Estimation with lossy measurements: jump estimators for jump systems

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Abstract

The problem of discrete time state estimation with lossy measurements is considered. This problem arises, for example, when measurement data is communicated over wireless channels subject to random interference. We describe the loss probabilities with Markov chains and model the joint plant / measurement loss process as a Markovian Jump Linear System. The time-varying Kalman estimator (TVKE) is known to solve a standard optimal estimation problem for Jump Linear Systems. Though the TVKE is optimal, a simpler estimator design, which we term a Jump Linear Estimator (JLE), is introduced to cope with losses. A JLE has predictor/corrector form, but at each time instant selects a corrector gain from a finite set of precalculated gains. The motivation for the JLE is twofold. First, the real-time computational cost of the JLE is less than the TVKE. Second, the JLE provides an upper bound on TVKE performance. In this paper, a special class of JLE, termed Finite Loss History Estimators (FLHE), which uses a canonical gain selection logic is considered. A notion of optimality for the FLHE is defined and an optimal synthesis method is given. In a simulation study for a double integrator system, performances are compared to both TVKE and theoretical predictions.

I. Introduction

This paper describes a state estimation scheme for discrete time plants with lossy measurements. The key idea inspiring the proposed scheme is that the recent history of lossiness should be sufficient to determine an appropriate corrective action for the estimator.

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We believe the estimation scheme is applicable when real-time data is sent across a lossy communication channel. Such scenarios might arise in an industrial facility [1] or in the coordinated motion of vehicles [2]. The estimation scheme can be motivated by both practical and theoretical considerations. Practically, the proposed scheme provides a computationally inexpensive way to handle lossy measurements. Theoretically, the proposed scheme provides bounds on the performance of time-varying Kalman estimation (TVKE).

For packet-based transmission of data over a wireless network, radio wave propagation (e.g. multipath fading) increases bit error rates relative to wired links by several orders of magnitude. There is an apparent lack (in the systems and control literature) of real-time-relevant wireless transmission error models, but empirical observations have been used to develop probabilistic characterizations of packet losses [3]. Markov chains are therefore used in what follows to describe probabalistic losses. The developed methods are thus generic to the extent that random processes can be described by Markov chains. The data packets are assumed to be transmitted at fixed time intervals and subject to transmission errors. It is also assumed that the transmitted information can be classified as lost (L) or received (R) at the receiving end of the channel (by virtue of unspecified coding and error correction algorithms). The real-time knowledge of packet losses is critical to the proposed estimator design; its actions are based on knowing the recent history of lossiness.

A system with measurement losses of this type can be modeled as a jump system, switching between modes with and without output. For an otherwise standard state estimation problem, the time-varying Kalman estimator (TVKE) gives optimal estimates for the states of this jump linear system. The TVKE will be optimal for all systems with known time-variation. The systems under consideration, however, are structured in two important ways:

1. The time-variation is not arbitrary – the system switches between two fixed modes.

2. The mode switching is governed by a random process with known characteristics.

The first feature underscores the combinatorial nature of this estimation problem. The number of potential loss scenarios which can occur in a finite time interval is an exponential function of the interval length. The design of a jump linear estimator, discussed in Section IV, uses the second feature to define optimality in terms of expected estimation errors.

Jump linear systems have been used to describe time-varying delays [4] as well as communication losses without delay [2]. A discussion of the stability of Markovian jump linear systems including a definition of mean-square stability can be found in [5]. Recently, control and estimation problems for jump linear systems have been the subject of several studies (e.g. [6], [7]). The most relevant to our results is the work by Ji and Chizeck on the LQG problem for Markovian jump linear systems [8], [9]. Optimal estimation for jump linear systems was part of their discussion of the LQG problem. Though not developed as such, our main results might readily be considered an extension of their work. The new/different characteristics of our study of the estimation problem are as follows:

1. Development/presentation is focused on the problem of lossy communication.
2. The steady-state distribution on Markov chain states is used as a fundamental component of our formulation.
3. Optimality for what we call Finite Loss History Estimators (FLHEs) is defined and an iterative synthesis method for optimal FLHEs is described.
4. The idea of "powering-up" the Markov chain to design more complex FLHEs and get tighter, more detailed bounds on TVKE performance is introduced.

Our results are in the spirit of standard steady-state optimal estimation for discrete LTI systems. While the estimators we propose are not time-invariant, their time-variation is related to the time-variation of the plant in a "steady-state" way.

In Section II, a standard estimation problem is revisited in the context of lossy measurements. Subsequent sections of the paper are, however, organized in a non-standard way. A simulation example of TVKE applied to a double integrator system with measurement losses is introduced in Section III. This example is intended to (further) arouse the reader's curiosity and facilitate introduction of our design concept in Section IV. There, we introduce the Jump Linear estimator (JLE) and Finite Loss History Estimator (FLHE), define optimality, and give an optimal synthesis procedure. Then, in Section V, the double integrator example is revisited, with comparisons of simulation results for FLHE and TVKE. Finally, our outlook for the future and a summary of the content of this paper are given in Section VI. A number of technical lemmas, found in Appendix A, were used to prove the main results.

II. A Standard Estimation Problem

In this section, we briefly review a standard estimation result as it applies to the lossy measurement scenario. This review will serve to introduce some notation and will also give some context to the results obtained in the following sections. Consider the following discrete, time-varying system:

$$x(k+1) = Ax(k) + Bw(k) \qquad\qquad y(k) = C_{\theta(k)}x(k) + v(k) \tag{1}$$

where $\theta(k) \in \{L, R\}$ describes the output mode[1], with $C_R = C$ and $C_L = 0$. The system is time-varying due only to the dependence of the output equation on $\theta(k)$. The initial condition, $x(0)$, is Gaussian with mean, x_0, and variance, M_0. The process and sensor noises are white and Gaussian with covariance matrices given by $W > 0$ and $V > 0$ respectively.

The following is a standard quadratic optimal estimation problem:

Problem 1: Given measurements $\mathcal{Y}_k = \{y(0), \ldots, y(k)\}$ and $\Theta_k = \{\theta(0), \ldots, \theta(k)\}$ for

[1] The system may have multiple outputs, though they will all be lost or received together when this model is used. Independent loss of multiple outputs can be captured by using more output modes.

the plant Eq (1), find the state estimate, $\hat{x}(k)$, which minimizes:

$$J_Z(k) = \mathrm{E}\left[\ \|x(k) - \hat{x}(k)\|^2\ \right] \tag{2}$$

The time-varying Kalman estimator (TVKE) [10] is the well-known solution to this estimation problem. The following notation is used to describe the TVKE:

Optimal Estimate:	$\hat{x}(k\|j) = \mathrm{E}[\ x(k)\ \|\ \mathcal{Y}_j, \Theta_j\ ]$
Estimation Error:	$\tilde{x}(k\|j) = x(k) - \hat{x}(k\|j)$
Error Covariances:	$Z(k) = \mathrm{E}\left[\ \tilde{x}(k\|k)\tilde{x}(k\|k)^T\ \right]$
	$M(k+1) = \mathrm{E}\left[\ \tilde{x}(k+1\|k)\tilde{x}(k+1\|k)^T\ \right]$

