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Revisiting the separation principle for improved transition control

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Feedback control can be used to suppress transient energy growth and delay turbulent transition in shear flows. The separation principle of modern control theory is commonly invoked to design observer-based control laws, whereby a full-state feedback controller and a state estimator are designed independently, then combined to achieve an output feedback flow control law. In previous work, we established that transient energy growth can never be fully eliminated by observer-based control, even when an associated full-state feedback law can fully suppress such growth. In this paper, we use a linearized channel flow to show that observer-based feedback will lead to higher levels of transient energy growth than if no control is used at all. Further, we show that transient energy growth can actually be reduced via optimal static output feedback controllers, thus overcoming the performance limitations of observer-based designs. We introduce a modified Anderson-Moore algorithm for efficiently computing optimal static output feedback controllers, then show that the resulting controllers reduce the worst-case transient energy growth relative to the uncontrolled system and to observer-based designs. Further, our results indicate that optimal static output feedback exhibits robustness to Reynolds number variations and modeling uncertainty. The result of this study highlight the advantages of optimal static output feedback control over observer-based designs and create opportunities for realizing improved transition control strategies in the future.

I. Introduction

An ability to delay transition to turbulence is of great interests, owing to the potential for drag reduction and energy savings in numerous engineering systems. In the present study, we consider the transition control problem from the standpoint of transient energy growth suppression in shear flow. In many linearly stable shear flows, it can be shown that small disturbances can be amplified significantly before eventual decay [1]. This undesired increase is called transient energy growth (TEG). TEG can trigger nonlinearities that give rise to turbulence, and so it is desirable to reduce or even fully suppress such growth.

Feedback controllers are often designed to reduce TEG in shear flows. Full-state feedback is usually the first choice for feasibility studies in numerical simulations. However, in practical applications, full-state feedback cannot be achieved because measurements of the full state are not directly available for feedback. Thus, in practice, only measured outputs from a limited set of sensors—typically confined to solid boundaries—can be used for output feedback [2]. One commonly used approach for output feedback design is to first estimate the full state of the flow from measured outputs using an observer, and then to use the estimated state for feedback through a full-state feedback law—i.e., observer-based feedback. The separation principle of modern control theory establishes that observer-based feedback will result in a stable closed-loop system if the full-state feedback controller is stabilizing and the observer is stable [2]. This strategy is commonly used because it greatly simplifies the design process: the controller and estimator can be designed separately, then combined to guarantee closed-loop stability.

Unfortunately, as we have shown in previous work [3], controller synthesis based on the separation principle leads to adverse consequences in terms of TEG performance. Specifically, if the uncontrolled system

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exhibits TEG, then observer-based designs are guaranteed to exhibit TEG from some disturbances as well. This result has fundamental significance to transition delay strategies, since observer-based feedback can never fully eliminate TEG if the uncontrolled flow exhibits TEG. In the present paper, we show that observer-based feedback control of a linearized channel flow results in significantly higher levels of TEG relative to full-state feedback. In fact, we find that observer-based feedback can even degrade TEG performance relative to the worst-case TEG response of the uncontrolled flow. These findings extend our previous results regarding the performance limitations of observer-based feedback.

The performance limitations of observer-based feedback control in the context of TEG reduction and transition delay suggest that alternative strategies for output feedback are advisable. For example, static and dynamic output feedback controllers can be designed to achieve minimal TEG [4, 5]. These approaches have been shown to yield reduced TEG in a number of illustrative examples; however, the synthesis techniques tend to be computationally demanding, and thus difficult to use in the context of fluid flow control—especially without the availability of sufficiently reliable reduced-order models. Further, the resulting controllers can lack robustness to model uncertainties and parameter variations, making these designs less likely to perform as desired when applied in off-design conditions on actual fluid flows.

In the present paper, we propose to use static output feedback (SOF) controllers for optimal regulation. Specifically, we solve the linear quadratic regulation (LQR) problem with an additional constraint that the control law have a SOF structure. As we will show, the resulting SOF-LQR controllers outperform observer-based designs and reduce the worst-case TEG relative to the uncontrolled flow. Indeed, SOF controllers constitute semi-proper control laws, thus satisfying a necessary condition for overcoming the TEG performance limitations of observer-based feedback [3]. We further introduce an accelerated gradient method—based on the Anderson-Moore algorithm [6]—for determining the unique SOF-LQR law, making the synthesis task computationally tractable for many shear flows. As we will show in the context of linearized channel flow, SOF-LQR controllers exhibit robust TEG reduction in the face of model uncertainties and parameter variations. As such, SOF-LQR stands as a viable candidate for transition delay via output feedback control in future studies.

