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Dynamic Mode Shaping for Fluid Flow Control: New Strategies for Transient Growth Suppression

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Sub-critical transition to turbulence is often attributed to transient energy growth that arises from non-normality of the linearized Navier-Stokes operator. Here, we introduce a new dynamic mode shaping perspective for transient growth suppression that focuses on using feedback control to shape the spectral properties of the linearized flow. Specifically, we propose a dynamic mode matching strategy that can be used to reduce non-normality and transient growth. We also propose a dynamic mode orthogonalization strategy that can be used to eliminate non-normality and fully suppress transient growth. Further, we formulate dynamic mode shaping strategies that aim to handle some of the practical challenges inherent to fluid flow control applications, namely high-dimensionality, nonlinearity, and uncertainty. Dynamic mode shaping methods are demonstrated on a number of simple illustrative examples that show the utility of this new perspective for transient growth suppression. The methods and perspectives introduced here will serve as a foundation for realizing effective flow control in the future.

I. Introduction

An ability to suppress transition to turbulence would enable dramatic reductions in skin-friction drag in a variety of engineering systems, inevitably leading to efficiency and performance enhancements across a broad range of application domains. In the context of wall-bounded flows, turbulent transition is often initiated by a *linear* transient growth mechanism [1–12]. The linearity of this transition mechanism has motivated numerous investigations on *linear* optimal and robust control strategies aimed at suppressing and understanding transition [13–23]. Optimal and robust controller synthesis strategies simplify the controller synthesis task by masking internal system design details from the user. Instead, the control engineer can simply focus on a set of "design knobs" that weigh intuitive measures of performance (e.g., balancing input energy with state-regulation error). In the context of fluid flow control, this simplification is greatly welcome, owing to the high-dimensionality of the state-space. We do note, however, that this simplification comes at a cost. By masking the internal system details from the user, optimal and robust controller synthesis frameworks do not easily lend themselves to leveraging (nor elucidating) physical insights. To this end, even with the simplifications that come with optimal and robust controller synthesis techniques, fluid flow controller design can still be quite daunting. Appropriately tuning the controller to realize an adequate control strategy remains something of an art.

Here, we propose an alternative perspective for flow control design—one that focuses on shaping the spectral properties of the linearized Navier-Stokes operator directly. It is the non-normality of the linearized Navier-Stokes operator that is responsible for transient growth [24]. While non-normality can be defined in various equivalent ways [25–27], a convenient definition is based on whether or not the modes of the operator are normal (orthogonal) to one another—i.e., if the modes are not orthogonal, then the operator is deemed to be non-normal. The possibility of transient growth due to non-normality of eigenmodes can be understood from a graphical comparison of the normal and non-normal systems shown in Figures 1(A) and 1(B), respectively. While both linear systems are stable and begin with the same initial state $\vec{x}(t_1)$,

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the non-normality of modal directions in Figure 1(B) gives rise to a future state $\vec{x}(t_2)$ that has a larger magnitude than it had at the initial time t_1.

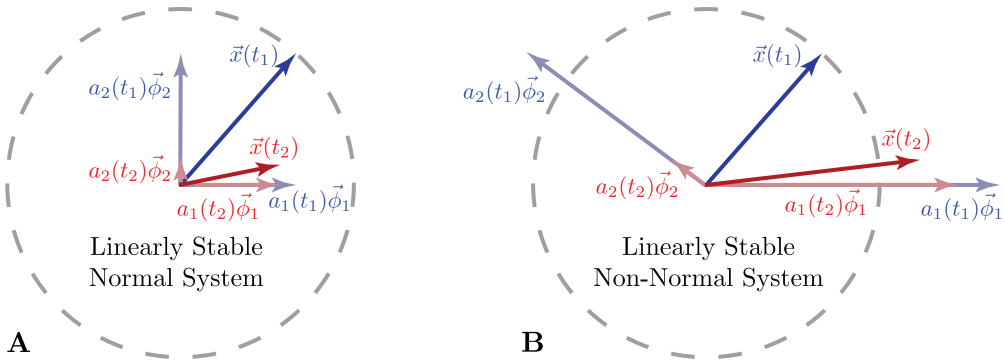


Figure 1: Non-normality between modes can lead to transient growth, even when individual modal components $a_i(t)\vec{\phi}_i$ are decaying. The normal system in (A) and the non-normal system in (B) both begin with the same initial condition $\vec{x}(t_1)$ and have the same stable eigenvalues; however, the state in (A) decays for all time, whereas the state in (B) initially grows before it decays to the origin. The dashed line in the figure corresponds to the magnitude of the state at the initial time t_1.

Of course, in the context of linear systems, stability implies that the state will ultimately decay to the origin; however, in the context of *linearized* fluid flows, transient growth allows disturbances to be amplified in the short-term, which can trigger nonlinear instabilities and transition—i.e., once the state moves beyond the basin of attraction, turbulent dynamics ensue [28]. It is also worth noting that the resulting transient growth becomes more pronounced as the non-normality between modes increases, less pronounced as non-normality decreases, and disappears completely in the case of normal modes. For instance, linearization of the Navier-Stokes equations for three-dimensional channel flow yields the Orr-Sommerfeld-Squire (OSS) operator. As Reynolds number (Re) increases, the modes of OSS become closer to parallel and transient growth becomes more pronounced. As a result, at higher Re, it becomes more likely that transient growth due to linear mechanisms will trigger the nonlinear processes that govern turbulent dynamics.

From this standpoint, it becomes apparent that a feedback control strategy aimed at mitigating transition to turbulence should be aimed directly at inhibiting transient growth due to non-normality—an objective that can be achieved by *shaping* the modes of the closed-loop system to be orthogonal to one another. One can understand this notion in the context of the simple system of Figure 1: can feedback control be leveraged to shape the non-normal system (B) into the normal system (A)? Indeed, addressing this general question will be the focus of the present study. In particular, we will view the control problem from the standpoint of a novel perspective for fluid flow control, which we call *dynamic mode shaping*. While some of the theoretical work concerning dynamic mode shaping has been considered previously within the controls community—under the name of *eigenstructure assignment* (ESA) [29, 30]—the present work addresses a number of issues that currently stand in the way of leveraging the dynamic mode shaping perspective for *fluid flow control*. In addition to our contributions toward a synthesis of dynamic mode orthogonalization controllers, we also address challenges of high-dimensionality, nonlinearity, and uncertainty that will be essential for relevance to practical fluid flow configurations.

As noted in [29], the solution to the pole placement problem of modern control theory is not unique, which offers additional flexibility to the control designer to specify the shape of the closed-loop modes as well. Although ESA is a familiar technique of modern control theory, the challenges associated with specifying desirable spectral characteristics for a closed-loop system (i.e., closed-loop eigenvalues and eigenvectors) has led control theorists to favor optimal and robust control methodologies, even with the demands of subsequent controller tuning.