For TVKE, the state estimate is computed using a predictor/corrector form:

$$\begin{aligned}\hat{x}(k|k) &= \hat{x}(k|k-1) + F(k)\left[y(k) - C_{\theta(k)}\hat{x}(k|k-1)\right] \\ \hat{x}(k|k-1) &= A\hat{x}(k-1|k-1)\end{aligned} \tag{3}$$

The time-varying corrector gain, $F(k)$, is computed recursively in real-time, as follows:

$$F(k) = M(k)C_{\theta(k)}^T\left[C_{\theta(k)}M(k)C_{\theta(k)}^T + V\right]^{-1} \tag{4}$$

$$Z(k) = M(k) - F(k)\left[C_{\theta(k)}M(k)C_{\theta(k)}^T + V\right]F(k)^T \tag{5}$$

$$M(k+1) = AZ(k)A^T + BWB^T \tag{6}$$

Initial conditions for the estimator are given by: $\hat{x}(0|-1) = x_0$ and $M(0) = M_0$.

III. Estimation over a Wireless Network

In this section, TVKE is applied to a system with probabilistic measurement loss. The system is modeled as a Markovian jump linear system. The simulation results show that the TVKE corrector gains behave in an approximately jump linear fashion. Presenting

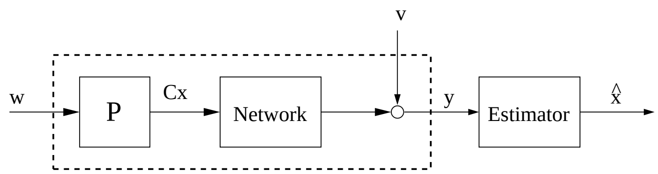

Fig. 1. Estimation using measurements communicated from a remote sensor

simulation results for the TVKE at this point allows the development of the Jump Linear Estimator (JLE), and more specifically the Finite Loss History Estimator (FLHE), to be interpreted in the context of TVKE's behavior.

Consider the estimation problem depicted in Figure 1. At each sample time, a remote sensor communicates measurements over a wireless network as packets which may be lost. The measurement mode of the packet (L-loss or R-reception) is assumed to be available to the estimator at each time instant. The combined plant/network system viewed by the estimator is taken as in Eq (1). All assumptions on the plant and disturbances made in Section II apply in what follows.

The packet loss process can be described with a Markov model [3]. Here, the packet losses are a random process modeled by the two state Markov chain in Figure 3(a). As indicated in the figure, the probability of a packet loss after a reception is given by γ while the probability of a loss following a loss is given by α ($0 \leq \alpha, \gamma \leq 1$). This Markov chain defines a probability distribution on sequences of $\theta(k)$. A network in which packet losses occur in infrequent but lengthy bursts can be effectively modeled with selections of γ small and α large. To represent a more complicated loss behavior, a Markov chain with more than two states can be used.

A double integrator system, of the form in Eq (1), with the following state matrices is

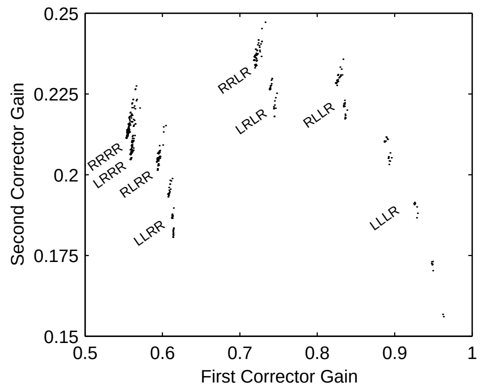

Fig. 2. Components of the 2×1 corrector gain used by TVKE when applied to a double integrator system with lossy measurements.

used as a pedagogical example throughout this paper:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad C_R = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad C_L = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{7}$$

The following covariances, describing the process noise, measurement noise, and initial estimation error are used:

$$W = 0.1, \qquad V = 1.0, \qquad M_0 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \tag{8}$$

The two state Markov chain is used to generate a random sequence of measurement loss/reception, with $\gamma = 0.3$ and $\alpha = 0.5$ (see Figure 3). A TVKE was applied to this system in simulation (using MATLAB) for 4000 samples. Figure 2 shows a scatter plot of the first vs. second component of the 2×1 corrector gains, $F(k)$ used by the TVKE.

The corrector gain is always set to zero when data is lost ($\theta(k) = L$). These zero gains are not shown in the figure. The first 100 gains are also not shown – they are in some sense transient behavior of this time varying system, depending more upon initial conditions than the stochastic disturbances. The clustering of the remaining gains is according to preceding sequences of loss/receptions. The label $RRR\mathbf{R}$, for example, indicates gains used when the current reception was preceded by three receptions ($[\theta(k-3), \theta(k-2), \theta(k-1), \theta(k)] =$

$[R, R, R, \mathbf{R}]$). The bold letter indicates the status of the most recent packet (e.g. $LLR\mathbf{R}$ is two losses followed by two receptions). The gain used by TVKE is not an exact function of the past four measurement modes. However, knowing the past four measurement modes does allow an educated guess for the TVKE gain to be made. To some degree then, the action taken by TVKE *is* a function of the preceding sequence of loss/reception. This is a major point of the paper. The proposed estimator design uses this observed characteristic, though with a formalized approach. In contrast with TVKE, the optimal synthesis of the proposed estimators depends upon the probability distributions assumed for $\theta(k)$.

IV. Jump Linear Estimation

In this section, estimators that reduce real-time computation relative to TVKE by estimating with pre-computed gains are introduced. This reduction in computational requirements is accompanied by a sacrifice in performance relative to TVKE. As discussed earlier, the jump systems under consideration have two features which can be exploited: they have a finite number of modes (two), and measurement lossiness is governed by a Markov chain. Based on Figure 2, it seems reasonable that a finite number of gains might be sufficient to mimic the behavior of the TVKE. The remainder of this section is organized as follows: First, a generic Jump Linear Estimator (JLE) is described. Then, the restriction to designs which use a finite history of the loss/reception are introduced, called Finite Loss History Estimators (FLHEs). A meaningful optimal design problem for FLHEs is then formulated and solved. Both the optimal estimation problem and solution can be extended to the more generic JLE structure, but the FLHE design is emphasized to simplify the technical details.

A. Jump Linear and Finite Loss History Estimators

An estimator of the following form, termed a Jump Linear Estimator, is proposed:

$$\begin{aligned}\hat{x}(k|k) &= \hat{x}(k|k-1) + F_{n(k)}[y(k) - C_{\theta(k)}\hat{x}(k|k-1)] \\ \hat{x}(k|k-1) &= A\hat{x}(k-1|k-1)\end{aligned} \tag{9}$$

where $n(k) \in \mathcal{N} = \{1, \ldots, N_n\}$. In the remainder of the paper, $\hat{x}(k|j)$ is used to denote the state estimate at time k using information up until time j (not necessarily using the optimal TVKE). At each time step, a corrector gain is chosen from a finite set of pre-computed gains, $F_{n(k)} \in \mathcal{F} = \{F_1, \ldots, F_{N_n}\}$. Restricted to this structure, the estimator design consists of:

1. Choose number of gains, N_n
2. Assign set of gains, $\mathcal{F}$
3. Describe switching logic used to select among the gains

This is a rich class of estimators and can lead to a variety of designs (a few can be found in [11]). For the sake of clarity, we consider designs based on a canonical switching logic which we call "finite loss history". The corrector gain is selected based on the last r measurement loss modes, so $N_n = 2^r$. We refer to these designs as Finite Loss History Estimators (FLHE).

