equation 4.7 yields two rigid body mode shapes and only one elastic mode shape. Additionally, planar motion also implies that the cross product term $\omega \times (J\omega)$ is zero.

Although the equations for the three mass example are simpler than the most general case, the coupling terms of interest are preserved. From here, many similarities can be observed between the structure and content of the full equations with linearized vibrations (Equation 4.9) and the small angle approximation of the nonlinear dynamics in Equation 4.3. Several differences may also be noted, however, which implies that the formulations are in fact distinct. Specifically, in Equation 4.1, the equation for the nonlinear bending with angle θ has several unique terms which depend on variable combinations such as $\theta^2\ddot{\theta}$ and $\dot{\theta}^2\theta$. Equation 4.9, with linear deformation, does not have analagous terms for these quantities.

4.4 The Decoupled Equations

As previously mentioned, it is common to neglect the inertial coupling terms. In this case, if the terms that are second-order in η or involve products of η, $\dot{\eta}$, or $\dot{\phi}$ are assumed to be sufficiently small, the equations simplify to:

$$\begin{aligned} m_{\text{tot}}\ddot{r}_B &= F_{ext} \\ J_{\text{rig}}\ddot{\phi} &= M_{ext} \\ \mathcal{M}\ddot{\eta} + \mathcal{K}\eta &= \mathcal{F} \end{aligned} \tag{4.10}$$

Equation 4.10 may also be calculated directly from the simplified general equations of motion (Equation 3.23). At this point, the simplified linear vibration equations of motion (Equation 4.10) may be compared to the nonlinear vibration equations of motion which assume small deformation and small angular rates (Equation 4.4). The translational and rotation equations are identical, with θ and η being the same except for some amplification. With the proper scaling of the mode shapes and modal coordinates, θ and η take on the same values and the equations of motion are equivalent.

Chapter 5

Three Mass Simulation and Analysis

5.1 Simulation Setup

Through simulation, the validity of the linear solution and the simplifying assumptions are explored. The simulation provides a time history of the states, the coupling terms, and the aerodynamic forces and moments. This allows for a detailed comparison of the nonlinear and linear results, as well as insight into how the coupling terms may affect those results. In order to provide meaningful simulations and assumption analysis, a simple aerodynamic force model was first created. Only lift acting on the wing masses (a and c) is considered in the model. Each wing mass has a control surface which may be deflected to generate lift. The surfaces are identical, and are assigned a lift coefficient and wing reference area consistent with the Mini-MUTT [14]. The model considers a constant forward velocity, and the control surfaces are trimmed accordingly. It is assumed that the lift always acts perpendicular to the wing, represented by the line connecting the fuselage mass and the wing mass in Figure 4.2. The angle of attack is a function of the wing mass velocity relative to the wing orientation and the control deflection.

$$L_a = \frac{1}{2}\rho_\infty V_\infty^2 \frac{S_\text{w}}{2} C_{L_\alpha}\alpha_a \qquad L_c = \frac{1}{2}\rho_\infty V_\infty^2 \frac{S_\text{w}}{2} C_{L_\alpha}\alpha_c \tag{5.1}$$

$$\alpha_a = \tan^{-1}(\frac{V_{\text{w},a}}{V_\infty}) + \xi_0 + \xi_a \qquad \alpha_c = \tan^{-1}(\frac{V_{\text{w},c}}{V_\infty}) + \xi_0 + \xi_c \tag{5.2}$$

where

$$
\begin{aligned}
L &= \text{lift on wing masses} \\
\rho_\infty &= \text{freestream air density} \\
V_\infty &= \text{freestream air velocity} \\
S_\text{w} &= \text{wing reference area for the entire span} \\
C_{L_\alpha} &= \text{coefficient of lift due to angle of attack} \\
\alpha &= \text{angle of attack} \\
V_\text{w} &= \text{component of the masses' inertial velocity perpendicular to the wing} \\
\xi_0 &= \text{trim control deflection} \\
\xi &= \text{control deflection relative to the trim}
\end{aligned}
$$

Specific numerical values for the aerodynamics are given in Appendix A.6.3.

The control surface input is selected in order to excite both the elastic deflection and the mean-axis rotation, which contribute to the neglected coupling terms and nonlinearities. The input consists of an asymmetric sinusoid superimposed on a higher frequency symmetric sinusoid. The asymmetric sinusoid excites the mean-axis rotation, and has a frequency of 6 rad/s, which was selected to achieve large angular rates without exceeding reasonable bank limits. The high frequency sinusoid excites the vibration mode, and has a frequency equal to the natural frequency of the bending mode, about 35 rad/s. Alternate control inputs, such as symmetric and asymmetric doublets were considered. The responses to most control inputs, however, did not manage to excite both the rotational and vibration degrees of freedom enough to produce interesting results.

5.2 Trajectory Analysis

A worst-case scenario of rotation and elastic coupling is generated for the simulation results. Using flight test data for the Mini-MUTT, upper bounds were estimated on the expected rotation rates and deflection. The upper bound on the rotation rates was determined to be around $\pm 290°$/s. The upper bound on the deflection corresponded to a bending angle of approximately $\pm 20°$. The largest coupling terms appear when the rotation rates and deformation achieve these upper limits simultaneously. In order to generate these conditions, the symmetric sinusoid is assigned an amplitude of 7.7°and the asymmetric sinusoid is assigned an amplitude of 11°. A short window of the resulting control inputs is given in

Figure 5.1. With these inputs, the deflection and rotation rates surpass the upper bounds

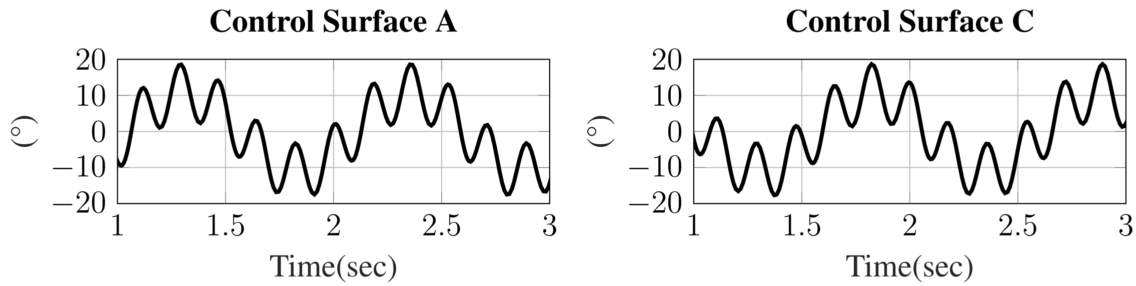

Figure 5.1: Control Surface Input to Generate Large but Expected Conditions.