We note here, however, that the challenge of spectral specification is less daunting in the context of transient growth suppression—at the very least we know that we should target an orthogonal set of closed-loop modes. For a stable linear system, a set of orthogonal modes is a *sufficient* condition to guarantee monotonic decay of trajectories. Further, although normality is not a *necessary* condition for monotonic decay, examination of the pseudospectrum for a normal operator reveals desirable robustness properties that make closed-loop orthgonalization an attractive choice for control in the context of uncertain systems.

Perturbations to a normal operator yield well-behaved perturbations in eigenvalues; in contrast, non-normal operators are prone to undesirable sensitivities to such perturbations [25]. Hence, although orthogonality of eigenmodes is not a *necessary* condition for monotonic decay, normal operators have desirable robustness properties and provide a desirable target for closed-loop dynamics.

Here, we formulate dynamic mode shaping controllers for closed-loop mode orthogonalization and investigate their utility for transient growth suppression. Beyond targeting desirable closed-loop modes, the dynamic mode shaping framework can also be used in a pole placement capacity, allowing the designer to prescribe the temporal response characteristics of a flow by specifying a set of desired closed-loop eigenvalues.

It is interesting to note that although pole placement has been considered in previous investigations of flow control [31], the notion of dynamic mode shaping by eigenstructure assignment has *never* formally been investigated in the context of fluid flow control. This last point comes as a big surprise given the predominance of modal decomposition techniques for fluid flow analysis: for instance, beyond transition control, the dynamic mode shaping framework to be developed and studied here is a natural one to consider in the context of models based on dynamic mode decomposition (DMD), which considers the dynamics of fluid flows in terms of the spectral properties of an approximate linear dynamical system that governs the flow evolution [32, 34–39, 42]. The ubiquity of modal decomposition techniques for fluid flow analysis suggests that dynamic mode shaping methodologies will offer a welcome perspective for flow control; practitioners can apply the same familiar principles they already use to *interpret* fluid dynamic behaviors to the task of *designing* reliable strategies for flow control and manipulation. The methods introduced here will inevitably enable modal decomposition techniques, such as DMD, to be developed beyond flow analysis and diagnostic techniques.

In the present study, we formulate and propose a number of controller synthesis strategies aimed at suppressing transient growth. The strategies proposed here will build off one another—each addressing a different practical challenge that establishes the necessary groundwork for ultimately realizing feedback fluid flow control. In Section II, we present the foundations of controller synthesis for dynamic mode shaping. In Section III, we show that the dynamic mode shaping framework can be used in a *dynamic mode matching* capacity. We apply the dynamic mode matching controller synthesis method to *match* the spectral properties of a non-normal system with the spectral properties of a system with a lesser degree of non-normality. In Section IV, we propose and formulate a *dynamic mode orthogonalization* technique to eliminate transient growth, when such a solution is admissible. We demonstrate the technique on a simple linear system and briefly consider the robustness properties of the control method to modeling uncertainties. In Section V, we extend the dynamic mode orthogonalization technique for controlling low-rank large-scale systems. We show that controller synthesis can be performed efficiently by working with a low-order representation of the system dynamics. Lastly, in Section VI, we introduce a nonlinear controller synthesis approach based on the notion of Koopman invariant subspaces and Carleman linearization. This final approach enables transient growth suppression in particular classes of nonlinear systems by means of dynamic mode shaping control.

II. Foundations of Dynamic Mode Shaping

Consider the finite-dimensional state-space representation of the linearized Navier-Stokes equations,

$$\begin{aligned} \dot{\vec{x}} &= A\vec{x} + B\vec{u} \\ \vec{y} &= C\vec{x}, \end{aligned} \tag{1}$$

where $\vec{x} \in \mathbb{R}^n$ is the vector of flow state variables, $\vec{u} \in \mathbb{R}^m$ the vector of actuator inputs, and $\vec{y} \in \mathbb{R}^p$ the vector of measured outputs. Here, $A \in \mathbb{R}^{n\times n}$ is determined by the dynamical characteristics of the fluid flow, $B \in \mathbb{R}^{n\times m}$ is determined by the specific arrangement of actuators and their influence on the flow dynamics, and $C \in \mathbb{R}^{p\times n}$ is determined by the specific arrangement and type of sensors. In essence, (A, B, C) define the given flow control configuration for a particular operating point.

The proposed dynamic mode shaping strategy for transient growth suppression is best understood from the standpoint of a modal representation of the flow response. Assuming A has n distinct eigenvalues[a] μ_i with associated eigenvectors $\vec{\xi}_i$ and reciprocal eigenvectors $\vec{\rho}_i$, the sum of the individual modal dynamics

[a]Modal decompositions and dynamic mode shaping can be employed in the case of confluent eigenvalues as well, though additional considerations need to be made.

yields a convenient expression for the system response,

$$\begin{aligned}\vec{x}(t) &= \sum_{i=1}^{n}\langle\vec{\rho}_i,\vec{x}(0)\rangle \mathrm{e}^{\mu_i t}\vec{\xi}_i + \int_0^t \sum_{i=1}^{n}\langle\vec{\rho}_i, B\vec{u}(\tau)\rangle \mathrm{e}^{\mu_i(t-\tau)}\vec{\xi}_i \mathrm{d}\tau \\ \vec{y}(t) &= C\vec{x}(t),\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes inner-product. The aim of dynamic mode shaping is to alter the spectral properties of the linearized flow, such that the individual modal contributions result in desirable spatiotemporal response characteristics for the full system. Moving the system eigenvalues from μ_i to λ_i will alter the temporal nature of the flow; shaping the system modes from $\vec{\xi}_i$ to $\vec{\phi}_i$ will alter the spatial response characteristics of the flow; and shaping the reciprocal eigenvectors from $\vec{\rho}_i$ to $\vec{\zeta}_i$ will modify the contribution of each mode to the overall flow response. As we will show next, desired closed-loop spectral properties can be achieved through a linear output feedback law of the form $\vec{u} = K\vec{y} = KC\vec{x}$, with the static controller gain matrix $K \in \mathbb{R}^{m\times p}$.

To see how we can shape the spectral properties of the closed-loop system via the feedback law $\vec{u} = K\vec{y}$, we begin by considering the eigendecomposition of the controlled (closed-loop) dynamics: $(A + BKC)\vec{\phi}_i = \lambda_i\vec{\phi}_i$, for $i = 1,\dots,n$. By re-writing this eigendecomposition as,

$$\underbrace{\begin{bmatrix} A-\lambda_i I & B \end{bmatrix}}_{G_i} \begin{bmatrix} \vec{\phi}_i \\ KC\vec{\phi}_i \end{bmatrix} = 0,$$

we immediately see that not all choices of closed-loop modes are admissible. The closed-loop mode $\vec{\phi}_i$ corresponding to the specified closed-loop eigenvalue λ_i will only be attainable if the vector $\begin{bmatrix} \vec{\phi}_i \\ KC\vec{\phi}_i \end{bmatrix}$ is in the nullspace of $G_i := \begin{bmatrix} A-\lambda_i I & B \end{bmatrix}$: i.e.,

$$\begin{bmatrix} \vec{\phi}_i \\ KC\vec{\phi}_i \end{bmatrix} \in \mathcal{N}(G_i). \tag{2}$$

The relationship in Eq. (2) highlights a key condition for modal admissibility and controller gain determination, which will be central to our approach here. In fact, given a flow control configuration (A, B, C) and a self-conjugate set of allowable closed-loop eigenvalues[a] $\{\lambda_i\}$, the relationship in Eq. (2) can be used to yield a complete spectral characterization of *all* closed-loop system realizations. In the event that a desired mode is not admissible, the "best" approximation of this mode can be achieved via projection onto the admissible modal subspace, as depicted in Figure 2. The admissibility constraint in Eq. (2) can be leveraged in determining admissibility conditions[b], both for dynamic mode matching and for dynamic mode orthogonalization.