Consider a sequence of losses (L) and receptions (R). Let $f : \{L, R\}^r \to \{1, \ldots, 2^r\}$ denote a numbering of the 2^r possible sequences of length r. The corrector gain logic can be represented as $n(k) = f(\Theta_{k-r+1,k})$ where $\Theta_{k-r+1,k} := [\theta(k-r+1), \ldots, \theta(k)]$. For example, the following numbering is used when $r = 2$: $f([R, R]) = 1$, $f([L, R]) = 2$, $f([R, L]) = 3$, and $f([L, L]) = 4$. At time k, the precomputed gains F_1, F_2, F_3, and F_4 are applied when $[\theta(k-1), \theta(k)] = [R, R]$, $[L, R]$, $[R, L]$, $[L, L]$, respectively. The function f for $r = 3$ is given in Appendix B. The numberings assign smaller indices to sequences with more recent receptions. The length of history used, r, will be referred to as the *order* of an FLHE.

While the corrector gain used by TVKE depends on initial conditions and all past history of $\theta(k)$, a FLHE uses a corrector gain dependent only on the past r loss process observations. Since the number of gains and the switching logic has been predefined as part of the FLHE structure, design of a FLHE for fixed r consists of computing the set of 2^r corrector gains, $\mathcal{F}$. For simple problems, this might be done by inspecting a scatter plot of the TVKE gains (e.g. Figure 2). For higher dimensional systems, FLHE gains might be obtained by averaging the TVKE gains after particular loss process sequences. These ad-hoc design methods, though potentially practical, will not be pursued further in this paper. Instead, we will formulate and solve an optimal FLHE problem.

B. An optimal FLHE design problem

In this section, we first formulate a meaningful performance cost for a FLHE. In completing this task, we also introduce notation and technical results which will be required later. At first glance, it seems natural to use the cost that was defined in Problem 1:

$$J_Z(k) = \mathrm{E}\left[\ \|x(k) - \hat{x}(k|k)\|^2\ \right] = \mathrm{Tr}\left[Z(k)\right] \tag{10}$$

The second equality follows from the definition of $Z(k)$. Instead, a cost based on the predicted error covariance will be considered:

$$J_M(k) = \mathrm{E}\left[\ \|x(k+1) - \hat{x}(k+1|k)\|^2\ \right] = \mathrm{Tr}\left[M(k+1)\right] \tag{11}$$

Working with $J_M(k)$ simplifies the proofs in the following section. Moreover, an FLHE design minimizes $J_Z(k)$ if and only if it minimizes $J_M(k)$, so both costs yield the same optimal design.

For any FLHE, the predicted error covariance, $M(k)$, evolves as follows:

$$M(k+1) = A(I - F_{n(k)}C_{\theta(k)})M(k)(I - F_{n(k)}C_{\theta(k)})^T A^T + AF_{n(k)}VF_{n(k)}^T A^T + BWB^T \tag{12}$$

Given the initial condition $M(0) = M_0$, we can compute $M(k+1)$ by iterating Eq (12) forward in time. However, this can not be done *a-priori* because the iteration requires knowledge of Θ_k and $\mathcal{N}_k = \{n(0), n(1), \ldots, n(k)\}$. Thus $M(k+1)$ depends on the particular instantiation of Θ_k and $\mathcal{N}_k$. Consequently, finding the FLHE that minimizes $\mathrm{Tr}\,[M(k+1)]$ is not a well-defined problem.

One way to create a well-defined problem of similar spirit is to consider the average performance across all sample paths, $\mathrm{Tr}\,[\mathrm{E}_{\mathcal{N}_k}\,[\ M(k+1)\ ]]$. Given a probability distribution on $\mathcal{N}_k$, the cost for FLHE can be computed offline[2]. Fortunately, the probability distribution on $\mathcal{N}_k$ is specified by the probability distribution on $\theta(k)$ and the switching logic used by the estimator. The choice of canonical switching logic implies that the probability distribution on $n(k) = f(\Theta_{k-r+1,k})$ evolves according to a 'powered up' Markov chain. Examples of powered up Markov chains for $r = 2, 3$ are shown in Figure 3(b)-(c).

The expected Markov chain state at time index k may be described in terms of a probability distribution on its state space – the aforementioned powered up Markov chains have 2^r states. This distribution is then represented as a row vector with entries $v_j(k) = \Pr\{\ n(k) = j\ \}$, for $j \in \mathcal{N} = \{1, \ldots, 2^r\}$. The one-step transition matrix of the Markov chain, $P = [p_{ij}]_{i,j\in\mathcal{N}}$, governs the evolution of these probability distributions: $v(k+1) = v(k)P$. The transition matrix, P, has the following properties:

1. $0 \le p_{ij} \le 1 \ \ \forall\ i, j \in \mathcal{N}$
2. $p_{ij} = \Pr\{\ n(k) = j \mid n(k-1) = i\ \} \ \ \forall\ k > 0$
3. $\sum_{j\in\mathcal{N}} p_{ij} = 1 \ \ \forall\ i \in \mathcal{N}$.

The entries of the transition matrix corresponding to the Markov chains in Figure 3 can be written in terms of the probabilities α and γ as given in Appendix B.

[2] The identical cost may also be computed offline for TVKE. The computational burdens depend differently on k: exponential for TVKE and linear for FLHE.

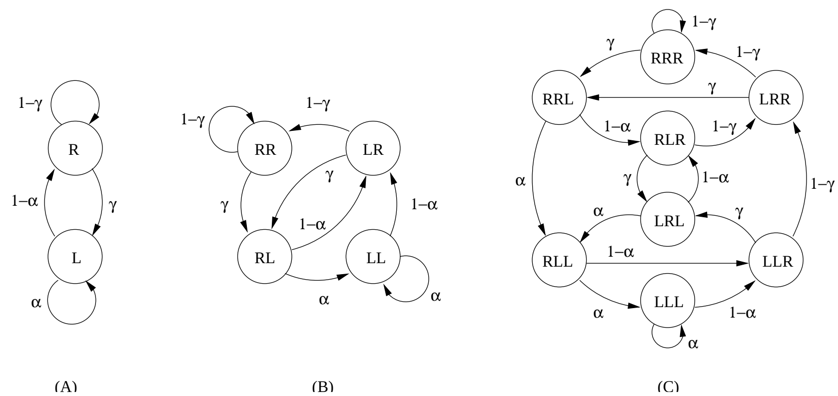

Fig. 3. Markov chain for communication loss process: $r = 1$ (a), $r = 2$ (b) and $r = 3$ (c)

Using this probability distribution on the sequence $n(k)$ inherited from that on $\theta(k)$, we can expand an expression for $\underset{\mathcal{N}_{k-1}}{\mathrm{E}}[\ M(k)\ ]$ in terms of conditional probabilities as follows:

$$\underset{\mathcal{N}_{k-1}}{\mathrm{E}}[\ M(k)\ ] = \sum_{i=1}^{N_n} \underbrace{\mathrm{E}_{\mathcal{N}_{k-2}}[\ M(k)\ |\ \ n(k-1) = i\ ]}_{:=M_i(k)} v_i(k-1) \tag{13}$$

Given a fixed set of corrector gains, the modal covariances defined above, $M_i(k)$, can be shown to satisfy a recursion (which will be given in Lemma 1).