simultaneously, as shown in Figure 5.2. A brief window of time from the simulation is shown. The window was selected for a time when the trajectories had most noticeable differences. Even with this conservative analysis, the trajectories of the rotation and elastic

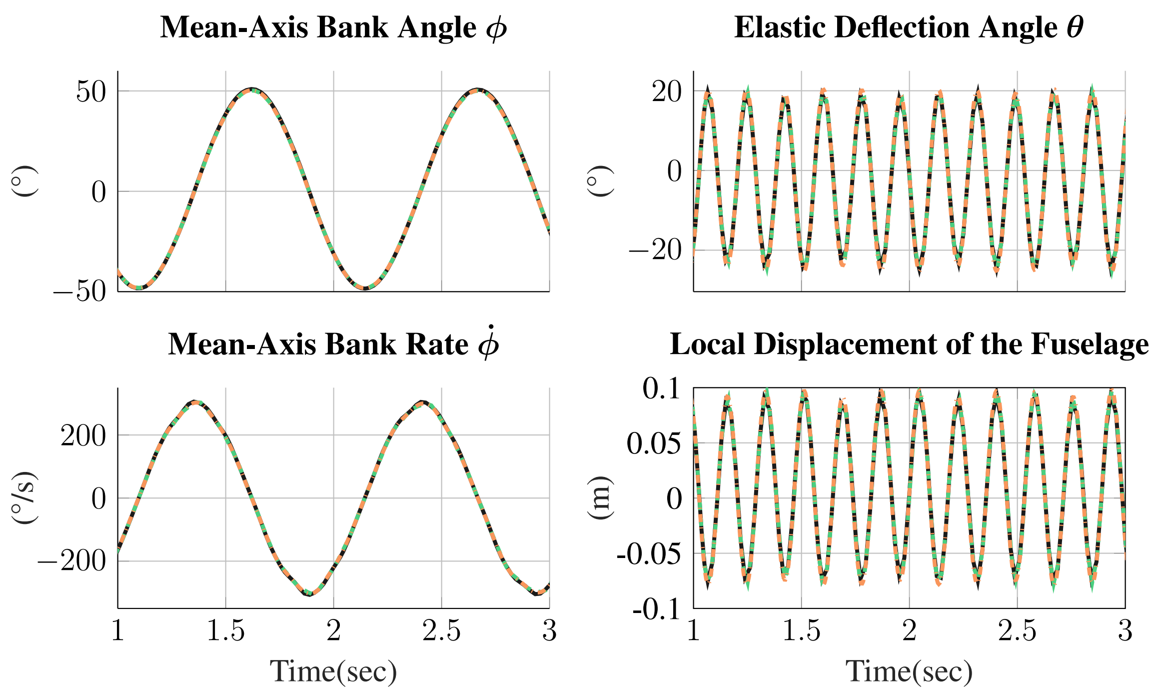

Figure 5.2: Bank and Deflection Trajectories
Exact 4.1 (——), Full Linear Vibration 4.9 (- - -), Simplified Linear Vibration 4.10 (-·-·)

degrees of freedom are nearly indistinguishable for the exact equations (Equation 4.1), full linear vibration equations (Equation 4.9), and simplified linear vibration equations (Equation 4.10). It should also be noted that the displacement of the fuselage from the center of mass never exceeds 10 cm. Even at the upper limits of deformation, the fuselage does not deform significantly relative to the mean-axis frame. Before drawing a conclusion about the effects of the coupling terms, the trajectory of the center of mass is presented in Figure 5.3. Upon an examination of the center of mass location, differences are more noticeable. The Y

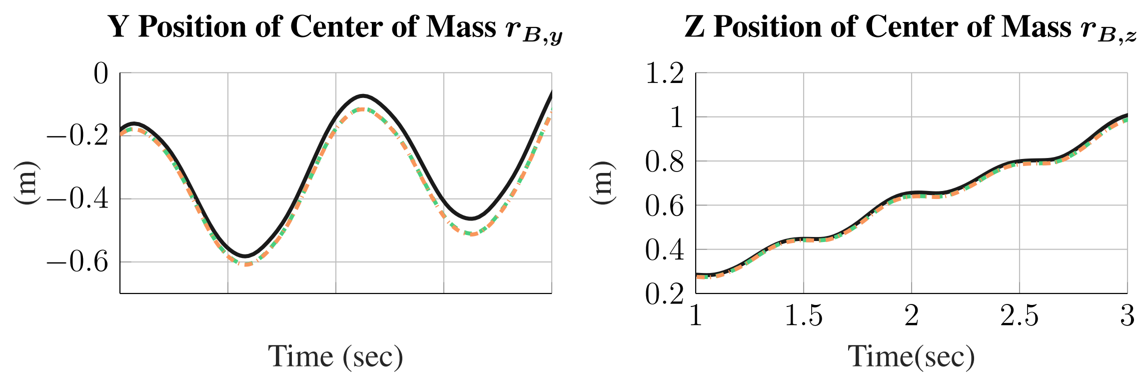

Figure 5.3: Center of Mass Trajectory
Exact 4.1 (—), Full Linear Vibration 4.9 (- - -), Simplified Linear Vibration 4.10 (-·-·)

component of the center of mass position has the most noticeable discrepancy, but it is still very small. Furthermore, the linear vibration solution with and without the coupling terms, Equations 4.9 and 4.10 respectively, appear identical. This would suggest that linearizing the vibrations has a small yet noticeable effect, but dropping the coupling terms produces negligible differences. As compared to the exact equations of motion (Equation 4.1), the root mean square of the error for selected quantities was averaged over the given time window for the full linear vibration solution (Equation 4.9) and the linear vibration solution without the coupling terms (Equation 4.10). The results are given in Table 5.1 .