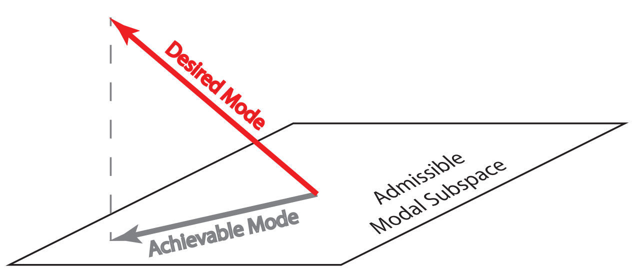


Figure 2: Given a flow control configuration and desired set of closed-loop eigenvalues, not all closed-loop modes are admissible; orthogonality and arbitrary shaping of modes may not be achieved exactly. A projection of the desired mode onto the admissible subspace can be performed to determine the "closest" achievable mode.

[a]Only controllable eigenvalues can be placed arbitrarily.

[b]The closed-loop modes must form a self-conjugate set to ensure the system remains real-valued. The self-conjugacy requirements can be relaxed for complex-valued systems.

In practice, one method for computing the admissible subspace of closed-loop modes and the necessary controller gains for achieving these desired modes is based on the singular value decomposition (SVD) of G_i,

$$G_i = U\Sigma \begin{bmatrix} \tilde{V}_i & V_i \\ \tilde{W}_i & W_i \end{bmatrix}^{\mathsf{T}},$$

where the last q_i right-singular vectors form an orthonormal basis for $\mathcal{N}(G_i)$, such that

$$\mathcal{N}(G_i) = \text{span}(\begin{bmatrix} V_i \\ W_i \end{bmatrix}).$$

Hence, any admissible mode $\vec{\phi}_i$ can be specified as a linear combination of columns of V_i—i.e., $\vec{\phi}_i = V_i\vec{\alpha}_i$, where $\vec{\alpha}_i \in \mathbb{R}^{q_i}$ is a vector of coefficients and q_i is the dimension of the associated admissible modal subspace. Further, in order to satisfy the constraint specified in Eq. (2), the following relation must also hold, $\vec{\psi}_i = KC\vec{\phi}_i = W_i\vec{\alpha}_i$. We can exploit this final expression to compute the controller gain required to achieve a *specified* set of admissible closed-loop modes. In particular, define the matrix of admissible closed-loop modes Φ and the associated auxiliary matrix Ψ needed to satisfy the nullspace constraint in Eq. (2),

$$\Phi = \begin{bmatrix} \vec{\phi}_1 & \vec{\phi}_2 & \cdots & \vec{\phi}_n \end{bmatrix} = \begin{bmatrix} V_1\vec{\alpha}_1 & V_2\vec{\alpha}_2 & \cdots & V_n\vec{\alpha}_n \end{bmatrix}, \tag{3}$$

$$\Psi = \begin{bmatrix} \vec{\psi}_1 & \vec{\psi}_2 & \cdots & \vec{\psi}_n \end{bmatrix} = \begin{bmatrix} W_1\vec{\alpha}_1 & W_2\vec{\alpha}_2 & \cdots & W_n\vec{\alpha}_n \end{bmatrix}, \tag{4}$$

then the control gain K can be determined from the relation

$$KC\Phi = \Psi. \tag{5}$$

In the remainder of the manuscript, we consider the case of full-state feedback (i.e., C=I). We introduce alternative strategies for transient growth suppression based on the dynamic mode shaping framework outlined here. In Section III, we make direct use of the dynamic mode shaping perspective for reducing the degree of transient growth by means of *dynamic mode matching.* In Section IV, we build upon the dynamic mode shaping perspective to formulate a *dynamic mode orthogonalization* control strategy, which is aimed at eliminating transient growth. We also demonstrate robustness properties of the resulting dynamic mode orthogonalization controller when applied to uncertain systems. Subsequently, in Section V, we show that dynamic mode orthogonalization can be formulated using model-reduction techniques in the context of low-rank large-scale systems. In Section VI, we extend these dynamic mode shaping approaches to controlling nonlinear systems by appealing to the notion of Koopman invariant subspaces. To this end, we introduce a new perspective for nonlinear feedback control based on our proposed linear controller synthesis techniques.

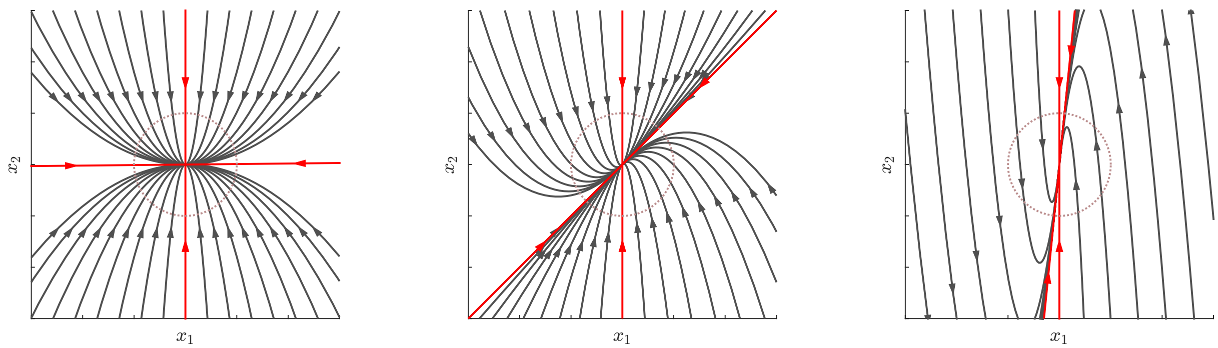
III. Reducing Transient Growth via Dynamic Mode Matching

In principle, transient growth can be *eliminated* by achieving an orthogonal set of closed-loop modes via feedback control, as will be discussed in Section IV; however, such outcomes are not necessarily admissible for a given flow control configuration. Here, we propose a dynamic mode matching strategy that aims to *reduce* the degree of transient growth. (We describe the dynamic mode matching strategy first because it follows from a direct application of dynamic mode shaping, introduced in Section II.) By noting that non-normality decreases with decreasing *Re*, then it seems reasonable to investigate under what conditions it is possible to reduce non-normality by controlling a high *Re* flow to "behave like" a lower *Re* flow. Indeed, if feedback control can be used to shape the spectral properties of a high *Re* flow to match the spectral properties of a lower *Re* flow, then the degree of non-normality and associated transient growth will be reduced.