To define this recursion, we will make use of the following conditional probability:

$$p^*_{ij}(k) = \Pr\{\ n(k-1) = j\ |\ \ n(k) = i\ \} \overset{(a)}{=} \frac{v_j(k-1)p_{ji}}{v_i(k)} \tag{14}$$

where equality (a) can be established using Bayes' Rule.

Define the affine operator $\mathcal{A}_{F_i}(\cdot)$ for each $i \in \mathcal{N}$ as (see Eq (12)):

$$\mathcal{A}_{F_i}(M) \doteq A(I - F_iC_{g(i)})M(I - F_iC_{g(i)})^TA^T + AF_iVF_i^TA^T + BWB^T \tag{15}$$

where $g : \{1, \ldots, 2^r\} \to \{L, R\}$ associates the appropriate measurement loss mode with a state of the Markov chain (See App B for examples). The iteration for the modal covariances is stated in the following lemma (the $\mathcal{A}$ is for affine):

Lemma 1 ($\mathcal{A}$-iteration) Assume a system with measurement losses, and FLHE design as above, having a fixed set of corrector gains, $\{F_i\}$. The modal covariances, $M_i(k)$, satisfy the following recursion $\forall i \in \mathcal{N}$:

$$M_i(k+1) = \mathcal{A}_{F_i}\left(M_{\mathrm{pre},i}(k)\right) \qquad \text{where} \qquad M_{\mathrm{pre},i}(k) := \sum_{j=1}^{N_n} p^*_{ij}(k) M_j(k) \tag{16}$$

Proof:

$$\begin{aligned} M_i(k+1) &= \mathop{\mathrm{E}}_{n(k-1)}\left[\mathop{\mathrm{E}}_{\mathcal{N}_{k-2}}\left[\ M(k+1)\ \middle|\ n(k)=i\ \right] \middle|\ n(k)=i \right] \\ &\overset{(a)}{=} \mathop{\mathrm{E}}_{n(k-1)}\left[\mathop{\mathrm{E}}_{\mathcal{N}_{k-2}}\left[\ \mathcal{A}_{F_i}\left(M(k)\right)\ \right] \middle|\ n(k)=i \right] \\ &\overset{(b)}{=} \sum_{j=1}^{N_n} p^*_{ij}(k) \mathop{\mathrm{E}}_{\mathcal{N}_{k-2}}\left[\ \mathcal{A}_{F_i}\left(M(k)\right)\ \middle|\ n(k-1)=j\ \right] \\ &\overset{(c)}{=} \mathcal{A}_{F_i}\left(M_{\mathrm{pre},i}(k)\right) \end{aligned}$$

Equality (a) follows from the error covariance recursion (Eq (12)) and the conditional knowledge that $n(k) = i$. Applying the outer conditional expectation and using the defined p^*_{ij}(k) yields equality (b). Equality (c) follows because $\mathcal{A}_{F_i}(\cdot)$ is affine and $\sum_{j=1}^{N_n} p^*_{ij}(k) = 1$. ■

Note the meaning of the iterates $M_i(k)$ and $M_{\mathrm{pre},i}(k)$:

$$M_i(k) = \mathop{\mathrm{E}}_{\mathcal{N}_{k-2}}\left[\ M(k)\ \middle|\ n(k-1)=i\ \right], \qquad M_{\mathrm{pre},i}(k) = \mathop{\mathrm{E}}_{\mathcal{N}_{k-1}}\left[\ M(k)\ \middle|\ n(k)=i\ \right]$$

The subscript 'pre' is used for the expected prediction error covariance at time k, once the measurement mode at time k is known. The 'pre' refers to these being expected error covariances *prior* to corrective action based on the measurement at time k. Since they are defined *after* the measurement mode at time k is known, one might prefer that these covariances were subscripted 'post'. For no particular reason, we use the former notation.

Given an fixed set of corrector gains, $\mathrm{Tr}\left[\mathrm{E}_{\mathcal{N}_k}\left[\ M(k+1)\ \right]\right]$ at time k can be computed directly with Eq (13) and the $\mathcal{A}$-iteration. However, minimizing this cost at a particular time instant, k, does not yield an estimation problem that we feel is of practical interest. We are interested in the performance of the estimator at all times rather than at a particular instant. The desired objective is therefore phrased in terms of the limit as $k \to \infty$ as follows:

Problem 2 (Optimal FLHE) Assume a given plant (as in Eq (1)), associated Markovian measurement loss structure, and a length r for an FLHE design. If the outputs $\mathcal{Y}_k = \{y(0), \ldots, y(k)\}$ and measurement losses $\Theta_k = \{\theta(0), \ldots, \theta(k)\}$ up until time k are given, find the set of gains for the given FLHE structure which minimizes:

$$J_\infty = \lim_{k\to\infty} \mathrm{Tr}\left[\mathop{\mathrm{E}}_{\mathcal{N}_{k-1}} [\ M(k)\ ] \right] \tag{17}$$

The limit in this problem formulation is proved to exist when the estimator is stable and (A, B) is a controllable pair. We will use the following definition of stability:

Definition 1: The estimator is *stable* if there exists $c \in \mathbb{R}$ such that:

$$\mathrm{Tr}\left[\mathrm{E}_{\mathcal{N}_{k-1}} [\ M(k)\ ]\right] \leq c < \infty \quad \forall k$$

A similar definition is given in [12]. The proof which shows that the Problem 2 is well posed relies on a simplifying assumption for the initial probability distribution on the state of the Markov chain. Refer to the Markov chains shown in Figure 3. When $0 < \alpha, \gamma < 1$ then the transition matrix, P, satisfies several standard conditions: regularity, aperiodicity, irreducibility [13]. This guarantees the existence of a unique steady-state probability distribution, v^{ss}, which satisfies $v^{ss}P = v^{ss}$ and $\lim_{k\to\infty} v_j(k) = v_j^{ss} \neq 0\ \forall j \in \mathcal{N}$ and $\forall v_j(0)$. Based on the cost given above, the following assumption is made:

Assumption: $v(0) = v^{ss}$

We highlight this assumption because it will be in effect for the remainder of the paper and has important consequences: $v(k) = v^{ss}$ and therefore $p^*_{ij}(k) = \frac{v_j^{ss} p_{ji}}{v_i^{ss}}$ for all k (p^*_{ij} denotes this time invariant $p^*_{ij}(k)$)[3]. The $\mathcal{A}$-iteration is then linear and time-invariant, simplifying proofs. This assumption is not expected to be restrictive because J_∞ stresses the average performance as $k \to \infty$, when $v_j(k)$ will be close to v_j^{ss}. The following theorem establishes that J_∞ is well posed:

[3] The probabilities p^*_{ij} are defined differently by Ji and Chizeck in [8], where the calculation is based on a uniform distribution on the state space rather than the steady-state distribution.