Quantity	Full Linear Solution 4.9	Simplified Linear Solution 4.10
ϕ RMS Error	0.40°	0.38°
$\dot{\phi}$ RMS Error	3.42°/s	2.31°/s
θ RMS Error	0.37°	0.53°
$r_{B,y}$ RMS Error	3.73 cm	3.91 cm
$r_{B,z}$ RMS Error	1.26 cm	1.36 cm

Table 5.1: RMS Error for Three Mass Example Solutions

5.3 Coupling and Forcing Analysis

In order to understand why the solutions have such similar trajectories under the worst-case conditions simulated, the coupling terms are compared to other terms of interest. In order to clarify the analysis of the coupling terms, the equations of motion for the three mass example are rewritten (originally Equation 4.9) as:

$$\begin{aligned} m_{\text{tot}}\ddot{r}_B &= F_{ext} \\ \left(J_{\text{rig}} + \Delta J\right)\ddot{\phi} + \Omega_1 &= M_{ext} \\ \mathcal{M}\ddot{\eta} + \Omega_2 + \mathcal{K}\eta &= \mathcal{F} \end{aligned} \tag{5.3}$$

where $\Delta J = \mathcal{M}\eta^2$, $\Omega_1 = 2\mathcal{M}\,\eta\,\dot{\eta}\,\dot{\phi}$, and $\Omega_2 = -\mathcal{M}\dot{\phi}^2\eta$. The coupling terms Ω_1 and Ω_2 are compared to the aerodynamic forces and moments. Additionally, the inertia tensor change ΔJ is compared to the rigid body inertia tensor. Finally, because the coupling term Ω_2 is linear in η, it suugests that it may be viewed as a stiffening/softening term. Hence, Ω_2/η may be compared to the generalized stiffness $\mathcal{K}$. Because the ratios of these terms span such large values, they are displayed on a logarithmic plot. Figure 5.4 provides validation for the reasoning behind the simplifying assumptions. The aerodynamic forces and moments are almost always much larger than the apparent forces and moments from the coupling terms. Although the coupling terms do become comparable with the aerodynamics at several times, these instances are brief and their effects appear to be negligible based on Figures 5.2 and 5.3. Additionally, the ratios of Ω_2/η to $\mathcal{K}$ and ΔJ to J_{rig} appear quite small. To more explicitly quantify these ratios, the terms are averaged over the given time window and then the ratio of the averages is calculated. This calculation is presented in Table 5.2.

Ω_1 / Aero Moment	0.070
Ω_2 / Aero Modal Force	0.058
Ω_2 / Stiffness Force	0.010
ΔJ / J_{rig}	0.010
$\frac{\Omega_2}{\eta}$ / $\mathcal{K}$	0.010

Table 5.2: Numerical Comparison of Neglected Terms to Forcing Terms

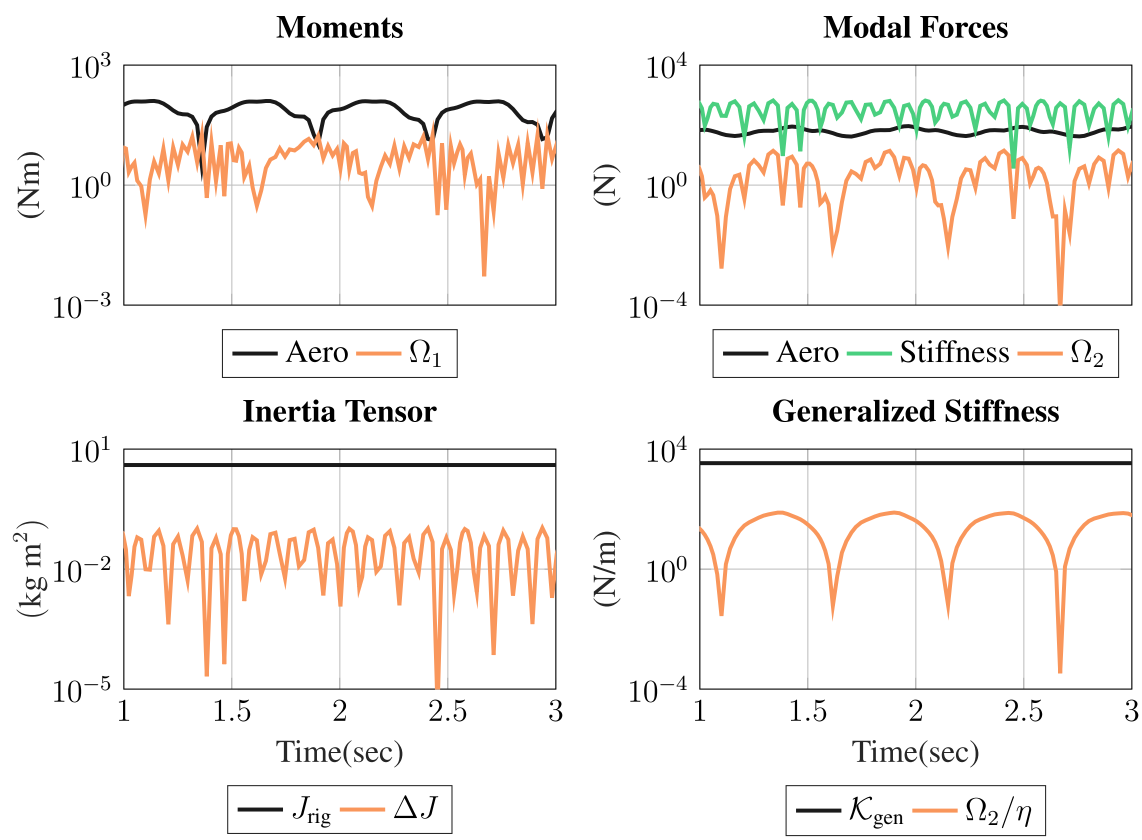

Figure 5.4: Coupling Term Comparison

5.4 Insight from the Three Mass Example

The simplicity of the three mass example allows for insight into a concept that can be challenging in general. One important point that the three mass example highlights is that the mean axes do not only apply to a body with linear vibration modes. The concept of the mean axes is very general, and its dynamics can be identified using equations 2.15 and 2.29, even if the body is deforming in some nonlinear manner. In general, the challenge is explicitly identifying the mean axes, visualizing it, and describing the elastic motion of the body with respect to the mean axes. A special property of the three mass example is that the mean axes can be explicitly identified and the elastic motion described without the use of linear vibration modes. The mean axes can be visualized as in figure 4.2 and the dynamics can be fully specified according to equation 4.1. A similar example of a body with nonlinear deformation that can be described using the mean axes is presented in [7].