In Section II, we summarize the shortcomings of observer-based feedback for TEG reduction and present an accelerated Anderson-Moore algorithm for designing SOF-LQR controllers that can overcome these limitations. In Section III, a linearized channel flow is used to illustrate the performance benefits of SOF-LQR control in comparison to standard observer-based designs. In Section IV, we discuss the results and draw conclusions.

II. Feedback flow control synthesis

II.A. Shortcomings of the separation principle for transient energy growth suppression

Consider the state-space representation of the linearized Navier-Stokes equations about a laminar equilibrium solution,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^p$ is the output vector, and $t \in \mathbb{R}$ is the time. For an initial flow perturbation $x(t_o) = x_o$, the system response is given in terms of the matrix exponential $x(t) = \mathrm{e}^{A(t-t_o)}x_o$. The associated perturbation energy is given as,

$$E(t) = x^T(t)Qx(t), \tag{2}$$

where $Q = Q^T > 0$. Further, the maximum TEG is defined as [3],

$$G = \max_{t \geq t_0} \max_{E(t_0) \neq 0} \frac{E(t) - E(t_0)}{E(t_0)}, \tag{3}$$

which results from a so-called *worst-case* or *optimal disturbance* [7]. The perturbation energy will never exceed its initial value when $G = 0$. In contrast, certain perturbations will result in non-trivial TEG whenever $G > 0$.

Feedback controllers have been shown to reduce TEG in various shear flows. In particular, the linear quadratic regulator (LQR) is a well-known design technique that has been proven to be successful at reducing

TEG in previous flow control studies [8–10]. LQR synthesis is based on solving,

$$\min_{u(t)} J = \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)dt \tag{4}$$

subject to the linear dynamic constraint

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5}$$

where $R > 0$. The resulting LQR controller is a full-state feedback law of the form $u(t) = -Kx(t)$, which can be determined from the solution of an algebraic Riccati equation [2]. LQR controllers are particularly appealing because they can be tuned to reduce TEG, while demonstrating robustness to parametric and modeling uncertainties. However, outside of numerical simulations, standard full-state feedback LQR controllers are typically not practically viable for flow control; standard LQR control requires knowledge of the full state of the flow, which is usually not directly available for feedback in practice.

When full-state feedback is not a viable option, an observer (i.e., state estimator) is usually designed to estimate the current state of the flow from available sensor measurements $y(t)$. A linear quadratic estimator (LQE) is often used to estimate the states of the flow. LQE provides the optimal (minimum variance) estimate of the state from noisy sensor measurements and an uncertain process model. Supplying LQE state estimates to the LQR feedback law yields a linear quadratic Gaussian (LQG) control law, which results in a stable closed-loop system. Considering the error signal $e(t) = x(t) - \hat{x}(t)$ to be the difference between the flow states and the state estimates, the closed-loop observer-based dynamics can be expressed as,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{6}$$

where L is the observer gain. The separation principle of modern control theory is straightforward to establish from this expression, since the eigenvalues of the closed-loop system $\tilde{A}$ are simply the union of the eigenvalues of $(A-BK)$ and $(A-LC)$; thus, the stability of the closed-loop system in (6) is guaranteed when the controller gain K is stabilizing (i.e., $A-BK$ is stable) and the estimator is stable (i.e., $A-LC$ is stable) [2]. Note, this result is general and is not restricted to linear quadratic synthesis. We emphasize here that the separation principle only guarantees closed-loop stability, *not* closed-loop performance. When closed-loop performance is a prime consideration, the separation principle must be invoked with caution; indeed, in the context of TEG suppression, the separation principle tends to be a poor design choice. Consider that TEG is typically attributed to being a byproduct of system non-normality [1, 11]. Then, upon inspection of $\tilde{A}$ in (6), it is clear that observer-based feedback yields a non-normal closed-loop system structure (i.e., $\tilde{A}\tilde{A}^T \neq \tilde{A}^T\tilde{A}$), creating the possibility of TEG. Indeed, as we proved in previous work [3], invoking the separation principle for observer-based feedback design is guaranteed to result in TEG whenever the uncontrolled system exhibits TEG. The growth in this case can still be viewed as a byproduct of system non-normality, but now due to an adverse "cyber-physical" interaction between the control system dynamics and the fluid dynamics [3]. In fact, even if the modes of the physical system under full-state feedback $(A-BK)$ can be orthogonalized [12], invoking the separation principle for observer-based control guarantees that the closed-loop system in (6) will be non-normal.

