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Stability and Performance Analysis of Nonlinear and Non-Normal Systems using Quadratic Constraints

Aniketh Kalur,* Peter Seiler,† and Maziar S. Hemati‡
Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, MN 55455, USA

We propose a system-theoretic approach for analyzing stability and transient energy growth performance of nonlinear fluid flow systems. The systems in consideration are composed of a non-normal linear element in feedback with a static and lossless nonlinearity—the Navier-Stokes equations being a special case. Specifically, we show that the input-output properties of the nonlinear element can be represented by a set of quadratic constraints. As a result, the nonlinear system can be analyzed by solving the Lyapunov inequalities of a linear system with a set of quadratic constraints that capture nonlinear behavior. Here, we investigate the proposed analysis framework on the Waleffe-Kim-Hamilton model—a low-dimensional mechanistic model of transitional and turbulent shear flows. Our proposed analysis framework can analyze global and local stability of a given equilibrium point of the nonlinear system. We show that nonlinear flow interactions have a destabilizing effect on the system response. The Lagrange multipliers in the proposed analysis provide additional information regarding the dominant nonlinear flow interaction terms.

I. Introduction

In shear flows, laminar-to-turbulent transition has been observed to occur well below the critical Reynolds number (Re) determined from linear stability analysis [1–3]. Since wall-bounded turbulent flows exhibit markedly greater skin-friction drag than laminar flows, the mechanisms responsible for transition have been of great research interest [4–7]. Many works attribute the cause of sub-critical transition to a linear non-modal mechanism for transient energy growth [4, 7, 8]. The high-degree of non-normality of the linearized Navier-Stokes equations (NSE) results in a transient amplification of small perturbations, even when the dynamics are linearly asymptotically stable. Much work in systems analysis and controller design in fluid mechanics has focused on the linearized NSE, without explicitly accounting for the nonlinear interaction terms [7, 9–13]. Transition can also be initiated by secondary instabilities arising from nonlinear flow interactions [14]. These transition scenarios cannot be analyzed without accounting for the nonlinear terms in NSE.

The exact interplay between non-normality and nonlinearity in the transition process is not fully understood. However, it is well accepted that the linear terms in NSE contribute to the non-modal growth, whereas the nonlinear terms contribute to mixing and vortex regeneration [15]. In [16], Waleffe highlights the importance of both the nonlinearity and non-normality in the transition process and for sustained turbulence. He introduces a low-dimensional non-linear model that mimics the behaviour of the NSE by capturing physical mechanisms central to the transition process. The low-dimensional Waleffe-Kim-Hamilton (WKH) model consists of a non-normal linear element and a lossless nonlinearity similar to that in NSE.

Transition can be viewed as an issue in system robustness rather than system stability [17]. The large amplification of disturbances and changes in spectra based on the ϵ-pseudospectral interpretation [18] extend themselves directly to system robustness issues. Frameworks for analyzing a linear element with nonlinearities and/or uncertainties acting in feedback have been well established in the robust-control community. Partitioning a system into the feedback interconnection of two positive operators—a so-called Luré decomposition—is advocated in absolute stability theory, which is now considered fundamental to nonlinear systems theory [19]. Such representations have recently been leveraged for analysis and control within fluid mechanics. Ahmadi et al. [20] perform a nonlinear input-output analysis of wall-bounded shear flows through a careful development of the dissipation inequality framework proposed by [21]. Heins et al. [22] design output-feedback control laws for turbulent channel flows, exploiting the passivity property of the nonlinearity [19]. Both of these methods build on the nonlinear NSE and have shown great promise both in terms of nonlinear analysis and controller synthesis in fluid mechanics.

*Graduate Student, Aerospace Engineering and Mechanics, AIAA Member.
†Associate Professor, Aerospace Engineering and Mechanics, AIAA Member.
‡Assistant Professor, Aerospace Engineering and Mechanics, AIAA Senior Member.

In this paper, we introduce a quadratic constraints framework for accommodating nonlinear effects for reduced-complexity stability and transient energy growth analyses. The quadratic constraints framework broadly combines concepts from input-output theory, absolute stability theory, and robust controls into a unifying framework [23]. We develop the framework on the nonlinear WKH system, but the approach can also be applied to NSE directly. We first express the WKH system in Luré form, with non-normal linear dynamics in feedback with a static lossless nonlinearity. The "global" lossless property of the nonlinearity can be expressed in terms of a quadratic constraint between its inputs and outputs. This quadratic constraint is then used to capture the influence of the nonlinearity on the linear dynamics, which is exploited for analyzing global stability and transient energy growth performance. Following the global analysis, we further extend the quadratic constraints framework to capture "local" input-output behavior of the nonlinear terms. The "local" constraints on the nonlinear terms enable "local" stability and transient energy growth analysis of the WKH system around a given equilibrium point.