Because the mean axes may be explicitly identified for the nonlinear elastic deformation of the three mass example, it offers an opportunity to compare the effects of linearizing the elastic deformation in this framework. In the context of the Mini-MUTT, the simulation results showed little difference between the nonlinear and linear deformation models. For an application with larger expected deflection, however, this type of example may be modified to approximate the deflection ranges for which the linear deformation model is sufficiently accurate.

Furthermore, this example is sufficiently simple to examine the coupling effects commonly neglected in the mean-axis approach. These assumptions are more difficult to validate for the full model created with finite element mode shapes and more complex aerodynamics. When the example is tailored to represent the more complicated system, it can offer some validation of the assumptions that the inertia tensor is constant and that the coupling terms are negligible.

Chapter 6

Conclusions and Future Work

A Newtonian derivation of the mean-axis equations of motion has been presented. These equations are consistent with the equations produced from Lagrangian derivations, but may offer new insight into the formulation of the mean-axis dynamics. The additional degrees of freedom resulting from the mean-axis frame were handled precisely, and simplifying assumptions with regard to the inertia tensor and additional coupling between rotation and deformation were discussed. Finally, an illustrative example of a highly simplified aircraft with lumped-mass and rolling motion was presented, and the assumptions were analyzed in the framework of this simple example. A time history of the simulated vehicle dynamics revealed that the solutions vary only slightly under conditions with aggressive maneuvering and large deformation. The simulation also allowed for a comparison of the coupling terms to the aerodynamic moments and generalized modal forcing. It was shown that it is reasonable to neglect the coupling terms for this example, even during extreme maneuvers and deflection. Furthermore, it was shown that use of the linearized vibration modes provides very close results to the equations of motion constructed with nonlinear deformation.

Some of the key assumptions made throughout this derivation of the mean-axis equations of motion are collinearity and a zero net internal moment of system. When these assumptions are invalid, several unanswered questions remain. Particularly, how should a non-zero internal moment be dealt with? Should it be ignored as a modeling or linearization error, or should it be incorporated into the flight dynamics model. In this case, what are the effects on conservation of angular momentum and conservation of energy? A model with deformation in multiple dimensions and more than one elastic mode shape will be required to investigate these effects. It may also be hypothesized that any of these errors are only significant if the model is being used outside of the range of validty for linearization, in which case these effects may be ignored. Future work in this area will require additional insight into structural modeling and the effects of linearization. Ultimately, this topic may provide more information on when it is valid to use the mean-axis modeling technique.

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Appendix A

Appendices

A.1 Useful Facts of Modal Matrices

Some useful facts regarding properties of the modal matrix are presented here for use throughout the Appendix.

1. $\Phi_T^T M \Phi_T = m_{tot} I_3$:
   Recall that $M = \text{diag}(m_1 I_3, \ldots, m_n I_3)$ and $\Phi_T := [I_3, \ldots, I_3]^T$ by definition of the translational mode shapes (Equation 3.3). Hence $\Phi_T^T M \Phi_T = \sum_{i=1}^n m_i I_3 := m_{tot} I_3$.

2. $\Phi_T^T M \Phi(s)\eta = m_{tot}\eta_T$:
   It follows from modal orthogonality that $\Phi_T^T M \Phi(s)\eta = \Phi_T^T M \Phi_T \eta_T$ (only the terms corresponding to the translational coordinates η_T remain). By Fact 1, $\Phi_T^T M \Phi_T \eta_T = m_{tot}\eta_T$. This is also true for derivatives of η_T. Hence, $\Phi_T^T M \Phi(s)\dot{\eta} = m_{tot}\dot{\eta}_T$ and $\Phi_T^T M \Phi(s)\ddot{\eta} = m_{tot}\ddot{\eta}_T$.

3. $\Phi_T^T M b = 0$ and $\Phi_T^T M s = 0$:
   Note that $\Phi_T^T M b = \sum_{i=1}^n m_i b_i$. This term is zero because the center of mass is assumed to be at the origin for all possible deformations. Equivalently stated, the mean-axis translational constraint (Definition A.2.A) implies that $\sum_{i=1}^n m_i b_i = 0$ by Theorem 1. Similarly, $\Phi_T^T M s = \sum_{i=1}^n m_i s_i$. This term is zero because the center of mass is assumed to be at the origin when the particles are in their undeformed positions.

4. $s + \Phi(s)\eta = s + \delta = b$:

   The deformation δ is described using a modal matrix computed with the undeformed shapes, $\Phi(s)$. Therefore, $\delta = \Phi(s)\eta$. Recall that $s + \delta = b$ and hence $s + \Phi(s)\eta = b$.

5. $\Phi_R(b) = \Phi_R(s) + \Phi_R(\delta)$:

   Recall that $\Phi_R(b)^T := [b_1^\times, \ldots, b_n^\times]^T$ by definition of the rotational mode shapes (Equation 3.5). Moreover, $b_i = s_i + \delta_i$ and $[b_1^\times, \ldots, b_n^\times]^T = [(s_1 + \delta_1)_1^\times, \ldots, (s_n + \delta_n)^\times]^T = [s_1^\times, \ldots, s_n^\times]^T + [\delta_1^\times, \ldots, \delta_n^\times]^T$. Hence $\Phi_R(b) = \Phi_R(s) + \Phi_R(\delta)$.