Given these observations, it is reasonable to expect that separation-principle-based designs will never outperform—in terms of TEG—the corresponding full-state feedback controllers from which they are designed. In the present study, we report results within the context of controlling a linearized channel flow that support this hypothesis (see Section III). Indeed, observer-based feedback degrades TEG performance and results in a higher degree of TEG than the associated full-state feedback controller. Furthermore, in the channel flow study, the performance degradation is so dramatic that TEG increases even relative to the worst-case TEG associated with the uncontrolled system. In the next section, we introduce an alternative static output feedback strategy in an effort to overcome the performance limitations of the separation principle and associated observer-based designs.

II.B. Optimal static output feedback synthesis

Given the shortcomings of observer-based feedback described above, we propose an alternate strategy for TEG reduction based on the standard LQR problem introduced in Section II.A, but now with an additional constraint that the resulting feedback law is to have a static output feedback (SOF) structure,

$$u(t) = Fy(t),\ F \in \mathbb{R}^{m\times p}. \tag{7}$$

The benefit of the SOF control structure is that the input is determined directly from the measured output, removing the need for an observer. However, this benefit comes at the cost of a slightly more involved controller synthesis procedure, which we describe next.

The closed-loop dynamics under SOF-LQR control are of the form,

$$\dot{x} = (A + BFC)x. \tag{8}$$

Thus, the standard LQR objective function in (4) can be rewritten to conform to a SOF structure,

$$J = \int_0^\infty x^T(t)[Q + (FC)^T R(FC)]x(t)dt. \tag{9}$$

The solution of this SOF-LQR problem can be calculated iteratively using Anderson-Moore methods [6]. To do so, define the set of all stabilizing SOF controllers $D_s = \{F \in \mathbb{R}^{m\times p} | Re\{\lambda(A + BFC)\} < 0\}$. Then, re-write the SOF-LQR design problem as [13],

$$\text{Minimize}\ \ J(F) = \mathrm{tr}[S(F)X]\ \ \text{subject to}\ \ F \in D_s, \tag{10}$$

where $S(F)$ is a solution to

$$S(F)[A + BFC] + [A + BFC]^T S(F) + C^T F^T RFC + Q = 0. \tag{11}$$

and $X = x(0)x(0)^T$[14]. The gradient of the cost function J with respect to the SOF control gain F can be expressed as,

$$\frac{\partial J}{\partial F} = 2[B^T S(F)L(F)C^T + RFCL(F)C^T] \tag{12}$$

where $L(F)$ is the solution to the Lyapunov equation,

$$L(F)[A + BFC]^T + [A + BFC]L(F) + X = 0. \tag{13}$$

Then, the optimal SOF gain $F^* \in D_s$ will be a minimizer of (10) and so must satisfy a zero gradient conditions, which reduced to

$$[B^T S(F^*)L(F^*)C^T + RF^*CL(F^*)C^T] = 0. \tag{14}$$

After some further manipulation, we find that a necessary condition for optimality is