The remainder of the paper is organized as follows. In Section II, we introduce the fluid flow model used in our analysis. In Section III, we briefly describe the global property of the nonlinear terms in the WKH model. We also formulate the global quadratic constraints and optimization problems for global analysis of stability and transient energy growth. We further discuss results from the global stability and transient energy growth analysis. In Section IV, we formulate quadratic constraints and optimization problems for local stability and transient energy growth analysis. The results obtained from local stability and transient energy growth analysis are also discussed here. Finally, we provide the concluding remarks of our study in Section V.

II. Waleffe-Kim-Hamilton Model

A simple model for transition and sustained turbulence was proposed by Waleffe-Kim-Hamilton, based on observations from direct numerical simulations of a plane Couette flow reported in [14]. This model is referred to as the Waleffe-Kim-Hamilton (WKH) model which is a low-dimensional model that mimics the behaviour of transitional shear flows governed by NSE. The WKH model was introduced to highlight the importance of nonlinear effects instead of focusing only on the transient amplification provided by the linear terms in NSE. The WKH model was studied in greater detail by Waleffe in [24].

The four state WKH model is given by,

$$\begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\\ \dot{m}\end{bmatrix} = \frac{1}{Re}\begin{bmatrix}0\\0\\0\\ \sigma\end{bmatrix} - \frac{1}{Re}\begin{bmatrix}\lambda u\\ \mu v\\ \nu w\\ \sigma m\end{bmatrix} + \begin{bmatrix}0 & 0 & -\gamma w & v\\ 0 & 0 & \delta w & 0\\ \gamma w & -\delta w & 0 & 0\\ -v & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}u\\ v\\ w\\ m\end{bmatrix}. \tag{1}$$

where u represents the amplitude of the spanwise modulation of streamwise velocity; v represents the amplitude of the streamwise rolls; w represents the amplitude of the inflectional streak instability, and m represents the amplitude of the mean shear [24]. The constants λ, μ, ν, σ are positive parameters corresponding to viscous decay rates. The constants γ and δ represent nonlinear interaction coefficients [24]. Note, the nonlinearity is skew-symmetric and quadratic.

The model admits a laminar equilibrium point at $(u, v, w, m)_e = (0, 0, 0, 1)$. For stability and performance analysis, we perform a change in coordinates to translate the equilibrium point of Eq. (1) to the origin. The change in coordinates gives us the new equilibrium point $\mathbf{x}_e = (0, 0, 0, 0)$ and the new state after transformation is $\mathbf{x} = (u, v, w, \bar{m})$, where $\bar{m} = m - 1$. The system in this translated coordinate system is then,

$$\underbrace{\begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\\ \dot{\bar{m}}\end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix}-\frac{\lambda}{Re} & 1 & & \\ & -\frac{\mu}{Re} & & \\ & & -\frac{\nu}{Re} & \\ & & & -\frac{\sigma}{Re}\end{bmatrix}\begin{bmatrix}u\\ v\\ w\\ \bar{m}\end{bmatrix}}_{A\mathbf{x}} + \underbrace{\begin{bmatrix}0 & 0 & -\gamma w & v\\ 0 & 0 & \delta w & 0\\ \gamma w & -\delta w & 0 & 0\\ -v & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}u\\ v\\ w\\ \bar{m}\end{bmatrix}}_{N(\mathbf{x})} \tag{2}$$

The WKH system in Eq. (2) can be represented as

$$\dot{\mathbf{x}} = A\mathbf{x} + N(\mathbf{x}), \tag{3}$$

where A is the asymptotically stable linear term, and $N(\mathbf{x})$ is a quadratic nonlinearity. Note that $N(\mathbf{x}) = Q(\mathbf{x})\mathbf{x}$ and $Q(\mathbf{x}) = -Q(\mathbf{x})^T \in \mathbb{R}^{4\times4}$. The linear and nonlinear terms can be partitioned into Luré form, with the two systems acting in feedback with each other:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{z} \tag{4a}$$

$$\mathbf{z} = N(\mathbf{x}) \tag{4b}$$

where $\mathbf{z} \in \mathbb{R}^4$ (see Figure 1). This Luré decomposition of WKH system is denoted as $F_u(L, N(\mathbf{x}))$.