Theorem 1 (Steady State Modal Covariances) Assume (A, B) controllable. If the FLHE is stable then $\lim_{k\to\infty} \mathrm{Tr}\left[\underset{\mathcal{N}_{k-1}}{\mathrm{E}}\left[\ M(k)\ \right]\right]$ exists.

Proof: The $\mathcal{A}$-iteration (Lemma 1) is governed by a collection of affine operators, $\{\mathcal{A}_{F_i}(\cdot)\}$. Thus the map from $\{M_i(k)\}$ to $\{M_i(k+1)\}$ is also affine. The $\mathcal{A}$-iteration is stable if the spectral radius of its linear part is strictly less than 1.

Let $\{M_i(k)\}$ be a solution to the $\mathcal{A}$-iteration started with initial conditions $M_i(0) = 0\ \forall\ i \in \mathcal{N}$. By Lemma 2 in Appendix A, the iterates are monotonic: $M_i(k+1) \geq M_i(k) \geq 0$ $\forall i \in \mathcal{N}$. By Eq (13), $\mathrm{Tr}\left[\mathrm{E}_{\mathcal{N}_{k-1}}\left[\ M(k)\ \right]\right] \geq v_i^{ss}\mathrm{Tr}\left[M_i(k)\right]$ and thus the assumption of stability implies that the modal covariances are uniformly bounded. The modal covariances are bounded, monotonic matrix sequences, hence they must converge to a limit. Denote the limit matrices by $M_i^{ss} \doteq \lim_{k\to\infty} M_i(k)\ \forall i \in \mathcal{N}$. $\{M_i^{ss}\}$ must be a fixed point of the $\mathcal{A}$-iteration and by Lemma 6, each M_i^{ss} is positive definite.

Let $\{\tilde{M}_i(k)\}$ denote an alternate solution to the $\mathcal{A}$-iteration with initial conditions satisfying $0 \leq \tilde{M}_i(0) \leq M_i^{ss}\ \forall i \in \mathcal{N}$. These A-iterates can be bounded above and below $\forall i \in \mathcal{N}$ as follows: $M_i(k) \leq \tilde{M}_i(k) \leq M_i^{ss}$. Both inequalities follow from Lemma 3. As shown above, $\lim_{k\to\infty} M_i(k) = M_i^{ss}$ and hence $\lim_{k\to\infty} \tilde{M}_i(k) = M_i^{ss}$.

Since the limit matrices are positive definite, the above sandwich argument shows that when the initial conditions deviate from the limit matrices in any negative semidefinite direction, the $\mathcal{A}$-iteration converges to the limit matrices. The set of symmetric matrices are an invariant set of the $\mathcal{A}$-iteration. Since a negative semidefinite basis exists for the set of symmetric matrices, the spectral radius of the linear portion of the $\mathcal{A}$-iteration restricted to symmetric matrices is strictly less than 1 and the $\mathcal{A}$-iteration will converge to the given limit matrices for any initial symmetric iterates. The theorem now follows from Eq (13). ■

C. Synthesis of Optimal FLHE Gains

In this section, the optimal FLHE design problem is solved with an iterative synthesis of the set of corrector gains. Theorem 2 gives a necessary and sufficient condition for the existence of a stable FLHE of given order. Assuming a stable FLHE exists, Theorem 3 gives a construction for a set of FLHE gains which are optimal with respect to the estimation cost, J_∞. This construction follows from the proof of Theorem 2. The proofs of these results rely heavily on monotonicity and relative ordering properties associated with both the $\mathcal{A}$-iteration and the $\mathcal{R}$-iteration that is introduced in the following theorem:

Theorem 2 (Existence of Stable FLHE) Assume (A,B) controllable. For each $i \in \mathcal{N}$,

$$\mathcal{R}_i(M) \doteq AMA^T + BWB^T - AMC_{g(i)}^T \left(C_{g(i)}MC_{g(i)}^T + V\right)^{-1} C_{g(i)}MA^T$$

The $\mathcal{R}$-iteration ($\mathcal{R}$ is for Riccati) is then defined as:

$$M_i(k+1) = \mathcal{R}_i(M_{\text{pre},i}(k)) \qquad M_{\text{pre},i}(k) = \sum_{j=1}^{N_n} p_{ij}^* M_j(k) \qquad \forall i \in \mathcal{N} \tag{18}$$

There exists a stable FLHE **iff** the $\mathcal{R}$-iteration converges.

Proof: ($\Leftarrow$)Assume that the $\mathcal{R}$-iteration fails to converge when started from zero initial conditions. Lemma 4 in the appendix states that this iteration is monotonically nondecreasing, $M_i(k+1) \geq M_i(k) \geq 0 \;\; \forall i \in \mathcal{N}$. The $\mathcal{R}$-iteration therefore diverges, and there exists at least one $i^* \in \mathcal{N}$ such that $M_{i^*}(k)$ grows unbounded as $k \to \infty$.

Using Lemma 5 in the appendix, the iterates of the $\mathcal{A}$-iteration for any $\mathcal{F}$ are bounded below by the iterates of the $\mathcal{R}$-iteration started with zero initial conditions. Therefore, one of the A-iterates diverges. Since $\text{Tr}\left[\text{E}_{\mathcal{N}_{k-1}}\left[\ M(k)\ \right]\right] \geq v_i^{ss}\text{Tr}\left[M_i(k)\right] \forall i$, this implies that $J_M(k)$ diverges to infinity. Hence the FLHE with corrector gains given by $\mathcal{F}$ is unstable. ■

Proof: ($\Rightarrow$) Assume the $\mathcal{R}$-iteration converges to a limit when started from zero initial conditions. Denote the limit matrices by $M_j^{ss} \doteq \lim_{k\to\infty} M_j(k)\ \forall j \in \mathcal{N}$. $\{M_j^{ss}\}$ must be a fixed point of the $\mathcal{R}$-iteration and by Lemma 4, all of the matrices be positive semidefinite. Choose $\{F_i\}$ in terms of $\{M_i^{ss}\}$ as follows:

$$F_i \doteq M_{\text{pre},i}^{ss} C_{g(i)}^T \left(C_{g(i)} M_{\text{pre},i}^{ss} C_{g(i)}^T + V \right)^{-1} \tag{19}$$

To complete the proof, this set of gains must be shown to yield a stable FLHE. By completing the square, $\{M_i^{ss}\}$ is a fixed point for the $\mathcal{A}$-iteration using $\mathcal{F}$ and each M_i^{ss} is positive definite (Lemma 6 and the controllability assumption).