$$F = -R^{-1}[B^T S(F)L(F)C^T][CL(F)C^T]^{-1}, \tag{15}$$

yielding a search direction for use in an Anderson-Moore method,

$$T = -F - R^{-1}[B^T S(F)L(F)C^T][CL(F)C^T]^{-1}. \tag{16}$$

Although the expressions above are sufficient for implementing the Anderson-Moore method, such methods tend to require a significantly large number of iterations. For high-dimensional fluid flows, each iteration can require a significant computational demand, and so it is desirable to reduce the total number of iterations through an accelerated technique. The step-size α along the gradient direction must be chosen with care, in order to balance precision with the total number of iterations. An inappropriate choice of α will either lead to slow convergence or to miss the optimal solution. In order to overcome this challenge, we formulate an accelerated Anderson-Moore algorithm that incorporates Armijo-type adaptations. Instead of using a fixed step-size α, we instead use an Armijo-rule [15] to adaptively update the step-size to achieve a better balance between precision and iteration count. A similar method to the one we propose here was originally presented in [16]; however, during our investigation on controlling linearized channel flow, we found that the specific algorithm in [16] would violate the condition that $F \in D_s$ during iterations. As such, we modified the approach to ensure convergence to a stabilizing optimal control law for the linearized channel flow configuration. Although our resulting method exhibits slower convergence than the method proposed in [16], it still outperforms the standard Anderson-Moore algorithm in terms of iteration count. Furthermore, the specific Armijo-type adaptation we propose guarantees convergence to a stabilizing SOF control law, and thus provides the optimal SOF-LQR solution for our flow control problem. The method we propose and use in this study is summarized below.

Anderson-Moore algorithm with Armijo-type adaptation

step 0: Initialize F_0 to be any $F_0 \in D_s$. Set $0 < \alpha < 1$, $0 < \sigma < 1/2$, and $\delta > 0$.
step 1: Solve (11) for $S(F_i)$.
step 2: Solve (13) for $L(F_i)$.
step 3: Use (16) to find the smallest integer $m_1 \geq 1$ such that $F_i + \alpha^{m_1} T_i \in D_s$.
step 4: Find the smallest integer $m_2 \geq m_1$ such that

$$J(F_i + \alpha^{m_2} T_i) \leq J(F_i) + \sigma \alpha^{m_2} \text{tr}(\frac{\partial J}{\partial F}^T T_i).$$

step 5: Find integer $n \in \{m_1, \cdots, m_2\}$ such that

$$J(F_i + \alpha^n T_i) = \min J(F_i + \alpha^j T_i), \text{ where } j \in \{m_1, \cdots, m_2\}$$

step 6: Set $F_{i+1} = F_i + \alpha^n T_i$, $i = i + 1$
step 7: Check $\|\frac{\partial J}{\partial F}\|_2 \leq \delta$, if true, stop. Otherwise, go to **step 1**.

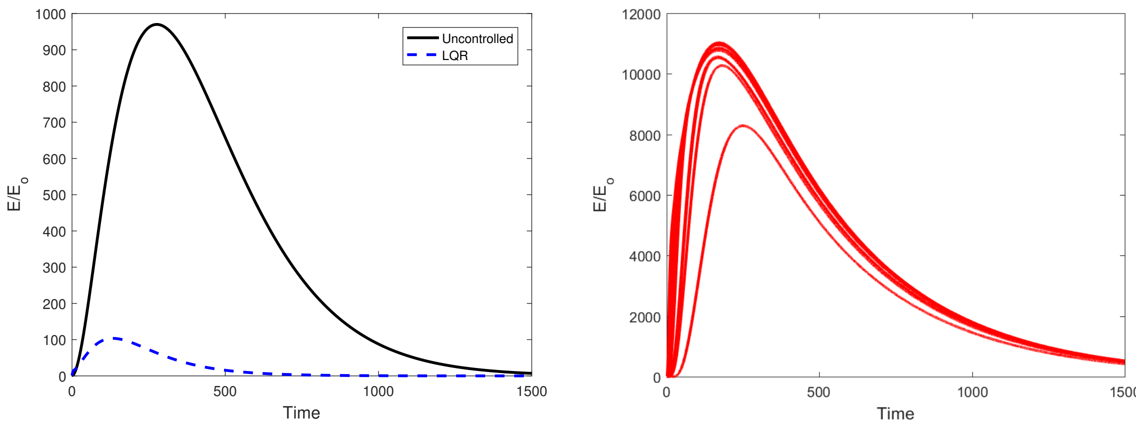
Note that all Anderson-Moore methods require initialization with a stabilizing SOF gain $F \in D_s$ [16]. For an asymptotically stable system, setting $F_0 = 0$ is a valid choice; however, if the linearized channel flow is not asymptotically stable, then setting $F_0 = 0$ as a starting value causes the algorithm to fail. In the present study, actuator dynamics are modeled via an integral term, and so the resulting linear system model will not be asymptotically stable. Thus, we first determine a stabilizing static output feedback gain using the ILMI method proposed in [17], then proceed to compute the optimal SOF-LQR controller using the accelerated Anderson-Moore algorithm above.