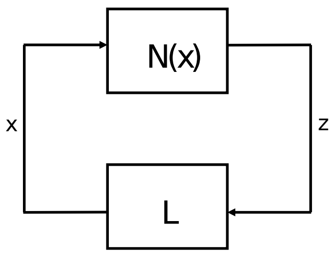

Fig. 1 Luré interconnection $F_u(L, N(\mathbf{x}))$ of WKH system.

In all that follows, the values for the decay rates are chosen to be $\lambda = \mu = \sigma = 10$ and $\nu = 15$. The nonlinear interaction coefficients are selected to be $\gamma = 0.1$ and $\delta = 1$. These values are chosen based on the system studied in [24]. In the remainder of this work, the only parameter that is varied for stability and transient energy growth analyses is Re.

III. Global Analysis

The skew-symmetry of the nonlinear term implies

$$\mathbf{x}^T\mathbf{z} = \mathbf{x}^T N(\mathbf{x}) = \mathbf{x}^T Q(\mathbf{x})\mathbf{x} = 0, \ \forall\ \mathbf{x}. \tag{5}$$

The physical interpretation of this property is that the nonlinearity conserves energy. In other words, the nonlinearity neither creates nor destroys energy and, instead, only redistributes energy between modes. This "lossless" property of the nonlinear term is also observed in shear flows like the channel flow system [25]. This lossless property is equivalent to the following quadratic constraint on the input-output signals of the nonlinearity:

$$\begin{pmatrix}\mathbf{x}\\ \mathbf{z}\end{pmatrix}^T \underbrace{\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{pmatrix}}_{M_0} \begin{pmatrix}\mathbf{x}\\ \mathbf{z}\end{pmatrix} = 0, \ \forall \mathbf{x} \in \mathbb{R}^4 \text{ and } \mathbf{z} = N(\mathbf{x}) \tag{6}$$

where $\mathbf{0}, \mathbf{I} \in \mathbb{R}^{4\times4}$ denote the zero and identity matrices respectively. The lossless property of the nonlinearity is thus defined by a quadratic constraint on $(\mathbf{x}, \mathbf{z})$ in terms of the block matrix $M_0 \in \mathbb{R}^{8\times8}$.

A. Global stability analysis

Lyapunov methods [19] can be used to assess the stability of the system in Eq. (4). Define a candidate Lyapunov function $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$ where $P > 0$. The time derivative of the Lyapunov function along trajectories of the system is given by:

$$\begin{aligned}\frac{d}{dt}V(\mathbf{x}(t)) &= 2\mathbf{x}(t)^T P(A\mathbf{x}(t) + \mathbf{z}(t))\\ &= 2\mathbf{x}(t)^T P(A\mathbf{x}(t) + N(\mathbf{x}(t))).\end{aligned} \tag{7}$$

If $\frac{d}{dt}V(\mathbf{x}(t)) < 0$ along all trajectories of the system, then the system is globally asymptotically stable. The analysis is complicated by the quadratic nonlinearity that appears in Eq. (7). The lossless property can be used to obtain a simple sufficient condition to verify stability. Specifically, the system is globally asymptotically stable if there exists a matrix $P = P^T > 0$ and scalar ξ such that:

$$2\mathbf{x}^T P(A\mathbf{x} + \mathbf{z}) + 2\xi\mathbf{x}^T\mathbf{z} < 0 \;\; \forall \mathbf{x}, \mathbf{z} \in \mathbb{R}^4. \tag{8}$$

The lossless property in Eq. (6) implies that the second term $2\xi\mathbf{x}^T\mathbf{z}$ is equal to zero along any trajectory of the system. Hence the Lyapunov function is decaying along system trajectories if Eq. (8) holds.