To build intuition for this strategy, consider a simple non-normal system that has been used previously to explain the role of non-normality in transient growth [5, 24] and to assess transient-growth-control strategies [20]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/\mathrm{R} & 0 \\ 1 & -2/\mathrm{R} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\vec{u}, \tag{6}$$

where R is a parameter that acts like Re and $\vec{u} \in \mathbb{R}^2$ is a vector of control inputs. As seen in the phase portraits for this system (see Figure 3), non-normality increases as the parameter R increases. All three tiles in Figure 3 are associated with non-normal systems; however, only trajectories in tile (c) can exhibit transient growth, thus indicating that non-normality is not a sufficient condition for transient growth. Indeed, the possibility of transient growth arises only beyond some threshold in the alignment of modal directions and in the value of the parameter R. Based on these observations, the task of transient growth suppression for the system shown in Figure 3(c) does not require modal orthogonalization; rather, the objective can be achieved by shaping the closed-loop spectral properties of the system with $R = 10$ to match the spectral properties of the system with $R = 0.01$ or $R = 1$, as in Figure 3(a) or (b), respectively. *Dynamic mode matching* controllers for this purpose can be formulated through a direct application of the dynamic mode shaping framework introduced in Section II: (i) determine the spectral properties of a desired low-R system, (ii) check for admissibility within the context of the control configuration, then (iii) design a feedback controller to match these spectral properties via dynamic mode shaping.



(a) $R = 0.01$ **(b)** $R = 1$ **(c)** $R = 10$

Figure 3: Phase portraits for the non-normal system in (6) show that as the parameter R increases, the modal directions (highlighted in red) approach one another, thus giving rise to the possibility of transient growth away from the origin. Tiles (a) and (b) show that non-normality is not a sufficient condition for transient growth—states beginning in the dashed circle remain in the dashed circle in both cases; however, beyond some threshold of non-normality, trajectories can exit the dashed circle prior to decaying to the origin, as in tile (c).

As an example of dynamic mode matching, again consider the model system in Eq. (6), now with $R = 500$. The goal is to *reduce* the degree of transient growth by matching the spectral properties of the uncontrolled system with $R = 100$. To do so, we simply compute the target spectral properties by setting $R = 100$, then perform dynamic mode shaping with these spectral properties as the target. For this example, the spectral properties for the open-loop system with $R = 100$ are admissible, so the controller is able to successfully match this exactly. The resulting control law $\vec{u} = \begin{bmatrix} -0.008x_1 & -0.016x_2 \end{bmatrix}^\mathsf{T}$ yields the desired closed-loop spectral properties and leads to a reduction in transient growth, as seen in the center tile in Figure 4.

Dynamic mode matching can be taken a step further: If desirable closed-loop spectral properties have already been achieved for a system at a particular parameter value, then these same spectral properties can be targeted for a different parameter value of interest with dynamic mode matching. As a simple demonstration of this notion, consider again the simple system in (6). Further, suppose that our objective is to design a control law that yields a set of orthogonal closed-loop modes without altering the system eigenvalues. In Section IV, we formulate a general approach for synthesizing such a feedback control law. For the simple system under consideration here, one simple strategy for synthesizing a control law that guarantees closed-loop orthogonalization and unaltered eigenvalues, for any R, consists of two steps: (i) retain the mode associated with the eigenvalue $-1/R$, then (ii) shape the remaining mode to yield an orthogonal set. Following this approach yields a control law $\vec{u} = \begin{bmatrix} 0 & -x_1 \end{bmatrix}^\mathsf{T}$, which will yield orthogonal modes with unaltered eigenvalues for all values of R. Interestingly, this result indicates that only a single input is required for closed-loop orthogonalization; in fact, it appears that system controllability is not a necessary condition for orthogonalizability! The open- and closed-loop responses for R = 100 are presented in Figure 4. Although the resulting controller here may seem like an obvious choice for this simple system, the systematic approach to computing this controller by the dynamic mode shaping framework enables a means of realizing an orthogonalizing controller in more complex scenarios for which a clear path to removal of non-normality is not obvious.

For the present discussion, we assume that a dynamic mode orthogonalization control law has already been realized for $R = 100$; then, the spectral properties of this orthogonalized closed-loop system can also be targeted via dynamic mode matching. Indeed, the spectral properties for the orthogonalized $R = 100$ system are admissible, and dynamic mode matching control successfully shapes the $R = 500$ system to match this system structure exactly (see bottom tiles in Figure 4). The associated control law is $\vec{u} = \begin{bmatrix} -.018x_1 & -x_1 - 0.006x_2 \end{bmatrix}^{\mathsf{T}}$.

This last example highlights an important point: while the pole placement problem can be incorporated to match the temporal character of a low-Re flow (e.g., controlling a flow at Re=500 to match eigenvalues of a flow at Re=100), doing so does not necessarily remove transient growth. However, by noting that the solution to the pole placement problem is not unique, additional considerations with regards to associated mode shapes can be made to target the transient response characteristics directly. Dynamic mode matching also creates a means of reducing computational demands in controller realization: once an orthogonalizing controller is realized by computationally demanding methods for one Re, the less computationally demanding dynamic mode matching approach can be used to inform the design of a controller for a different Re.