III. Results

The controller synthesis approaches described above will be applied to a linearized channel flow. Blowing and suction actuation is applied at the upper- and lower-walls of the channel, with the blowing and suction rate at each wall serving as a system input. Shear-stress measurements at the upper- and lower-walls are used for output feedback. The channel flow dynamics are modeled by means of a Fourier-Fourier-Chebyshev spectral collocation method. In the present study, all results correspond to the streamwise and spanwise wave number pair $(\alpha, \beta) = (0, 1)$ for $Re = 3000$. Full details of this model are outlined in [18].

In studying control system performance, we will primarily consider the worst-case response of the system by determining the associated optimal disturbance x_o that leads to the maximum TEG. Here, the optimal disturbance is calculated by means of the method described in [19]. In the context of observer-based LQG control, some additional care is required in performing these calculations, since we are interested in the worst-case disturbance that maximizes the physical energy of the system. Since the observer-based feedback system has a "cyber-physical" state $\tilde{x} = (x, \hat{x})$, we define the physical energy in terms of a weighting matrix $W_\epsilon = \text{diag}(Q, \epsilon Q)$, where Q is the same weighting matrix specified in equation (2). Then, by choosing a sufficiently small ϵ, the optimal disturbance calculated with weighting matrix W_ϵ will approximate the optimal disturbance that maximized the physical TEG of the system. Further, note that the state estimates for these optimal disturbances tend to be several orders of magnitude larger than the physical states of the system. Although it is possible to encounter such disturbances in practice, we opt to consider the response of the LQG controller to an alternative *sub-optimal disturbance* in the results presented here. After computing the optimal disturbance for the LQG controller, we retain the physical disturbance from this solution, but initialize the estimator states to zero—as is commonly done in practice. Thus, the LQG responses presented here are not the worst-case responses; whereas, all other responses reported here do correspond to the worst-case.

Results from different control approaches are shown in Figure 1. Figure 1a shows the worst-case response of the uncontrolled channel flow (solid), which is linearly stable at $Re = 3000$ and exhibits a high degree of TEG. Also plotted in Figure 1a is the worst-case response of the LQR controlled system (dashed), which reduces TEG relative to the uncontrolled response. The responses of twelve different LQG controlled systems subject to sub-optimal disturbances are shown in Figure 1b. Each LQG controller is designed using the same controller gain as is used for LQR control case, but differ from one another in that the LQE observer gains used for state estimation are designed by varying the associated weighting matrices in the objective function specification. Comparing the responses in Figure 1, it is clear that even the best performing LQG controller degrades TEG performance relative to the LQR controller and even relative to the uncontrolled channel flow. Finally, recall that the LQG performance reported here does not even consider the worst-case response to an optimal disturbance. In response to an optimal disturbance, LQG will result in even higher levels of TEG. Thus, in practice, even the smallest of flow disturbances can lead to undesirable TEG owing to errors in the state estimates. The results here suggest that LQG and other observer-based feedback strategies based on the separation principle can be surprisingly poor candidates for TEG reduction and transition delay.



(a) LQR and uncontrolled response to optimal disturbance **(b)** LQG response to sub-optimal disturbance

Figure 1: Spanwise wave case $\alpha = 0$, $\beta = 1$ at $Re = 3000$. Under critical Reynolds number, the open-loop system is linearly stable with high TEG (solid). A selected LQR controller can reduce the amount of the maximum TEG to around 103.5 (dashed). But LQG result in high levels of TEG.

We next investigate the worst-case response of SOF-LQR controllers. Figure 2 clearly shows that the worst-case SOF-LQR response reduces TEG relative to the worst-case uncontrolled response. This result alone highlights that SOF-LQR can overcome the performance limitations of observer-based feedback. Still, full-state feedback LQR control is able to reduce TEG more effectively than SOF-LQR; however, greater TEG reductions can be achieved with SOF-LQR if additional sensors are available for feedback.