6. $J\omega = \sum_{i=1}^n m_i b_i \times (\omega \times b_i) = \Phi_R(b)^T M \Phi_R(b)\omega$
   $J = \sum_{i=1}^n m_i(b_i^T b_i I_3 - b_i b_i^T) = \Phi_R(b)^T M \Phi_R(b)$:

   Recall that the inertia tensor can be written in several ways. Let $[x_i, y_i, z_i]^T$ be the components of the vector $\boldsymbol{b}_i$ expressed in the body frame. The inertia tensor is defined as

   $$J = \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{bmatrix} \tag{A.1}$$

   where $J_{xx} = \sum_{i=1}^n m_i(y_i^2 + z_i^2)$, $J_{xy} = -\sum_{i=1}^n m_i(x_i y_i)$, etc. This can be written more concretely as $J = \sum_{i=1}^n m_i(b_i^T b_i I_3 - b_i b_i^T)$.

   The inertia tensor J can be multiplied by the angular velocity ω to write $J\omega = \sum_{i=1}^n m_i \left(|b_i|^2\omega - (b_i \cdot \omega)b_i\right)$. It follows from the vector triple product identity that this can equivalently written as $J\omega = \sum_{i=1}^n m_i b_i \times (\omega \times b_i)$. Finally, this term can be written in matrix form. Using the cross product property that $(a \times b) = -(b \times a)$, it can be seen that $\sum_{i=1}^n m_i b_i \times (\omega \times b_i) = -\sum_{i=1}^n m_i b_i \times (b_i \times \omega)$. This form for $J\omega$ can be written using the modal matrices as follows:

   $$J\omega = [-b_1^\times, \ldots, -b_n^\times] \begin{bmatrix} m_1 I_3 & & \\ & \ddots & \\ & & m_n I_3 \end{bmatrix} [b_1^\times \omega, \ldots, b_n^\times \omega] \tag{A.2}$$

   Thus it follows from Fact 5 that $J\omega = \Phi_R(b)^T M \Phi_R(b)\omega$. From the form of this term, it can be concluded that an additional form for the inertia tensor is $J = \Phi_R(b)^T M \Phi_R(b)$.

   Similarly, $J\dot{\omega} = \sum_{i=1}^n m_i \left(|b_i|^2\dot{\omega} - (b_i \cdot \dot{\omega})b_i\right) = \sum_{i=1}^n m_i b_i \times (\dot{\omega} \times b_i)$.

7. $J_{rig} = \Phi_R(s)^T M \Phi_R(s)$:
   From the matrix form of J given in the explanation of Fact 6, it can be concluded that substituting s for b in $J = \Phi_R(b)^T M \Phi_R(b)$ will yield $J_{rig} = \Phi_R(s)^T M \Phi_R(s)$, where J_{rig} is the rigid body (or undeformed) inertia tensor.

8. $J_{rig}\dot{\eta}_R = \Phi_R(s)^T M \Phi(s)\dot{\eta} = \sum_{i=1}^n m_i s_i \times \dot{\delta}_i = \sum_{i=1}^n m_i s_i \times \dot{\delta}_{R,i}$:
   Written in summation form, $\Phi_R(s)^T M \Phi(s)\dot{\eta} = \sum_{i=1}^n m_i s_i \times \dot{\delta}_i$. It follows from modal orthogonality that $\Phi_R(s)^T M \Phi(s)\dot{\eta} = \Phi_R(s)^T M \Phi_R(s)\dot{\eta}_R$. By Fact 7, this term is $J_{rig}\dot{\eta}_R$. Similarly, $\Phi_R(s)^T M \Phi(s)\ddot{\eta} = J_{rig}\ddot{\eta}_R$.

   Additionally, $\sum_{i=1}^n m_i s_i \times \dot{\delta}_{R,i} = \Phi_R^T(s) M \Phi_R(s)\dot{\eta}_R$. By Fact 7, this can be recognized as $J_{rig}\dot{\eta}_R$.

9. $\dot{J}\omega = \sum_{i=1}^n m_i b_i \times (\omega \times \dot{\delta}_i) + \sum_{i=1}^n m_i \dot{\delta}_i \times (\omega \times b_i)$:
   From Fact 6, the inertia tensor J can be written as $J = \Phi_R(b)^T M \Phi_R(b)$. Taking the derivative of this expression results in $\dot{J} = \Phi_R(b)^T M \Phi_R(\dot{\delta}) + \Phi_R(\dot{\delta})^T M \Phi_R(b)$. By the explanation for Fact 6, it can be seen that $\dot{J}\omega = \sum_{i=1}^n m_i b_i \times (\omega \times \dot{\delta}_i) + \sum_{i=1}^n m_i \dot{\delta}_i \times (\omega \times b_i)$.

A.2 Simplification of Translational Dynamics in Mean Axes

The body-referenced equations associated with multiplication by Φ_T^T are given Equation 3.10a and repeated below:

$$\Phi_T^T \left(Ma + K\Phi(s)\eta\right) = \Phi_T^T F \tag{3.10a}$$

where a is given by:

$$a := \Phi_T\, \ddot{r}_B + \Phi(s)\ddot{\eta} + \dot{\Omega}^\times(s + \Phi(s)\eta) + \Omega^\times\Omega^\times(s + \Phi(s)\eta) + 2\Omega^\times\Phi(s)\dot{\eta} \tag{A.3}$$

The various terms in this ODE can be simplified as follows:

1. $\Phi_T^T M \Phi_T \ddot{r}_B = m_{tot}\ddot{r}_B$: This is true by Fact 1.

2. $\Phi_T^T M \Phi(s)\ddot{\eta} = m_{tot}\ddot{\eta}_T$: This is true by Fact 2.

3. $\Phi_T^T M \dot{\Omega}^\times(s + \Phi(s)\eta) = m_{tot}\dot{\omega}^\times\eta_T$: Recall that $\dot{\Omega}^\times := \text{diag}(\dot{\omega}^\times, \ldots, \dot{\omega}^\times)$. Thus $\Phi_T^T M \dot{\Omega}^\times$ can be equivalently written as $\dot{\omega}^\times \Phi_T^T M$. Next, $\Phi_T^T M s = 0$ by Fact 3.