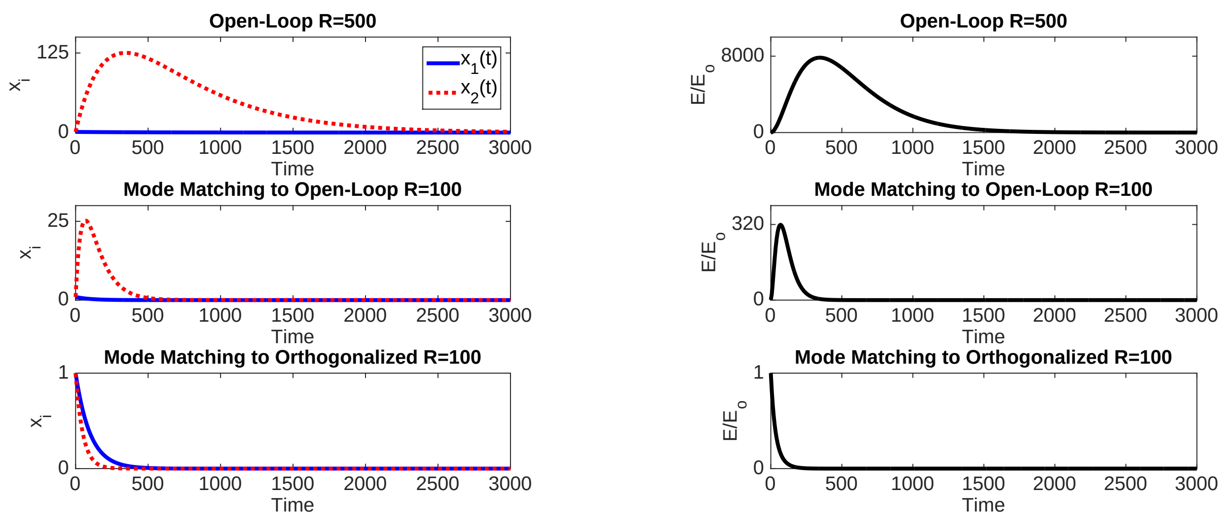


Figure 4: Dynamic mode shaping can be used in a "model matching" capacity by shaping the spectral properties of the closed-loop system to match the spectral properties of a different system. Here, we show three variations of the response of the model system in Eq. (6) with $R = 500$. Individual state variable responses are plotted on the left and associated energies on the right. **(Top)** The open-loop response exhibits relatively large transient growth. **(Center)** Dynamic mode matching control is used to achieve closed-loop dynamics that match the open-loop dynamics for $R = 100$, which exhibits a lesser degree of transient growth compared to the open-loop $R = 500$ case. **(Bottom)** Dynamic mode matching control is used to suppress transient growth completely, by matching the closed-loop dynamics to those achieved via dynamic mode orthogonalization for $R = 100$ (to be discussed in Section IV). The initial condition for all state variables was set to unity here. Energy $E(t) = \vec{x}(t)^{\mathsf{T}}\vec{x}(t)$ is normalized by the initial energy $E_o = E(t_o)$ in the plots.

IV. Eliminating Transient Growth via Dynamic Mode Orthogonalization

Non-normality of the linearized Navier-Stokes operator has been shown to be a *necessary condition* for sub-critical transition to turbulence [7]. Dynamic mode shaping offers a foundation for designing controllers that eliminate this non-normality and yield orthogonal closed-loop modes, thus offering a means of inhibiting transient growth and suppressing the linear mechanism for transition. Further, owing to the nature of the pseudospectra associated with normal operators [25], the resulting controllers are expected to possess an inherent robustness to modeling errors. Here, we formulate a feedback orthogonalization strategy based on mode shaping, then demonstrate the technique on a simple example.

IV.A. A Framework for Dynamic Mode Orthogonalization

Here, we extend the notion of dynamic mode shaping and present a framework for realizing an orthogonalizing controller (or family of controllers), if one exists. The same framework can be used to realize an approximately orthogonalizing controller as well, in the event that exact orthogonalization is not admissible for the particular flow control configuration.

The condition for closed-loop orthogonalization can be asserted in terms of the expression for the admissible set of closed-loop modes in (3). In particular, we constrain the Gram matrix of inner-products associated with Φ to be diagonal—i.e., the closed-loop modes must be chosen from the admissible set such that they are orthogonal. We will make use of the Euclidean inner-product here. Without loss of generality, we assume that the state has been transformed such that a weighted inner-product can be expressed with unit weighting. Then, the requirement for an *orthonormal* set of closed-loop modes amounts to $\Phi^{\mathsf{T}}\Phi = I$, or

$$\begin{bmatrix} \vec{\alpha}_1^{\mathsf{T}} V_1^{\mathsf{T}} V_1 \vec{\alpha}_1 & \vec{\alpha}_1^{\mathsf{T}} V_1^{\mathsf{T}} V_2 \vec{\alpha}_2 & \cdots & \vec{\alpha}_1^{\mathsf{T}} V_1^{\mathsf{T}} V_n \vec{\alpha}_n \\ \vec{\alpha}_2^{\mathsf{T}} V_2^{\mathsf{T}} V_1 \vec{\alpha}_1 & \vec{\alpha}_2^{\mathsf{T}} V_2^{\mathsf{T}} V_2 \vec{\alpha}_2 & \cdots & \vec{\alpha}_2^{\mathsf{T}} V_2^{\mathsf{T}} V_n \vec{\alpha}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{\alpha}_n^{\mathsf{T}} V_n^{\mathsf{T}} V_1 \vec{\alpha}_1 & \vec{\alpha}_n^{\mathsf{T}} V_n^{\mathsf{T}} V_2 \vec{\alpha}_2 & \cdots & \vec{\alpha}_n^{\mathsf{T}} V_n^{\mathsf{T}} V_n \vec{\alpha}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{7}$$

The matrix equation in (7) represents a system of n^2 bilinear vector equations in the n unknown vectors $\vec{\alpha}_i$; however, owing to symmetry of the Gram matrix, many of these equations will be redundant. In general, the vectors $\vec{\alpha}_i$ will have different dimensions q_i, determined by the dimension of the corresponding admissible subspace. Thus, letting $m = \sum_{i=1}^{n} q_i$, then (7) corresponds to a system of at most $m(m+1)/2$ independent bilinear scalar equations in m scalar unknowns. A closer examination of (7) provides a means of characterizing the existence and uniqueness of solutions. Such notions are essential for determining whether an orthogonal set of closed-loop modes is admissible. Dynamic mode orthogonalization will be exactly achievable if (7) admits at least one solution. In the event that (7) does not admit a solution, the condition can be recast as a minimization problem that achieves an approximate solution corresponding to a set of closed-loop modes that are "as close as possible to being orthogonal."

In the present study, we compute a least-squares solution to (7) iteratively using the Levenberg-Marquardt method [46]. In particular, we compute a locally minimizing solution for the coefficients vectors $\{\vec{\alpha}_i\}$,

$$\{\vec{\alpha}_i\}^{\text{opt}} = \arg\min_{\vec{\alpha}_i} \sum_{i=1}^{n} \sum_{j=1}^{i} (\delta_{ij} - \vec{\alpha}_i^{\mathsf{T}} V_i^{\mathsf{T}} V_j \vec{\alpha}_j)^2, \tag{8}$$

where δ_{ij} denotes the Kronecker delta function. Subsequently, the solution to (8) can be substituted into (5) to compute the controller gain matrix K that achieves the associated set of orthogonal closed-loop modes. Indeed, the resulting set of closed-loop modes will be exactly orthogonal when feedback orthogonalization is admissible; otherwise, the solution will achieve modal orthogonality in a least-squares sense.

Next, we demonstrate the dynamic mode orthogonalization control strategy through a simple example.

IV.B. An example of dynamic mode orthogonalization

To validate the controller synthesis techniques described above for dynamic mode orthogonalization, we consider a non-normal system that admits an orthogonal set of closed-loop modes. We construct B such that $\text{rank}(B) = n$, which guarantees that the system is orthogonalizable. To see this, consider that the non-normal operator A can be decomposed into a normal component A_n and a non-normal component A_{nn} as $A = A_n + A_{nn}$. If $\text{rank}(B) = n$, then there will always exist a controller $K = -B^{+}A_{nn}$ that makes the closed-loop system matrix $(A + BK)$ orthogonal. In fact, as is made clear here, an orthogonalizing controller can be computed by means of alternative methods as well; however, we continue with this example simply as a validation of the dynamic mode orthogonalization approach formulated here, which is more generally applicable and useful for synthesizing orthogonalizing controllers in more complex scenarios.