Let $\{\tilde{M}_i(k)\}$ denote the A-iterates for the corrector gains in Eq (19). Assume initial conditions for the $\mathcal{A}$-iteration satisfying $0 \leq \tilde{M}_i(0) \leq M_i^{ss}$ $\forall i \in \mathcal{N}$. The A-iterates can be bounded above and below $\forall i \in \mathcal{N}$ as follows: $M_i(k) \leq \tilde{M}_i(k) \leq M_i^{ss}$. The first inequality follows from Lemma 5. The second inequality follows through application of Lemma 3 using $\{\tilde{M}_i(k)\}$ and the $\mathcal{A}$-iteration started at the fixed point $\{M_i^{ss}\}$. By assumption, $\lim_{k\to\infty} M_i(k) = M_i^{ss}$ and hence $\lim_{k\to\infty} \tilde{M}_i(k) = M_i^{ss}$.

The above sandwiching of iterates leads to the same argument used in the proof of Theorem 1. The $\mathcal{A}$-iteration will converge to the given limit matrices for any initial symmetric iterates. By Eq (13), $\mathrm{E}_{\mathcal{N}_{k-1}}[\ M(k)\ ]$ is then uniformly bounded and thus the FLHE is stable.

■

Theorem 3 (Optimality) For a given plant, Markov chain, and order of FLHE, assume that the $\mathcal{R}$-iteration with initial conditions $M_i(0) = 0\ \forall\ i \in \mathcal{N}$ converges to $\{M_i^{ss}\}$. Let $M_{\text{pre},i}(k) \doteq \sum_{j=1}^{N_n} p_{ij}^* M_j(k)$ and choose the corrector gains as follows:

$$F_i \doteq M_{\text{pre},i}^{ss} C_{g(i)}^T \left(C_{g(i)} M_{\text{pre},i}^{ss} C_{g(i)}^T + V \right)^{-1} \ \ \forall i \in \mathcal{N} \tag{20}$$

These gains are stable and optimal in terms of the cost, J_∞. Moreover, for any other set of stable corrector gains, $\tilde{\mathcal{F}}$, $\lim_{k\to\infty} \tilde{M}_j(k) \geq M_j^{ss}$ $\forall j \in \mathcal{N}$.

Proof: Stability of the FLHE with the gains defined in Eq (20) follows from the proof of Theorem 2. It follows from that same proof and Lemma 5 that $\lim_{k\to\infty} \tilde{M}_j(k) \geq M_j^{ss}$

$\forall j \in \mathcal{N}$. The optimality with respect to the cost J_∞ follows from Eq (13). ■

The gains constructed in the proof of Theorem 3 are optimal with respect to J_∞. Note that the $\mathcal{R}$-iteration is a deterministic method for finding these optimal corrector gains, and will fail only if no set of stable gains exists. In the next section, this $\mathcal{R}$-iteration based design is used to design an optimal FLHE for the double integrator example.

V. Simulation Study

The estimation problem parameters introduced in Section III are again used in this section. The plant is a double integrator, Eq (7), with Markovian measurement losses described earlier ($\alpha = 0.5$ and $\gamma = 0.3$ in Figure 3). The gains used by TVKE when applied to this system in simulation were shown in Figure 2. In this section, further simulation results for TVKE and various FLHEs are used to illustrate the following points:

1. The optimal FLHE corrector gains roughly agree with an intuitive design based on the TVKE gains in Figure 2.
2. As the amount of loss history, r, used by the FLHE increases, the averaged performance becomes more like that of TVKE.
3. There are diminishing returns, in terms of J_∞, associated with increasing the amount of history used by the FLHE.

These are expected to be characteristic of more complex scenarios as well.

Optimal FLHE designs were carried out for this example system with $r = 1$, $r = 2$ and $r = 3$. For each design, the corresponding Markov chain in Figure 3 was used. Recall that following a lost measurement, the corrector gain will always be zero. In Figure 4, the optimal nonzero corrector gains (those which follow receptions) computed for these designs are shown overlaying the TVKE gains from Figure 2. For $r = 1$, a single nonzero gain is used whenever the measurement is received. As shown in the figure, this single gain makes a compromise

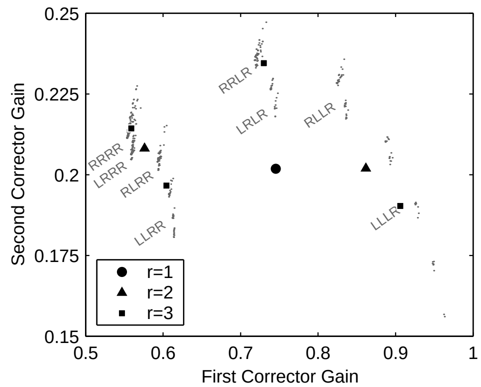

Fig. 4. Components of computed FLHE gains for $r = 1, 2, 3$ compared with gains applied by TVKE.

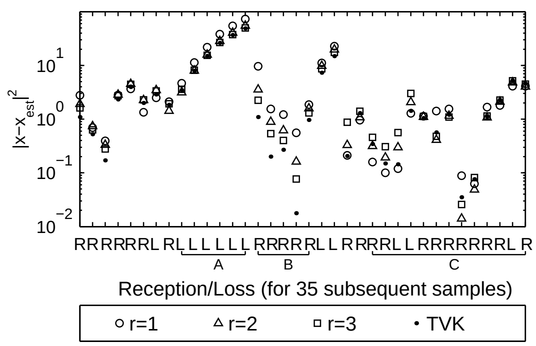

Fig. 5. Comparison of TVKE and FLHE ($r = 1, 2, 3$) performance for one instantiation of process noise, sensor noise and measurement loss.

among all the nonzero gains used by the TVKE. In contrast, the design for $r = 2$ leads to two nonzero gains, F_{LR} and F_{RR}. Again, the figure shows that F_{RR} makes a compromise between all TVKE gains for loss sequences ending with $R\mathbf{R}$. Similar interpretations can be made for F_{LR} and the four nonzero gains used in the $r = 3$ design.

In Figure 5, representative time traces of norm squared estimation error are shown using a logarithmic vertical scale. The reception/loss sequence is indicated on the horizontal axis with one label per sample instance (‘L’ for loss, ‘R’ for reception). The simulations for the different estimators are carried out with the same instantiation of measurement loss, process noise, and measurement noise as well as the same initial conditions. During the long string of losses (labeled ‘A’), the norm squared error increases at roughly the same rate for all the estimators. All estimators are operating in open loop – the corrector gain is zero. This string of losses is long (six samples) relative to the order of the FLHEs. Referring back to Figure 4, the TVKE applies a much wider variety of gains following sequences of three or more losses. Therefore, when receptions follow this long string of losses, it is expected that the FLHE designs will not recover accurate estimates as quickly as the TVKE. It can also be expected that higher order FLHE designs will recover faster. While no general conclusions can be drawn from a single instantiation, Figure 5 does support these expectations. In recovering from the losses (section labeled 'B') the TVKE recovers estimation errors the fastest, followed by the FLHEs ordered from $r = 3$ to $r = 1$.