Upon rearranging terms, we arrive at the following quadratic matrix inequality,

$$\begin{bmatrix}\mathbf{x}\\ \mathbf{z}\end{bmatrix}^T \left\{\begin{bmatrix}A^T P + PA & P\\ P & \mathbf{0}\end{bmatrix} + \xi M_0\right\}\begin{bmatrix}\mathbf{x}\\ \mathbf{z}\end{bmatrix} < 0. \tag{9}$$

Now, the stability of the nonlinear system i.e., linear system along with input-output behaviour constraints on the nonlinearity can be formulated as a problem of finding an energy weight $P > 0$ and ξ such that the matrix inequality is satisfied for all $\mathbf{x},\ \mathbf{z} \in \mathbb{R}^4$. Thus the system $F_u(L, N(\mathbf{x}))$ is globally asymptotically stable if there exists $P > 0$ and ξ such that the following linear matrix inequality (LMI) holds:

$$\begin{bmatrix}A^T P + PA & P\\ P & \mathbf{0}\end{bmatrix} + \xi M_0 < 0. \tag{10}$$

Equation (10) is simply an LMI in the variables (P, ξ). This is a convex optimization problem that can be solved with standard software. Feasibility of the LMI condition implies the system is globally asymptotically stable. The condition is only sufficient as it only relies on the lossless property and does not depend on any other specific details of the nonlinearity. Numerical tools are used to solve the optimization problems stated with LMI constraints. In the remainder of this work we use CVX [26, 27], which is a package for specifying convex optimizations, combined with the solver SDPT$_3$ [28].

To analyze the global stability of the WKH system, we solve the LMI in Eq. (10) for variables P and ξ at different values of Re. On performing the global stability analysis using the lossless constraints, we note that the WKH model for the given parameters values is globally asymptotically stable for $Re \leq 20$. This finding is consistent with the critical Re for global asymptotic stability reported by Waleffe [24], suggesting that the lossless constraint is sufficient for characterizing global stability of the WKH system.

B. Global maximum transient energy growth analysis

First consider a stable autonomous linear system $\dot{\mathbf{x}}(t) = A\mathbf{x}$. By assumption, A is Hurwitz so that $\mathbf{x}(t) \to 0$ when starting from any initial condition. If A is non-normal then the system can undergo a transient growth before asymptotically decaying to the origin. The maximum transient energy growth (MTEG) is defined as:

$$\text{MTEG} := \max_{t\geq 0} \|\mathbf{x}(t)\|^2 \;\; \text{subject to: } \|\mathbf{x}(0)\|^2 \leq 1. \tag{11}$$

A bound on the MTEG for this linear system can be solved via the following optimization [24,25]:

$$\begin{aligned} q^* := \min \quad & q \\ \text{subject to: } & \mathrm{I} \leq P \leq q\mathrm{I}\,, \\ & A^T P + PA < 0. \end{aligned} \tag{12}$$

This optimization has LMI constraints and a linear cost involving variables (P, q). This optimization is known as a semidefinite program (SDP). The LMI constraints and lossless property imply that $V(\mathbf{x}) := \mathbf{x}^T P\mathbf{x}$ is a Lyapunov function for the system so that $V(\mathbf{x}(t)) \leq V(\mathbf{x}(0))$ for all $t \geq 0$. The bounds on P further imply that $\|\mathbf{x}(t)\|^2 \leq q^*\|\mathbf{x}(0)\|^2$. These LMI constraints are conservative in general and hence q^* is a (possibly non-tight) upper bound on the MTEG.

A similar optimization can be formulated to study the MTEG for the nonlinear WKH system. The lossless property for the nonlinear term can again be used as a global constraint. This leads to the following optimization:

$$\begin{aligned} q^* := \min \quad & q \\ \text{subject to: } & \mathrm{I} \leq P \leq q\mathrm{I}\,, \\ & \begin{bmatrix}A^T P + PA & P\\ P & \mathbf{0}\end{bmatrix} + \xi M_0 < 0. \end{aligned} \tag{13}$$

Equation (13) is now a SDP in the variables (P, q, ξ). As before, the LMI constraints imply that $V(\mathbf{x}) := \mathbf{x}^T P\mathbf{x}$ is a Lyapunov function for the system so that $V(\mathbf{x}(t)) \leq V(\mathbf{x}(0))$ for all $t \geq 0$. The bounds on P further imply that $\|\mathbf{x}(t)\|^2 \leq q^* \|\mathbf{x}(0)\|^2$.

Finally, the SDP in Eq. (19) is used to obtain MTEG bounds on the WKH system in the globally stable regime of $Re \leq 20$. This yields an MTEG bound of 1. This implies that the energy of the system is monotonically decaying for $Re \leq 20$. Note that Lemma 2 in [29] states that a linear system has MTEG equal to one if and only if $A + A^T < 0$. The linear part of the WKH model satisfies this condition for $Re \leq 20$ and hence this verifies that the MTEG is equal to one if the nonlinearity is neglected. The results using Eq. (19) are stronger. Specifically, they imply that the system $F_u(L, N(x))$ has MTEG equal to one in the globally stable regime $Re \leq 20$ even if the nonlinearity is included.