Moreover, $\Phi_T^T M \Phi(s)\eta = m_{tot}\eta_T$ by Fact 2. Combining these facts yields the desired simplification for this term.

4. $\Phi_T^T M \Omega^\times \Omega^\times (s + \Phi(s)\eta) = m_{tot}\omega^\times \omega^\times \eta_T$: Similar to the simplification of Term 3, this term can be rewritten and simplified usings Facts 2 and 3.

5. $2\Phi_T^T M \Omega^\times \Phi(s)\dot{\eta} = 2m_{tot}\omega^\times \dot{\eta}_T$: This involves similar steps to those given for the simplification of Term 3, using Facts 2 and 3.

6. $\Phi_T^T K \Phi(s)\eta = 0$: This term first simplifies to $\Phi_T^T K \Phi_T \eta_T$ by the modal orthogonality (similar to Fact 2). Then use the fact that rigid body translations yield no restoring force to conclude that $K\Phi_T = 0$ and this term must be zero.

7. $\Phi_T^T F = \sum_{i=1}^n F_i := F_{ext}$: This follows from the structure of $\Phi_T = [I_3, \ldots, I_3]^T$.

Combining these simplifications finally yields the following form for the translational dynamics:

$$m_{tot} \left(\ddot{r}_B + \ddot{\eta}_T + \dot{\omega}^\times \eta_T + \omega^\times \omega^\times \eta_T + 2\omega^\times \dot{\eta}_T \right) = F_{ext} \tag{A.4}$$

A.3 Simplification of Rotational Dynamics in Mean Axes

The body-referenced equations associated with multiplication by $\Phi_R(b)^T$ are given Equation 3.10b and repeated below:

$$\Phi_R(b)^T \left(Ma + K\Phi(s)\eta \right) = \Phi_R(b)^T F \tag{3.10b}$$

where a is again given by:

$$a := \Phi_T\, \ddot{r}_B + \Phi(s)\ddot{\eta} + \dot{\Omega}^\times (s + \Phi(s)\eta) + \Omega^\times \Omega^\times (s + \Phi(s)\eta) + 2\Omega^\times \Phi(s)\dot{\eta} \tag{A.5}$$

The various terms in this ODE can be simplified as follows:

1. $\Phi_R(b)^T M \Phi_T \ddot{r}_B = 0$: Recall that $M = \text{diag}(m_1 I_3, \ldots, m_n I_3)$ and $\Phi_R(b)^T := [b_1^\times, \ldots, b_n^\times]^T$ by definition of the rotational mode shapes (Equation 3.5). Hence $\Phi_R(b)^T M \Phi_T \ddot{r}_B = \sum_{i=1}^n m_i b_i \times \ddot{r}_B$. This can be rewritten as $-\ddot{r}_B \times \sum_{i=1}^n m_i b_i$, which is zero by Fact 3.

2. $\Phi_R(b)^T M\Phi(s)\ddot{\eta} = J_{rig}\ddot{\eta}_R + \sum_{i=1}^{n} m_i\delta_i \times \ddot{\delta}_i$: By Fact 5, this term may be rewritten as $\Phi_R(s)^T M\Phi(s)\ddot{\eta} + \Phi_R(\delta)^T M\Phi(s)\ddot{\eta}$. By Fact 8, $\Phi_R(s)^T M\Phi(s)\ddot{\eta} = J_{rig}\ddot{\eta}_R$. Finally, $\Phi_R(\delta)^T M\Phi(s)\ddot{\eta}$ can be rewritten as $\sum_{i=1}^{n} m_i\delta_i \times \ddot{\delta}_i$.

3. $\Phi_R(b)^T M\dot{\Omega}^\times(s+\Phi(s)\eta) = J\dot{\omega}$: By Fact 4, this term can be written as $\Phi_R(b)^T M\dot{\Omega}^\times b$. By the form of $\dot{\Omega}^\times$, this can be written as $\sum_{i=1}^{n} m_i b_i \times (\dot{\omega} \times b_i)$. By Fact 6, this term is equal to $J\dot{\omega}$.

4. $\Phi_R(b)^T M\Omega^\times\Omega^\times(s + \Phi(s)\eta) = \omega \times (J\omega)$: This term can be written as the summation $\sum_{i=1}^{n} m_i b_i \times (\omega \times (\omega \times b_i))$, which can be rewritten using the Jacobi identity[1] to yield $\sum_{i=1}^{n} m_i \left[(\omega \times b_i) \times (\omega \times b_i) + \omega \times (b_i \times (\omega \times b_i))\right]$. The first term in the summation is zero since a vector cross product with itself is zero. The expression may now be rewritten as $\omega \times \sum_{i=1}^{n} m_i (b_i \times (\omega \times b_i))$ which, by Fact 6, is known to be $\omega \times (J\omega)$.

5. $2\Phi_R(b)^T M\Omega^\times\Phi(s)\dot{\eta} = \dot{J}\omega + \omega \times (J_{rig}\dot{\eta}_R) + \omega \times \sum_{i=1}^{n} m_i \left(\delta_i \times \dot{\delta}_i\right)$: This term can be written in summation form to show that $2\Phi_R(b)^T M\Omega^\times\Phi(s)\dot{\eta} = 2\sum_{i=1}^{n} m_i b_i \times (\omega \times \dot{\delta}_i)$. This term can be split into two identical terms, and then one of the terms may be rewritten using the Jacobi Identity (defined previously) to yield $\sum_{i=1}^{n} m_i b_i \times (\omega \times \dot{\delta}_i) + \sum_{i=1}^{n} m_i b_i \times (\omega \times \dot{\delta}_i) = \sum_{i=1}^{n} m_i b_i \times (\omega \times \dot{\delta}_i) + \sum_{i=1}^{n} m_i\dot{\delta}_i \times (\omega \times b_i) + \omega \times \sum_{i=1}^{n} m_i(b_i \times \dot{\delta}_i)$. Substituting $s_i + \delta_i$ for b_i in the last additive term, this expression can be written as $\sum_{i=1}^{n} m_i b_i \times (\omega \times \dot{\delta}_i) + \sum_{i=1}^{n} m_i\dot{\delta}_i \times (\omega \times b_i) + \omega \times \sum_{i=1}^{n} m_i(s_i \times \dot{\delta}_i) + \omega \times \sum_{i=1}^{n} m_i(\delta_i \times \dot{\delta}_i)$. By Facts 8 and 9, this expression can be further rewritten as $\dot{J}\omega + \omega \times J_{rig}\dot{\eta}_R + \omega \times \sum_{i=1}^{n} m_i(\delta_i \times \dot{\delta}_i)$.