In this example, we construct an arbitrary non-normal system, then compute a control law to orthogonalize the closed-loop modes and to retain the open-loop eigenvalues in closed-loop. Specifically, the system considered has eigenvalues $\lambda_i \in \{-0.1, -0.2, -0.3, -0.4, -0.5\}$ with,

$$A = \begin{bmatrix} -20.34 & -5.22 & 1.63 & 4.25 & 13.57 \\ 15.42 & 3.72 & -1.20 & -3.31 & -10.46 \\ -0.78 & -0.17 & -0.31 & 0.20 & 0.57 \\ 19.45 & 5.06 & -1.54 & -4.34 & -13.12 \\ -29.74 & -7.90 & 2.44 & 6.32 & 19.77 \end{bmatrix} \tag{9}$$

$$B = \begin{bmatrix} 0.10 & -1.02 & 1.74 & -1.92 & -0.04 \\ -0.52 & -0.75 & -0.57 & 0.42 & -0.92 \\ -0.12 & 1.37 & -0.72 & 0.93 & -1.39 \\ -0.72 & -0.51 & 0.67 & -1.47 & -0.92 \\ -0.97 & 0.71 & 0.96 & 1.70 & 0.87 \end{bmatrix}. \tag{10}$$

Applying the dynamic mode orthogonalization strategy described above will yield an orthogonalizing control law. We note that an infinity of controllers exist here because B is non-singular; here, we report a locally minimizing solution to (8),

$$K = \begin{bmatrix} 25.64 & 6.64 & -1.89 & -5.21 & -17.35 \\ -1.14 & -0.33 & 0.15 & 0.09 & 0.79 \\ 30.18 & 7.88 & -2.34 & -6.35 & -20.38 \\ 19.05 & 4.99 & -1.46 & -3.90 & -12.84 \\ -6.79 & -1.76 & 0.44 & 1.41 & 4.63 \end{bmatrix}. \tag{11}$$

(Note: Individual elements in the controller gain K are truncated to two decimal places for reporting purposes here.) Quantifying non-normality as $\|\Phi^\mathsf{T}\Phi - I\|_2$, the controller reduces the level of non-normality from 3.99 in the open-loop system to 1.27×10^{-12} in closed-loop, indicating that the controller synthesis procedure successfully achieves its objective. The open- and closed-loop responses for this system from unity initial condition are shown in Figure 5. The time histories clearly show that transient growth is fully suppressed by the orthogonalizing controller, as intended.

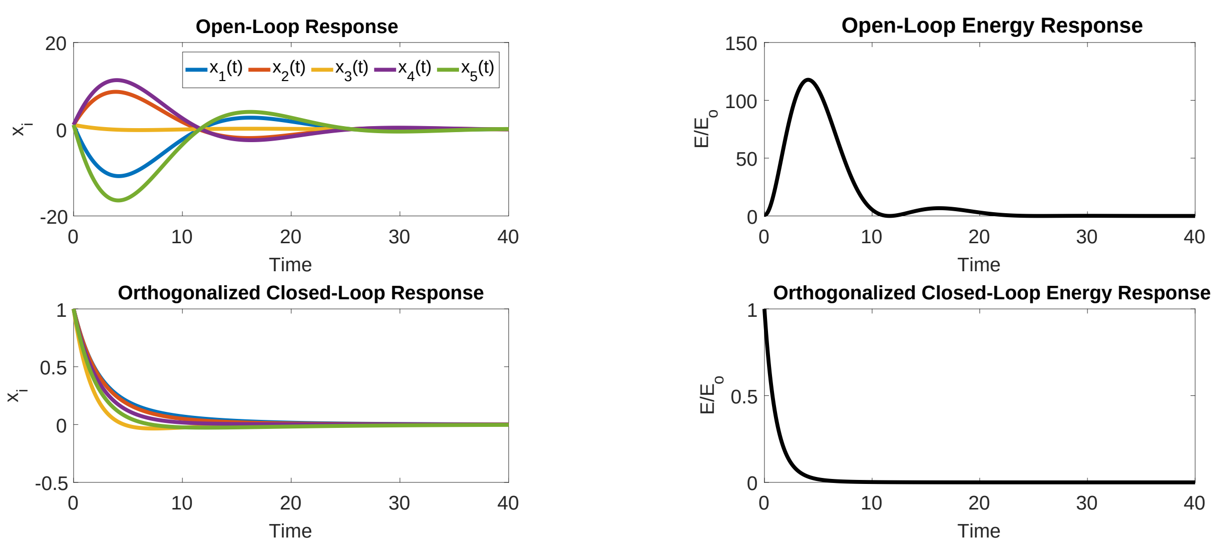


Figure 5: The dynamic mode orthogonalzation controller successfully suppresses transient growth. The plots on the left show individual state responses for the open- and closed-loop systems. Plots on the right show the transient energy responses for the open- and closed-loop systems. The initial condition for each state variable is set to unity here. Energy $E(t) = \vec{x}(t)^\mathsf{T}\vec{x}(t)$ is normalized by the initial energy $E_0 = E(t_0)$ in the plots.

We next perform a brief study of the robustness properties of the orthogonalizing controller to modeling uncertainties. The simple demonstration above assumes that the orthogonalizing controller is designed with exact knowledge of the system; however, this is seldom the case in practice. For non-normal operators, eigenvalues can be highly sensitive to perturbations; however, eigenvalue sensitivity to perturbations of normal operators are "well-behaved"—as is revealed by the pseudospectra of these operators [25]. Thus, an orthogonalizing controller computed based on an inexact model $A = \bar{A} + \Delta A$ of the actual system $\bar{A}$ can be expected to perform reasonably well when applied to the actual system. (Here, ΔA represents an unknown modeling error.) Indeed, robustness to modeling uncertainties was part of the motivation for seeking an orthogonalizing controller, rather than one that only suppresses transient growth.

To illustrate the robustness of dynamic mode orthogonalization controllers to modeling uncertainty, we consider the orthogonalizing controller computed for the same system above. We apply the controller to a disturbed version of the system $\bar{A} = A - \Delta A$, where each element of ΔA is drawn from $\mathcal{N}(0, \sigma^2)$, with $\sigma^2 = 0.05 \min(|\lambda_i|)$. The controlled response from unity initial condition for each of the 200 realizations is shown in Figure 6. Not only does energy decay monotonically for each realization, but the character of the controlled system response is strikingly similar between system realizations. Although this simple study does not serve as a comprehensive demonstration of robustness to modeling uncertainty, it does suggest that the control scheme does not necessarily require an exact model to be successful.