IV. Local Analysis

The nonlinear terms can also be described using "local" properties, such that the properties of each individual nonlinear term can be identified around a local neighborhood of $\mathbf{x} = 0$. These local properties can be leveraged to identify additional constraints on each of the nonlinear terms.

Figure 2 shows a simplified diagram illustrating the use of local constraints. The figure displays a scalar quadratic function $\mathbf{z} = \mathbf{x}^2$. (This function does not satisfy the lossless property and is shown only for illustration). Let $R > 0$ define a local region such that the input satisfies $|\mathbf{x}|^2 \leq R^2$. In this region the output satisfies $\mathbf{z}^2 = \mathbf{x}^4 \leq R^2\mathbf{x}^2$. This is a local quadratic inequality satisfied by $\mathbf{x}$ and $\mathbf{z}$. The size of the region R determines the amount of nonlinear behavior captured by the local constraint. The quadratic constraint corresponds closely to linear analysis for small values of R. On the other hand the constraint captures more nonlinear behavior for larger values of R. The remainder of this section generalizes this basic concept to the multivariable quadratic terms that appear in the WKH model.

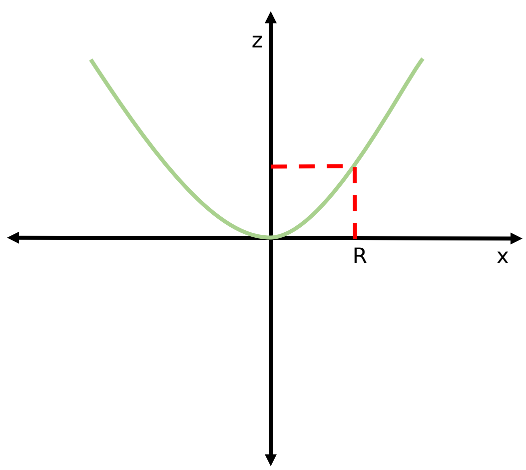

Fig. 2 2D illustration of the quadratic functions showing the local region R, such that R captures the local nonlinear behavior.

From Eq. (2) and Eq. (4b), each individual nonlinear term can also be expressed as a quadratic function:

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} \mathbf{x}^T Q_1 \mathbf{x} \\ \mathbf{x}^T Q_2 \mathbf{x} \\ \mathbf{x}^T Q_3 \mathbf{x} \\ \mathbf{x}^T Q_4 \mathbf{x} \end{bmatrix}. \tag{14}$$

Each $Q_i \in \mathbb{R}^{4\times 4}$ is a symmetric matrix. Hence it has real eigenvalues and the spectral radius $\rho(Q_i)$ denotes the largest (magnitude) of these eigenvalues [30]. Moreover, quadratic terms with symmetric matrices are upper bounded as follows [30]:

$$|z_i| = |\mathbf{x}^T Q_i \mathbf{x}| \leq \rho(Q_i)\mathbf{x}^T \mathbf{x} \tag{15}$$

Next, assume the state $\mathbf{x}$ remains with a circle of radius R, i.e. $\mathbf{x}^T\mathbf{x} \leq R^2$. We can then square Eq. (15) to obtain the following constraint:

$$z_i^2 \leq \underbrace{\rho(Q_i)^2 R^2}_{\alpha_i(R)^2} \mathbf{x}^T\mathbf{x}. \tag{16}$$

This is a constraint involving squares of $\mathbf{x}$ and z_i. It can be written in a more useful quadratic constraint form. Let $E_i \in \mathbb{R}^{4\times4}$ denote the matrix with the diagonal (i,i) entry equal to one and all other entries equal to zero. The constraint in Eq. (16) is equivalent to:

$$\begin{bmatrix}\mathbf{x}\\ \mathbf{z}\end{bmatrix}^T \underbrace{\begin{bmatrix}\alpha_i(R)^2\mathbf{I} & \mathbf{0}\\ \mathbf{0} & -E_i\end{bmatrix}}_{M_i} \begin{bmatrix}\mathbf{x}\\ \mathbf{z}\end{bmatrix} \geq 0. \tag{17}$$

This quadratic constraint provides a bound on the nonlinear term z_i that holds over the local region $\mathbf{x}^T\mathbf{x} \leq R^2$. A local bound can be obtained for each of the four quadratic nonlinearities in Eq. (17). We will make use of these local properties to study stability and MTEG of the WKH system in sections IV.A and IV.B, respectively.