6. $\Phi_R(b)^T K\Phi(s)\eta = 0$: This term represents the moment due to internal forces in the body. This term can be separated into two terms as $\Phi_R(s)^T K\Phi(s)\eta + \Phi_R(\delta)^T K\Phi(s)\eta$. The first term simplifies to $\Phi_R(s)^T K\Phi_R(s)\eta_R$ by modal orthogonality. Because the rigid body rotation yields no restoring force, $K\Phi_R = 0$ and this term must be zero. Note that the second term may be written as $\sum_{i=1}^{n} \left(\delta_i \times \sum_{j=1}^{n} K_{ij}\delta_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} K_{ij} (\delta_i \times \delta_j)$. If the collinearity assumption is satisfied, this expression becomes $\sum_{i=1}^{n}\sum_{j\neq i}^{n} K_{ij} (\delta_i \times \delta_j)$. Moreover, because $K_{ij} = K_{ji}$ and $(\delta_i \times \delta_j) = -(\delta_j \times \delta_i)$, this expression simplifies to zero and hence $\Phi_R(b)^T K\Phi(s)\eta = 0$.

   Regardless of collinearity, it can be noted that the internal moment M_{int} should be zero according to first principles. As discussed prior to the presentation of Equation 2.1, the internal force on particle i due to particle j is denoted by $\boldsymbol{F}_{ij}$. By

[1]The Jacobi identity states that $a \times (b \times c) = c \times (b \times a) + b \times (a \times c)$.

Newton's Third Law, the internal forces between particles i and j are assumed to be equal and opposite: $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. Moreover, the internal forces are assumed to act along the line between the two particles: $\boldsymbol{F}_{ij} = |\boldsymbol{F}_{ij}|(\boldsymbol{r}_j - \boldsymbol{r}_i)$. Because the forces have equal magnitude and opposite direction, as well as collinearity, the moment created by $\boldsymbol{F}_{ij}$ will be exactly opposite to that created by $\boldsymbol{F}_{ji}$. Therefore, if the model is reasonably accurate and captures this fundamental property that follows from Newton's Third Law, the net internal moment M_{int} will be zero.

Conversely, if the model results in a non-zero net internal moment, i.e. $M_{int} \neq 0$, then conservation of angular momentum will be violated. A non-zero internal moment will result in a non-zero rate of change of angular momentum, even if no moment is generated by outside forces. Hence the system will not satisfy conservation of angular momentum.

In reality, however, if collinearity is not satisfied then a non-zero internal moment may exist due to modeling errors. For completeness, therefore, a non-zero net internal moment may be considered in the form of $\Phi_R(\delta)^T K \Phi(s)\eta = -M_{int}$. If the internal moment is retained, it must be added to the forcing of the mean-axis rotational dynamics in Equation 2.29.

7. $\Phi_R(b)^T F = \sum_{i=1}^n M_i := M_{ext}$: This follows from the structure of $\Phi_R(b)^T := \left[b_1^\times, \ldots, b_n^\times\right]$. Note that M_{ext} is the total moment due to external forces and does not account for any net internal moment. Hence $\Phi_R(b)^T F = \sum_i b_i \times F_i$ and this term is equal to $M_{ext} := \sum_i M_i$ as claimed.

A.4 Internal Angular Momentum and its Derivative in Mean Axes

In component form, the internal angular momentum in the mean-axis reference frame is:

$$H_{int} = \sum_{i=1}^{n} m_i b_i \times \dot{\delta}_i \tag{A.6}$$

Using Fact 4 and 8, the expression for internal angular momentum in the mean axes becomes:

$$H_{int} = J_{rig}\dot{\eta}_R + \sum_{i=1}^{n} m_i \delta_i \times \dot{\delta}_i \tag{A.7}$$

Applying the transport theorem, the rate of change of the internal angular momentum is:

$$\dot{H}_{int} = J_{rig}\ddot{\eta}_R + \omega \times (J_{rig}\dot{\eta}_R) + \sum_{i=1}^{n} \delta_i \times m_i\ddot{\delta}_i + \omega \times \sum_{i=1}^{n} \left(\delta_i \times m_i\dot{\delta}_i\right) \tag{A.8}$$

A.5 Simplification of Elastic Dynamics in Mean Axes

The body-referenced equations associated with multiplication by Φ_E^T are given Equation 3.10c and repeated below:

$$\Phi_E^T \left(Ma + K\Phi(s)\eta\right) = \Phi_E^T F \tag{3.10c}$$

where a is again given by:

$$a := \Phi_T\,\ddot{r}_B + \Phi(s)\ddot{\eta} + \dot{\Omega}^\times(s + \Phi(s)\eta) + \Omega^\times\Omega^\times(s + \Phi(s)\eta) + 2\Omega^\times\Phi(s)\dot{\eta} \tag{A.9}$$

Before simplifying the terms of the elastic dynamics, a useful property that follows from collinearity of elastic deformation will be derived. Recall that collinearity implies $\sum_{i=1}^{n} m_i\delta_{E,i} \times \dot{\delta}_{E,i} = 0$. To proceed the following new notation is needed:

$$\delta_E^\times(\eta_E) = \begin{bmatrix} -\delta_{E,1}^\times \\ \vdots \\ -\delta_{E,n}^\times \end{bmatrix} \tag{A.10}$$

where the superscript $\times$ indicates the skew symmetric matrix corresponding to the cross product. (η_E) is included here to emphasize that $\delta_E^\times$ is a function of the elastic modal coordinates, but it will subsequently be dropped. Now the collinearity expression can be rewritten in modal form as $\delta_E^{\times T}(\eta_E)M\Phi_E\dot{\eta}_E = 0$. This equality must hold for all possible solutions $\eta_E(t)$ and $\dot{\eta}_E(t)$. In particular, it must hold for all possible initial conditions $\eta_E(0)$ and $\dot{\eta}_E(0)$. These initial conditions can be specified independently and hence it must be true that:

$$\delta_E^{\times T} M\Phi_E = 0 \tag{A.11}$$

It can similarly be shown that

$$\dot{\delta}_E^{\times T} M\Phi_E = 0 \tag{A.12}$$

The various terms in the ODE 3.10c can now be simplified as follows:

1. $\Phi_E^T M \Phi_T \ddot{r}_B = 0$: Recall that the mode shapes are orthogonal through the mass matrix. By definition, this means that $\Phi_E^T M \Phi_T = 0$.