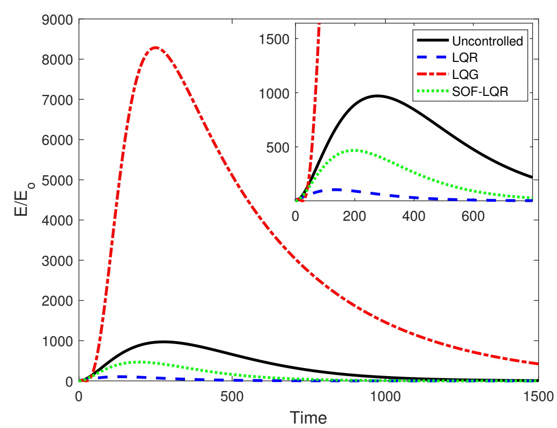
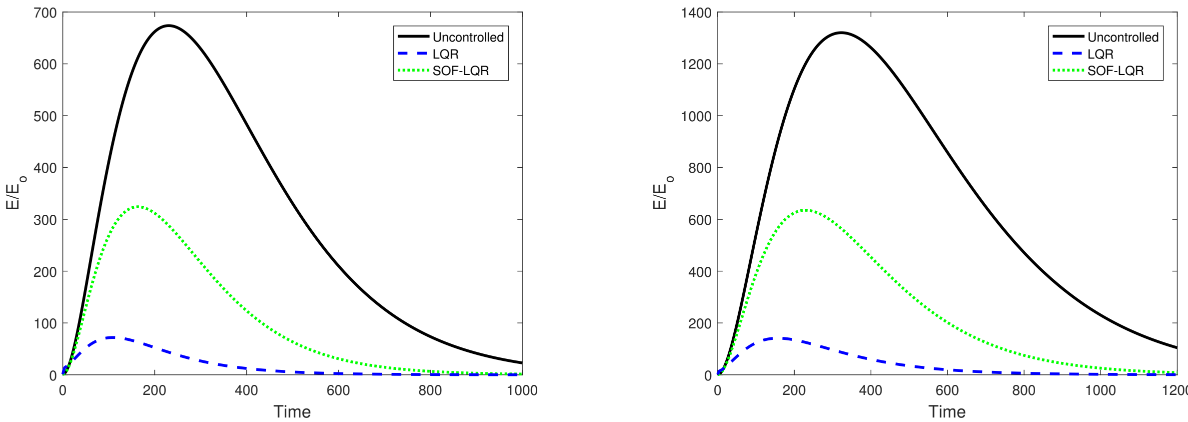


Figure 2: Spanwise wave case $\alpha = 0$, $\beta = 1$ at $Re = 3000$. Though cannot work as well as LQR, SOF-LQR still performances better than LQG controllers regarding to TEG suppression.

A final study that demonstrates the value of SOF-LQR control, beyond simply reducing the worst-case TEG, is to consider performance in off-design conditions. Figure 3 shows the worst-case response of the linearized channel flow under SOF-LQR control for $Re = 2500$ and $Re = 3500$, whereas the SOF-LQR controller was designed for $Re = 3000$. Here, the wave number pair $(\alpha, \beta) = (0, 1)$ is the same for both design and testing. The responses in these off-design conditions clearly show that SOF-LQR exhibits robustness to parameter variations and modeling uncertainty, as does the full-state feedback LQR controller. Indeed, these robustness results establish another potential advantage of SOF-LQR over LQG synthesis, which lacks robustness guarantees [20].



(a) $Re = 2500$

(b) $Re = 3500$

Figure 3: LQR and SOF-LQR both exhibit robust TEG reduction with Reynolds number variations.

IV. Discussion and conclusion

This paper considered the problem of TEG reduction in a linearized channel flow using output feedback control strategies. Standard observer-based feedback strategies predicated on the separation principle of modern control were found to exhibit dramatic performance degradation with regards to TEG. Subjecting the LQG controlled system to sub-optimal disturbances led to significantly higher TEG than the worst-case response of the uncontrolled system. We proposed an alternative SOF-LQR feedback strategy and proposed an accelerated Anderson-Moore algorithm to compute the associated optimal feedback gains. SOF-LQR control of the linearized channel flow was found to reduce the worst-case TEG relative to the uncontrolled flow. Further, this method was found to exhibit robust TEG reduction in the face of modeling uncertainties and Reynolds number variations.

The present study was restricted to a limited portion of the parameter space, i.e., $(\alpha, \beta, Re) = (0, 1, 3000)$. Future investigations will focus on designing controllers for other wave number pairs and Reynolds numbers, to establish whether or not the results of this study are universal. Nevertheless, the ability of SOF-LQR control to robustly and effectively reduce TEG makes this a viable alternative to observer-based feedback strategies, which have demonstrable performance limitations. Indeed, future investigations will be conducted on devising SOF-LQR strategies for improved transition control.

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