A. Local stability analysis

The lossless property in Eq. (6) captures the global behaviour of the quadratic nonlinearity, and hence is also considered to be a global property. Also, the WKH system is not globally stable for $Re > 20$ but it is still useful to understand the local stability properties. The linearization around $\mathbf{x} = 0$ is stable for all $Re \geq 0$ because A is Hurwitz. A more quantitative local stability analysis can be performed around $\mathbf{x} = 0$ using the local constraints derived above. This local analysis is performed with the following LMI feasibility problem:

$$\begin{aligned}
&P \geq \mathrm{I}\\
&\xi_{p_i} \geq 0 \qquad (\text{for } i = 1 \text{ to } 4)\\
&\begin{bmatrix}A^T P + PA & PB\\ B^T P & \mathbf{0}\end{bmatrix} + \xi_{p_0} M_0 + \sum_{i=1}^{4} \xi_{p_i} M_i < 0,
\end{aligned} \tag{18}$$

here ξ_{p_i} ($i = 1$ to 4) are Lagrange multipliers for local constraints. These Lagrange multipliers also provide information on how the objective function is changing with respect to the constraints.

Towards local analysis we select a local region of size R, and then analyze the stability of the nonlinear system for the selected R. To isolate the effects of the nonlinear term on stability, our goal is to find a critical Re for a given R. The LMI in Eq. (18) are solved for for P, ξ_{p_0} and ξ_{p_i} using convex optimization methods subject to the constraints. A Re_c can be obtained by performing a bisection search over various Re. The obtained critical Re for a given R are shown in Figure 3.

The results in Figure 3, shows that as size of the region R is increased, then the critical Re decreases. This result is intuitive, the size of R determines the amount of the nonlinearity being captured, therefore, for a small R value we can expect Re_c to be higher. Similarly, for a large R the nonlinear effects are large and hence the system is destabilized at a lower Re_c. The linear term in the WKH model is asymptotically stable for all Re, however, including the nonlinear terms "local" effects is found to destabilize the system.

It should also be noted that the plot converges to $Re_c = 20$ as $R \longrightarrow \infty$. The "local" constraint provide no value as $R \longrightarrow \infty$, and hence the Lagrange multipliers $\xi_{p_i} \longrightarrow 0$. This shows that the local constraints reduce the problem to one with only global constraints as $R \longrightarrow \infty$.

B. Local maximum transient energy growth analysis

A formulation similar to Eq. (13) can be used to study the effect of nonlinearity on MTEG in the nonlinear system. To perform the "local" MTEG analysis, additional local constraints are added to the optimization problem listed in Eq. (13). The local constraints that capture input-output property of the nonlinear term are captured by the M_i matrix. The addition of these constraints, now facilitates the study of "local" MTEG on the nonlinear system. The local MTEG for the nonlinear system is computed via the following convex optimization:

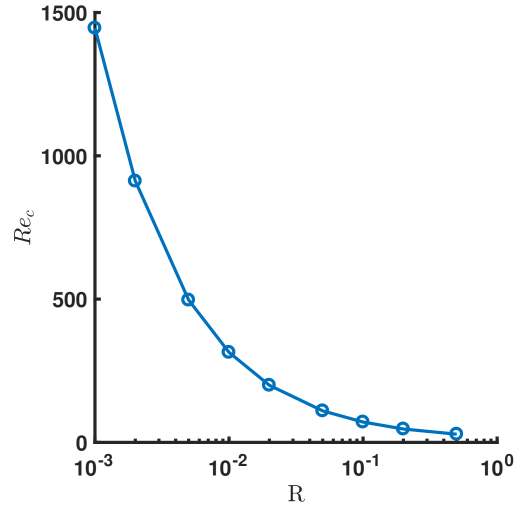

Fig. 3 Critical Reynolds number (Re_c) decreases as size of R is increased.