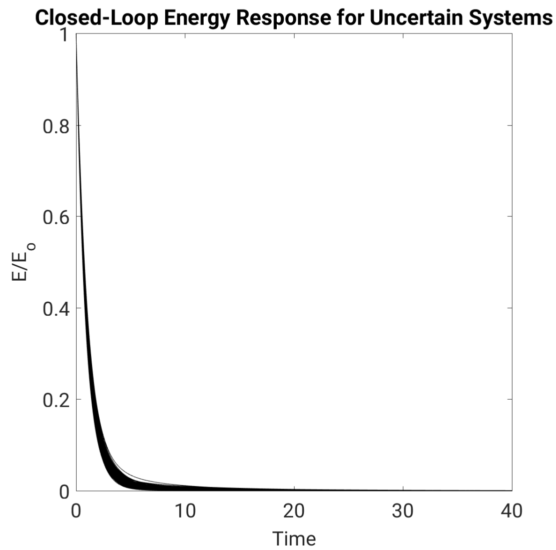


Figure 6: The dynamic mode orthogonalization controller possesses some robustness to modeling uncertainty. The orthogonalizing controller reported in 5 is applied to 200 perturbed versions of the original system, $\bar{A} = A - \Delta A$, where $\Delta A \sim \mathcal{N}(0, \sigma^2)$ and $\sigma^2 = 0.05 \min(|\lambda_i|)$. Here, we present the energy response for each of these 200 realizations, which have strikingly similar behavior. The initial condition for each state variable is set to unity for all 200 responses. Energy $E(t) = \vec{x}(t)^\mathsf{T} \vec{x}(t)$ is normalized by the initial energy $E_0 = E(t_0)$ in the plots.

V. Dynamic mode orthogonalization of low-rank large-scale systems

Upon discretization, the state dimension associated with a fluid flow tends to be quite large, which can make controller synthesis by direct computation impractical (or, at least, undesirable) in many situations. Dynamic mode orthogonalization can be applied in a more computationally tractable manner for certain classes of high-dimensional systems. In particular, consider a large-scale discrete-time system,

$$\vec{x}_{k+1} = A\vec{x}_k + B\vec{u}_k, \tag{12}$$

whose dynamics evolve on an r-dimensional subspace of the full n-dimensional state-space, where $r < n$. Suppose that we have determined an orthogonal basis for this r-dimensional subspace. Then, a reduced-order representation of the system dynamics can be obtained via the change of variables $\vec{x} = T\vec{z}$, where $\vec{z} \in \mathbb{R}^r$ and the r orthogonal columns of $T \in \mathbb{R}^{n \times r}$ span the associated r-dimensional subspace:

$$\begin{aligned} \vec{z}_{k+1} &= T^\mathsf{T} A T \vec{z}_k + T^\mathsf{T} B \vec{u}_k \\ &= \tilde{A}\vec{z}_k + \tilde{B}\vec{u}_k. \end{aligned} \tag{13}$$

We would now like to show that a dynamic mode orthogonalization controller synthesized on the low-dimensional representation in (13) can be used to yield a dynamic mode orthogonalization controller for the original large-scale system in (12). To do so, we will consider designing a controller such that the dynamics of the closed-loop system,

$$\vec{x}_{k+1} = \underbrace{(A + BK)}_{A_{\text{cl}}} \vec{x}_k, \tag{14}$$

evolve on the same r-dimensional subspace as the open-loop system,

$$\vec{z}_{k+1} = \underbrace{(\tilde{A} + \tilde{B}\tilde{K})}_{\tilde{A}_{\text{cl}}} \vec{z}_k, \tag{15}$$

where $\tilde{K} := KT$. Note that the reduced closed-loop dynamics are related to the full dynamics as $\tilde{A}_{\text{cl}} = T^{\mathsf{T}} A_{\text{cl}} T$. It is straightforward to show that the full-order and the reduced-order closed-loop systems will share non-zero eigenvalues, and that the associated modes of these systems are related as $\vec{\phi} = T\vec{\theta}$, where $\vec{\theta}$ is an eigenmode of the reduced-order system. Then, it follows, that an orthogonalizing controller on the reduced-order representation—if one exists—can be used to yield an orthogonalizing controller for the full-order system. Indeed, if $\vec{\theta}_i^{\mathsf{T}}\vec{\theta}_j = \delta_{ij}$, then

$$\vec{\phi}_i^{\mathsf{T}}\vec{\phi}_j = \vec{\theta}_i^{\mathsf{T}} T^{\mathsf{T}} T \vec{\theta}_j = \vec{\theta}_i^{\mathsf{T}}\vec{\theta}_j = \delta_{ij}. \tag{16}$$

Hence, the low-dimensional system representation in (13) can be used to synthesize a dynamic mode orthogonalization control strategy for the large-scale system. The resulting control law for the input actuation can be expressed equivalently in terms of the reduced-order state $\vec{z}$ or the full-order state $\vec{x}$, as $\vec{u} = \tilde{K}\vec{z} = \tilde{K}T^{\mathsf{T}}\vec{x}$.

As a simple demonstration, consider the low-rank large-scale system,

$$\vec{x}_{k+1} = T \begin{bmatrix} -1.1718 & -1.1384 \\ 3.8616 & 3.0218 \end{bmatrix} T^{\mathsf{T}} \vec{x}_k + T \begin{bmatrix} -0.9835 & 0.3949 \\ 0.6006 & 1.3580 \end{bmatrix} \vec{u}_k, \tag{17}$$

where $\vec{x} \in \mathbb{R}^{250}$, $\vec{u} \in \mathbb{R}^2$, and $T \in \mathbb{R}^{250\times 2}$ is a randomly chosen matrix with orthonormal columns. Using the methods described above, we find that this system admits an orthogonal set of modes for all non-zero eigenvalues. An orthogonalizing gain,

$$K = \begin{bmatrix} -2.7577 & -1.5115 \\ -1.6172 & -0.8584 \end{bmatrix} T^{\mathsf{T}}, \tag{18}$$

leads to full suppression of transient growth in closed-loop, as shown in Figure 7. It is important to note that the above approach can also be applied to continuous-time systems as in (1) as well; however, in the context of continuous-time systems, neither the open-loop nor the closed-loop response will decay to zero due to the "constant forcing" effect of zero-eigenvalues—an artifact of the "low-rank" nature of the A-matrix and its role in the continuous-time dynamics.

VI. Dynamic Mode Shaping in Nonlinear Systems

The dynamic mode shaping and dynamic mode orthogonalization perspectives offer a potential path for suppressing transient growth in nonlinear systems without resorting to linearization. By choosing an appropriate change of coordinates, a nonlinear model may admit a finite-dimensional linear representation [47, 48]. Such ideas are the subject of research on the Koopman operator and data-driven Koopman spectral analysis [36, 40–45]. Here, we demonstrate the utility of dynamic mode shaping for nonlinear controller synthesis by way of a simple example.