2. $\Phi_E^T M \Phi(s)\ddot{\eta} = \mathcal{M}_E \ddot{\eta}_E$: Modal orthogonality through the mass matrix implies that $\Phi_E^T M \Phi(s)\ddot{\eta} = \Phi_E^T M \Phi_E \ddot{\eta}_E$. Moreover, $\Phi_E^T M \Phi_E := \mathcal{M}_E$ where $\mathcal{M}_E$ is the generalized mass matrix associated with the elastic modal coordinates.

3. $\Phi_E^T M \dot{\Omega}^\times (s + \Phi(s)\eta) = 0$: This expression can be rewritten as $\Phi_E^T M \dot{\Omega}^\times s +$ $\Phi_E^T M \dot{\Omega}^\times \Phi(s)\eta$. Due to the form of $\dot{\Omega}^\times$, the first term can be written as $\Phi_E^T M \begin{bmatrix} s_1^\times \dots s_n^\times \end{bmatrix}^T \dot{\omega} = \Phi_E^T M \Phi_R(s)\dot{\omega}$. From modal orthogonality, $\Phi_E^T M \Phi_R(s) = 0$, and hence the first time is eliminated. The second term is simplified if collinearity is assumed and $\eta_R = 0$ (which requires the additional assumption that $M_{int} = 0$ as described in Theorem 4). Under the assumption that $\eta_R = 0$, the second term becomes $\Phi_E^T M \dot{\Omega}^\times \Phi_E \eta_E$. This can be written as $\Phi_E^T M \delta_E^\times \dot{\omega}$. It follows from equation A.11 that $\Phi_E^T M \delta_E^\times \dot{\omega} = \delta_E^{\times T} M \Phi_E = 0$, and hence the second term is also eliminated.

4. $\Phi_E^T M \Omega^\times \Omega^\times (s + \Phi(s)\eta) = \Phi_E^T M \Omega^\times \Omega^\times (s + \Phi_E \eta_E)$: This term simplifies slightly under the assumption that $\eta_R = 0$. $\Phi(s)\eta$ becomes $\Phi_E \eta_E$.

5. $2\Phi_E^T M \Omega^\times \Phi(s)\dot{\eta} = 0$: Under the assumption that $\eta_R = 0$, the expression becomes $2\Phi_E^T M \Omega^\times \Phi_E \dot{\eta}_E$. Due to the form of $\Omega^\times$, the expression can be written as $2\Phi_E^T M \dot{\delta}_E^\times \omega$. From equation A.12, $\Phi_E^T M \dot{\delta}_E^\times = 0$ and the term is eliminated.

6. $\Phi_E^T K \Phi(s)\eta = \mathcal{K}_E \eta_E$: This term represents the restoring modal forces due to the stiffness in the body. Due to modal orthogonality, $\Phi_E^T K \Phi(s)\eta = \Phi_E^T K \Phi_E \eta_E$. Moreover, $\Phi_E^T K \Phi_E := \mathcal{K}_E$ where $\mathcal{K}_E$ is the generalized stiffness matrix associated with the elastic modal coordinates.

7. $\Phi_E^T F := \mathcal{F}_E$: This term is defined as the elastic modal forcing $\mathcal{F}_E$.

A.6 Three Mass Example Supplementary Material

A.6.1 Numerical Values for the Three Mass Structure

Specific numerical values for the structure that makes up the simplified aircraft in the three mass example are:

m_f	5 kg
m_w	2 kg
l	1 m
k	692.9 Nm/rad

Table A.1: Numerical Values for the Three Mass Structure

A.6.2 Linearized Stiffness

In order to use linear equations to describe the deformation of the three mass example, the stiffness matrix must be calculated. The textbook technique involves taking partial derivatives of the elastic potential energy with respect to the motion of each particle, but an alternate approach is given here. Recall from Equation 4.6 that $\theta \approx \frac{z'_a - 2z'_b + z'_c}{l}$. This equation can be used to write theta in matrix form as:

$$\theta \approx \frac{1}{l} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} z' \tag{A.13}$$

where $z' = \begin{bmatrix} z'_a & z'_b & z'_c \end{bmatrix}^T$. Note also that the potential energy due to deformation may be written as:

$$V = \theta^T k \theta \approx (z')^T K z' \tag{A.14}$$

where V is the potential energy, k is the bending stiffness, and K is the linearized stiffness matrix. Note that θ is unaffected by the transpose since it is a scalar. Substituting Equation A.13 for θ in Equation A.14, it can be shown that:

$$(z')^T \frac{k}{l^2} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} z' = (z')^T K z' \tag{A.15}$$

From this equation, it can be concluded that:

$$K = \frac{k}{l^2} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} = \frac{k}{l^2} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \tag{A.16}$$

A.6.3 Numerical Values for the Three Mass Aerodynamics

Specific numerical values for the structure that makes up the simplified aircraft in the three mass example are:

ρ_∞	1.2266 kg/m^2
V_∞	27.432 m/s
S_W	1.068 m^2
C_{L_α}	4.5
ξ_0	$\frac{2g\left(2m_w+m_f\right)}{\rho_\infty V_\infty^2 S_W C_{L_\alpha}}$

Table A.2: Numerical Values for the Three Mass Aerodynamics