$$\begin{aligned} &\min \quad q \\ &\text{subject to: } \mathrm{I} \leq P \leq q\mathrm{I}\,, \\ &\qquad \xi_{p_i} \geq 0 \qquad (\text{for } i = 1 \text{ to } 4)\,, \\ &\qquad \begin{bmatrix} A^T P + PA & P \\ P & \mathbf{0} \end{bmatrix} + \xi_{p_0} M_0 + \sum_{i=1}^{4} \xi_{p_i} M_i < 0. \end{aligned} \tag{19}$$

We will identify MTEG for the system about a local equilibrium point $\mathbf{x} = 0$ by solving this optimization for P, q, ξ_{p_0}, and ξ_{p_i} (for $i = 1$ to 4). It is well known that TEG originates from non-modal mechanisms in the linear system, and TEG is a necessary condition for transition in shear flows. In this work, we want to analyze the effect the nonlinear term has on the MTEG of the WKH system.

For a given region R, we investigate the effect of the nonlinearity on MTEG by solving the convex optimization problem in Eq. (19) over various Re. Figure 3 indicates that the MTEG for the nonlinear system converges to the linear system for small values of R, e.g. $R = 10^{-3}$. This agrees with the MTEG for the linear system as expected. However, once the size of R is increased to $R = 10^{-2}$, we observe that the nonlinear terms exacerbate the MTEG values for $Re \geq 160$. Similarly, further increasing the region size to $R = 10^{-1}$ shows that for $Re > 50$, the MTEG values are higher than those observed in the case with $R = 10^{-2}$. This further shows that as nonlinear effects are increased, a significant increase in MTEG shows up at a much lower Re. It should be noted that for the case with $R = 10^{-1}$, the convex optimization problem becomes infeasible beyond $Re = 70$. From this analysis we note that the nonlinear term in the WKH model cannot be ignored, as it has significant influence on both local stability and MTEG of the system.

C. Drawing physical insights from this framework

Apart from providing a framework to analyze the WKH model, the quadratic constraints based methods can also be used to gain insights into the physics and dominating mechanisms of the system. We analyze the Lagrange multipliers obtained after solving the convex optimization problem for a given Re and R. The Lagrange multipliers provide information on how the objective function is changing with respect to the constraints, which highlights the importance of the corresponding constraints in the optimization problem.

Dominant Lagrange multipliers are then identified by plotting their values over various Re. In Figure 5, we show the Lagrange multipliers obtained during the MTEG analysis for $R = 0.01$, each Lagrange multiplier is associated with their corresponding nonlinear terms. Here, we see that the ξ_{p_2} associated with the nonlinear term δw^2 is approximately 100 times more dominant than ξ_{p_3} which is associated with the nonlinear term $\gamma wu - \delta wv$. Similar trends are observed for other values of R. This points to the fact that the nonlinear term δw^2 and $\gamma wu - \delta wv$ are the most dominant term in

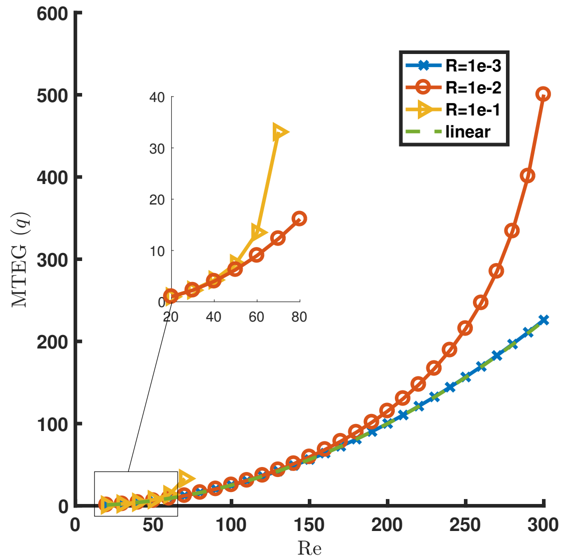

Fig. 4 As R increases, nonlinear effects become more substantial and MTEG increases.

comparison with other nonlinear terms in the model. We obtain similar findings related to dominating terms when comparing Lagrange multipliers in stability analysis results as well. Waleffe discusses similar findings in his paper by pointing to these terms as important. Also, the WKH model captures nonlinear effects feeding into the $\dot{v}$ and $\dot{w}$ terms and also identifies their roles in sustaining turbulence and energy conservation respectively.