Consider the nonlinear system,

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 + a x_2^2 \\ \lambda_2 x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \tag{19}$$

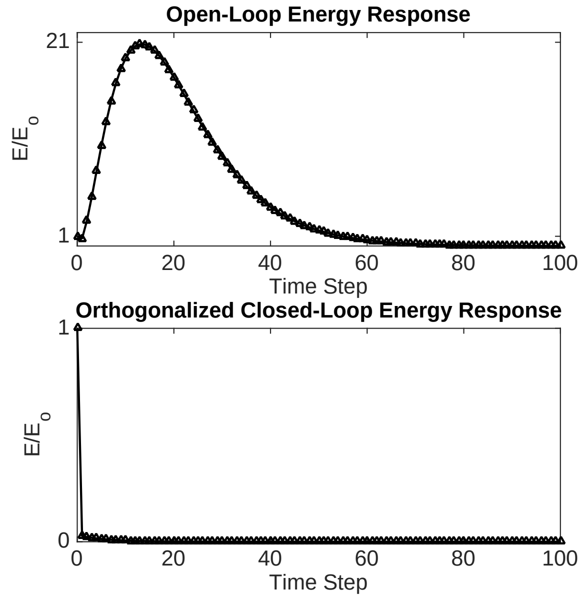


Figure 7: Dynamic mode orthogonalization suppresses transient growth in a low-rank large-scale system with $n = 250$. **(Top)** The open-loop dynamics exhibit transient energy growth, whereas **(Bottom)** dynamic mode orthogonalization completely suppresses transient energy growth. The underlying dynamics of the open-loop system are governed by (17). The initial condition associated with these results was selected randomly. Energy $E_k = \vec{x}_k^{\mathsf{T}}\vec{x}_k$ is normalized by the initial energy $E_o = E_{k=0}$ in the plots.

which was first used to demonstrate Koopman-based optimal control in [40]. This nonlinear system admits a finite-dimensional linear representation in terms of the extended set of variables $\{y_i\}$ for $i = 1, 2, 3$, where $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_2^2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & a \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 2\lambda_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u. \tag{20}$$

The linear representation admits an eigendecomposition, with eigenvalues $(\lambda_1, \lambda_2, 2\lambda_2)$ and corresponding modes,

$$\phi_{\lambda_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \phi_{\lambda_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \phi_{2\lambda_2} = \begin{bmatrix} -a \\ 0 \\ \lambda_1 - 2\lambda_2 \end{bmatrix}.$$

Modes ϕ_{λ_1} and ϕ_{λ_2} always constitute an orthogonal pair; however, depending on the specific values of a, λ_1, and λ_2, the third mode $\phi_{2\lambda_2}$ can lead to non-normality, indicating that the uncontrolled nonlinear system can exhibit transient growth.

Interestingly, the same dynamic mode shaping perspectives introduced previously can also be applied here; by leveraging the linear representation of this nonlinear system, the dynamic mode shaping approach is valid in this context. For example, transient growth exhibited by this *nonlinear* system can be eliminated by applying dynamic mode orthogonalization in the space of extended variables $\{y_i\}$. Of course, the linear control law determined in this linear setting of Koopman observables can be transformed back into the nonlinear setting, yielding a nonlinear control law that can be implemented on the original system. An orthogonalizing nonlinear control law will be $u = -ax_2^2$, which effectively acts to eliminate the term a from the closed-loop system. Although this choice is, perhaps, obvious from the analytical expressions for the modes of this system, the dynamic mode orthogonalization perspective provides a means of computing such solutions in more complicated systems for which an orthogonalizing control law may not be as obvious.

Dynamic mode shaping can also be employed to shape the Koopman spectral properties of the nonlinear system above. Let $\lambda_1 = -1/2$, $\lambda_2 = -3/4$, and $a = 2$, and suppose we wish retain the Koopman eigenvalues while shaping the closed-loop Koopman modes to be

$$\phi_{\lambda_1}^{\text{desired}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \phi_{\lambda_2}^{\text{desired}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \phi_{2\lambda_2}^{\text{desired}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The nonlinear control law $u = -3x_2^2$ that achieves this can be computed by invoking the dynamic mode shaping approach on (20). A comparison of open- and closed-loop responses for this system are shown in Figure 8.

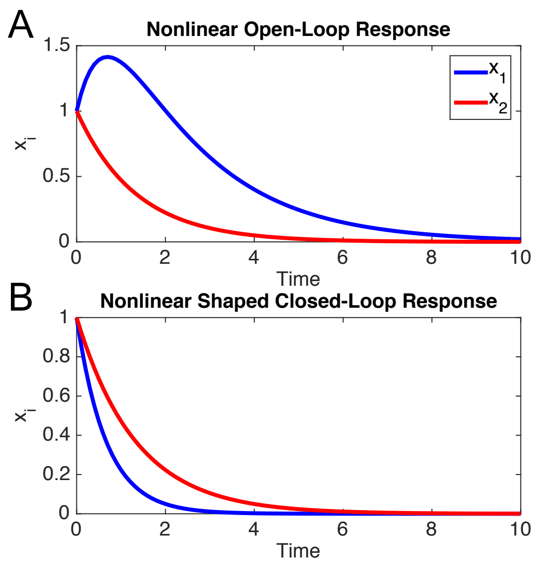


Figure 8: Dynamic mode shaping can be used to inhibit transient growth arising due to nonlinear mechanisms in special classes of nonlinear systems. Here, we consider the exact finite-dimensional linear representation in Eq. (20) of the nonlinear system in Eq. (19). (A) The open-loop response exhibits transient growth. (B) Dynamic mode shaping control is used to inhibit transient growth by shaping the closed-loop modes in the extended space of observables. For this demonstration, the control law is designed to keep the eigenvalues in this extended space unaltered. The desired closed-loop modes are reported in the text.

VII. Conclusions

In this paper, we have introduced dynamic mode shaping as a promising feedback control technique for use in fluid flow control applications. Indeed, dynamic mode shaping offers additional flexibility beyond alternative linear control techniques for fluid flow control, allowing the user to prescribe *both* the temporal response characteristics (associated with system eigenvalues) *as well as* spatial response characteristics (associated with system modes). Further, the dynamic mode shaping framework can be used to shape the reciprocal modes (i.e., left-eigenvectors) of the closed-loop system as well, thus offering a means of altering the contributions of individual modal dynamics to the overall fluid response.

The particular investigation here focused on developing dynamic mode shaping strategies for controlling transient energy growth in linear systems—a phenomenon that arises due to system non-normality and that is commonly associated with sub-critical transition to turbulence in various contexts. In particular, we proposed two controller synthesis strategies: (1) a *dynamic mode matching* strategy aimed at reducing transient energy growth by shaping the spectral properties of a system to match the spectral properties of a "more desirable" system, and (2) a *dynamic mode orthogonalization* strategy aimed at suppressing transient energy growth by shaping a non-normal system to attain orthogonal closed-loop modes. Additionally, we presented various techniques for addressing the usual challenges that arise in the context of fluid flow control: high-dimensionality, nonlinearity, and system uncertainty. The proposed strategies were demonstrated on simple examples to illustrate the utility and promise of the dynamic mode shaping perspective.

Although much remains to be done to make dynamic mode shaping suitable for practical fluid flow control, the perspectives and methods introduced here promise to serve as a foundation upon which such techniques can be further developed and refined. Owing to the predominance of modal analysis techniques in fluid flow diagnostics, we believe that the dynamic mode shaping perspective will be a welcome tool to add to the arsenal of fluid flow control techniques.

VIII. Acknowledgments

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