In the absence of a long string of losses (e.g. the sequence labeled 'C') the estimation errors using FLHE designs for $r = 2$ and $r = 3$ are not noticeably larger than those associated with TVKE. A more precise statement about performance can be made using the average behavior of the designs. The FLHE design minimizes the size of expected estimation error covariances: $\text{Tr}\,[M]$. For comparison with the TVKE, we will consider $\text{Tr}\,[Z]$ for each FLHE design. The corresponding modal covariances can be computed using the result of the iterative design as follows:

$$Z_i = M_i - F_i \left[C_{g(i)} M_i C_{g(i)}^T + V \right] F_i^T \qquad \text{for} \;\; i \in \mathcal{N}, \quad g(i) \in \{L, R\} \tag{21}$$

As described later, the results of many simulations, when averaged appropriately, are expected to generate empirical values which agree with these Z_i.

The following mode data indexed by $i \in \{1, \ldots, 2^r\}$ for FLHE designs with $r = 1, 2, 3$ is shown in Tables I, II, and III:

1. The optimal corrector gain, F_i, found from the procedure in Section IV.C.
2. The conditional expected squared estimation error, $\text{Tr}\, Z_i = \text{E}\,[\ \|x - \hat{x}\|^2 \mid n(k) = i\ ]$.
3. The probability, v_i^{ss}, of being in state i of the powered up Markov Chain.

TABLE I
FLHE DESIGN FOR $r = 1$

	R	L
F_i	$\begin{bmatrix} 0.745 \\ 0.202 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\text{Tr}\, Z_i$	0.94	4.31
v_i^{ss}	0.625	0.375

TABLE II
FLHE DESIGN FOR $r = 2$

	$R\mathbf{R}$	$L\mathbf{R}$	$R\mathbf{L}$	$L\mathbf{L}$
F_i	$\begin{bmatrix} 0.576 \\ 0.208 \end{bmatrix}$	$\begin{bmatrix} 0.862 \\ 0.202 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\text{Tr}\, Z_i$	0.759	1.05	1.64	6.72
v_i^{ss}	0.437	0.188	0.188	0.188

The limit of the total cost according to $J_Z(k)$ defined in Eq (10) can be computed as follows:

$$\lim_{k\to\infty} J_Z(k) = \sum_{i=1}^{2^r} v_i^{ss} \text{Tr}\, Z_i = \begin{cases} 2.205 & \text{for} \quad r = 1 \qquad 2.057 \quad \text{for} \quad r = 4 \\ 2.096 & \text{for} \quad r = 2 \qquad 2.052 \quad \text{for} \quad r = 5 \\ 2.069 & \text{for} \quad r = 3 \qquad 2.049 \quad \text{for} \quad r = 6 \end{cases} \tag{22}$$

This predicted cost decreases as r increases, though the returns diminish for larger r.

The Z_i computed from the limits of the $\mathcal{R}$-iteration are compared with averaged data from MATLAB simulations. The simulation data is averaged as follows to get empirical estimates of Z_i. First, a single instantiation of loss/reception (1000 samples) is generated using the Markov process. Then the systems (FLHE and TVKE) were simulated with 1000 instantiations of measurement and process noise using the same loss/reception sequence. For each time instant, the average squared estimation error over the 1000 different realizations of noise was computed. The average estimation errors were then classified according to finite loss history (e.g. the error at time 10 is classified as $R\mathbf{R}$ when $\theta(9) = \theta(10) = R$ and $r = 2$). The average of estimation errors with the same classification is an empirical estimate of $\text{Tr}\, Z_i$,

TABLE III
FLHE DESIGN FOR $r = 3$

	$RR\mathbf{R}$	$LR\mathbf{R}$	$RL\mathbf{R}$	$LL\mathbf{R}$
F_i	$\begin{bmatrix} 0.559 \\ 0.214 \end{bmatrix}$	$\begin{bmatrix} 0.604 \\ 0.197 \end{bmatrix}$	$\begin{bmatrix} 0.73 \\ 0.235 \end{bmatrix}$	$\begin{bmatrix} 0.906 \\ 0.19 \end{bmatrix}$
$\mathrm{Tr}\, Z_i$	0.732	0.791	0.906	1.1
v_i^{ss}	0.306	0.131	0.0938	0.0938
	$RR\mathbf{L}$	$LR\mathbf{L}$	$RL\mathbf{L}$	$LL\mathbf{L}$
F_i	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\mathrm{Tr}\, Z_i$	1.54	1.81	3.08	10.2
v_i^{ss}	0.131	0.0563	0.0938	0.0938

denoted $\mathrm{Tr}\, \hat{Z}_i$. Comparison of the estimators using the same instantiation of loss/reception is meaningful because the distribution on losses was not changed, though powered up Markov chains might have been used for the design. An instance of loss/reception 1000 samples long was deemed sufficiently long to capture a significant number of each pattern of loss/reception.

Figure 6 displays the averaged information in three subplots corresponding to $r = 1, 2, 3$. The horizontal axis of each subplot is labeled with the 2^r finite loss histories corresponding to the appropriate value of r. The vertical axes use a logarithmic scale. The analytically computed values of $\mathrm{Tr}\, Z_i$ for the optimal FLHE design are shown as circles. Averaged simulation results grouped by loss history are shown for TVKE (left) and FLHE (right) in each subplot. The small solid dots are the estimation errors at particular time instances, averaged over the different realizations of process noise. The horizontal lines indicate the average of a group of estimation errors, the empirical estimates $\mathrm{Tr}\, \hat{Z}_i$. Therefore, the FLHE horizontal lines and circles are expected to indicate identical estimation errors.

The same TVKE data (small dots) appears in all the subplots, though grouped differently. For example, consider the averaged TVKE error covariances for R in the $r = 1$ subplot. These points are divided into two groups in the $r = 2$ subplot: TVKE($R\mathbf{R}$) and

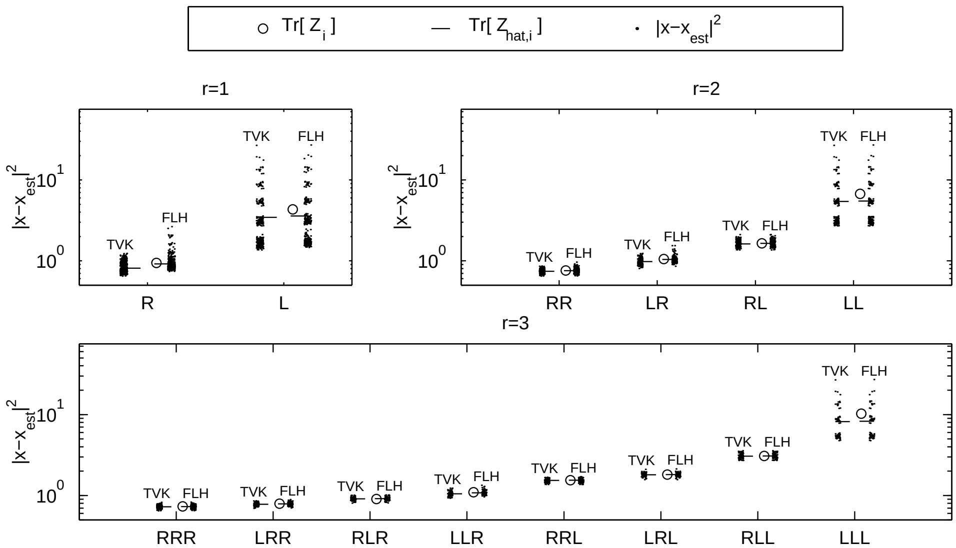

Fig. 6. Comparison of TVKE and FLHE ($r = 1, 2, 3$) performance for one instantiation of measurement loss and averaged over many instantiations of process and sensor noises.