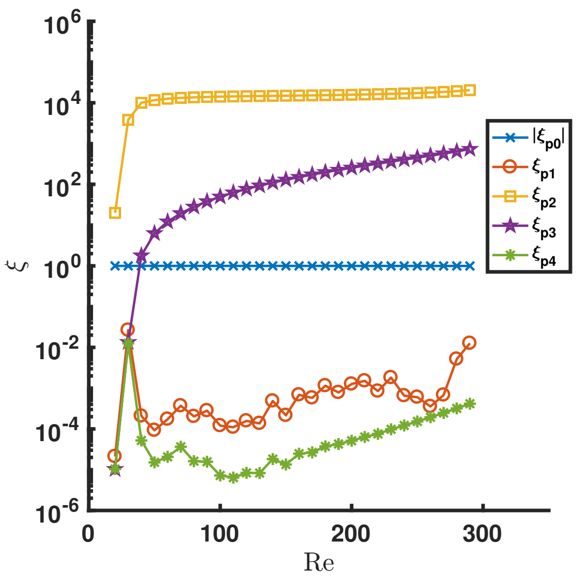

Fig. 5 The Lagrange multipliers shown against various Re. The two dominating nonlinear terms can be identified by analyzing the dominant Lagrange multipliers.

Therefore, from our analysis the effect of the dominating nonlinear terms is pronounced. A question that immediately arises is: if only the constraints associated with the two dominating nonlinear terms are retained in our MTEG analysis along with global lossless constraints, how would the predicted MTEG performance of the system change? Towards answering this question we choose an $R = 0.005$, we use only the lossless constraint along with constraints associated with the two dominating nonlinear term (see pink line in Figure 6) and compare results with the case where all the constraints are retained (see black line in Figure 6). We also analyze the MTEG trend with the global constraint and only the most dominant nonlinear term, i.e., δw^2 (shown by the green line in Figure 6). It is observed that only the δw^2 term captures the MTEG trend up to $Re \approx 150$, after which the estimates of the optimization method become conservative. Even though we report the case for $R = 0.005$, we observe similar qualitative trends for any other value of R for which the optimization problem is feasible.

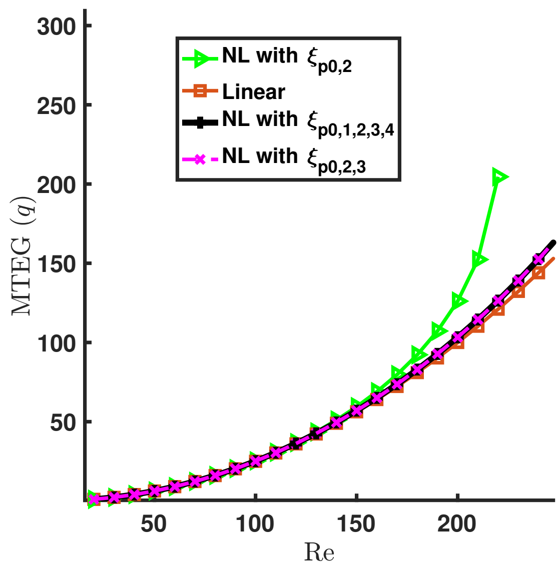

Fig. 6 Local MTEG analysis with global lossless constraint and two most dominant constraints compared against the MTEG of linear system as well as the system with global and all local constraints for $R = 0.005$.

V. Conclusion

In this work, we have presented a quadratic constraints framework to perform global and local stability and transient energy growth analysis of nonlinear systems. The framework uses exact information from the linear dynamics, while nonlinear terms are replaced by quadratic constraints that capture input-output properties of the nonlinearity. The proposed method is demonstrated on the WKH model. We first study the stability of the WKH model, for which the linear part is asymptotically stable for all Re. However, the nonlinear terms are destabilizing. It is found that the WKH system is not globally stable for $Re \geq 20$, consistent with previous results found in the literature. We also introduced a method for conducting transient energy growth analysis when the system is globally asymptotically stable. It was found that the maximum transient energy growth was unity below the critical Reynolds number.

In order to assess stability and transient energy growth performance above the critical Reynolds number, we introduced a new "local" analysis framework. Using this "local" analysis framework, we find that the nonlinear terms destabilize the system and also increase the maximum transient energy growth relative to the linear system. Lastly, analyzing the Lagrange multipliers associated with each "local" nonlinear constraints provides further insights into the physics. By comparing the relative magnitudes of the Lagrange multipliers, we are able to identify the dominating nonlinear effects in the system. The dominant nonlinear terms identified by this analysis are in agreement with the physical mechanisms originally described in [16].

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