TVKE($L\mathbf{R}$). These are the same data points, merely reclassified for comparison with the $r = 2$ FLHE. The TVKE $\operatorname{Tr} \hat{Z}_i$ is below the FLHE $\operatorname{Tr} \hat{Z}_i$ in every case as expected (TVKE is optimal). With $r = 1$, there is a noticeable difference between the performances of the TVKE and FLHE in the averaged sense (compare the horizontal lines). For $r = 2$ and 3, the average performance of the FLHE and TVKE is not as distinguishable.

Though $\operatorname{Tr} Z_i$ and $\operatorname{Tr} \hat{Z}_i$ for the FLHE are in close agreement in most cases, there appears to be a mismatch between $\operatorname{Tr} Z_i$ (circle) and $\operatorname{Tr} \hat{Z}_i$ (horizontal line) when i corresponds to a finite loss histories L, $L\mathbf{L}$, or $LL\mathbf{L}$. This is explained by the finite length (1000 samples) of our simulations. Rare loss sequences are not included in the simulation, but accounted for when solving the $\mathcal{R}$-iteration – they occur with a finite though small probability.

Finally, the banding of errors should be noted. This banding agrees with the grouping of gains seen in Figure 4. These effects are attributed to the combinatorial nature of the problem. In the $r = 1$ subplot, the errors associated with (L) form 6 (perhaps 7) identifiable bands. The lowest of these bands contains the the errors associated with ($R\mathbf{L}$). The remainder of the bands having errors associated with $(L\mathbf{L})$. Similar interpretations are possible comparing $r = 2$ to $r = 3$.

The simulation study results, though simple, encourage us to apply the proposed ideas in experimental settings.

VI. Conclusions

The Jump Linear Estimator (JLE) is able to deal with dropped measurements in a more cost effective manner than TVKE. Finite Loss History Estimators (FLHE) were emphasized as a simple, but useful form of the more generic JLE. The main theoretical results given in the paper are summarized as follows:

- (Theorem 1) An optimal quadratic estimation problem was shown to be well-posed for the system with Markovian measurement losses.
- (Theorem 2) The convergence of an iteration was shown to be a necessary and sufficient condition for the existence of a stable FLHE.
- (Theorem 3) A method of selecting optimal FLHE gains was given.

These results are analogous to standard results for steady-state Kalman estimation of LTI system states. Several of the Lemmas used in the proofs of the main results are generalizations of Lemmas used to prove the standard results. In [11], we referred to this generalization as pseudo-steady-state estimation. If control is collocated with estimation, the control signals are not subject to losses and state-feedback control can be based upon the state estimates without loss of stability.

The simulation study illustrated, in the particular example given, that the formalized approach generated FLHE gains which are intuitive when compared to the actions of TVKE. Moreover, the predicted and simulated FLHE performance agree and compare favorably with TVKE considering their relative real-time simplicity.

The theoretical results extend to more general Markovian Jump Linear Systems through appropriate modification of the $\mathcal{A}$-iteration and $\mathcal{R}$-iteration. Selected generalizations include: 1) different Markov chains for the loss process, 2) generic jump systems (A, B, C, W, V all vary with Markov state) [8], and 3) more general switching logic in the JLE (partially described in [11]). These generalizations do not alter the iterations in a way which interferes with the proofs of the theorems. These generalizations allow, for example, a scenario where multiple measurements are independently lost/received to be put in this framework.

The main results lead to several lines of future research. First, the existence of stable FLHE designs provided a sufficient condition for TVKE stability. A related open question is whether or not this is also a necessary condition for TVKE stability. We are only aware of sufficient conditions applicable to this problem in the literature. Second, the robustness/sensitivity of the optimal FLHE designs to perturbations of the Markov transition matrix has not been quantified. This is seen as a valuable area for future investigation because insensitivity to Markov chain parameters would justify the use of FLHE designs in scenarios where the probabilities of losses do not admit a stationary Markov chain model. For example, time-varying probabilities of communication loss are likely if the communicating agents are in a dynamic environment, as is the case with coordinated vehicle motion problems. Third, an alternative to actually carrying out the iterations in our development would be to directly solve the coupled $\mathcal{R}$-iteration equations for a steady state solution. This is analogous to solving the algebraic Riccati equation for the steady state Kalman estimator, where the eigenvalue decomposition of a Hamiltonian matrix can be used. A similar approach for the set of coupled algebraic Riccati equations has yet to be found [14], [15].

We were initially motivated to know when the state estimation error using TVKE remains bounded in spite of probabilistic measurement losses. This question remains unanswered, but in attempting to characterize TVKE performance with lossy measurements we were led to a class of estimators which leverage underlying Markov jump system structure to achieve comparable estimation to TVKE at a lesser real-time cost. The existence of such designs is itself a commentary on the problem's difficulty. In essence, we have endeavored to capitalize on the structure of the system in forming our estimator, which is philosophically similar (in a rough analogy) to both internal model and linear-parameter-varying approaches.

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Appendices

A. Auxiliary Results

Lemma 2 (Monotonicity of $\mathcal{A}$-iteration) The $\mathcal{A}$-iteration (Lemma 1) started with initial conditions $M_i(0) = 0 \; \forall \; i \in \mathcal{N}$ satisfies $M_i(k+1) \geq M_i(k) \geq 0 \; \forall i \in \mathcal{N}$.

Proof: The proof relies on the following fact which follows from the definition of $\mathcal{A}_{F_i}(\cdot)$:

$$M_1 \geq M_2 \geq 0 \;\Rightarrow\; \mathcal{A}_{F_i}(M_1) \geq \mathcal{A}_{F_i}(M_2) \geq 0 \;\; \forall i \in \mathcal{N} \tag{23}$$

The lemma is proved by induction. If $M_i(0) = 0 \; \forall i \in \mathcal{N}$, then $M_{\text{pre},i}(0) = 0 \; \forall i \in \mathcal{N}$. Hence $M_i(1) = AF_iVF_i^TA^T + BWB^T \geq 0 \; \forall i \in \mathcal{N}$ and the lemma statement holds for $k = 0$. Assume the lemma statement holds for $k-1$: $M_i(k) \geq M_i(k-1) \geq 0 \; \forall i \in \mathcal{N}$. It follows that $M_{\text{pre},i}(k) \geq M_{\text{pre},i}(k-1) \geq 0 \; \forall i \in \mathcal{N}$. Using Eq (23), $\mathcal{A}_{F_i}(M_{\text{pre},i}(k)) \geq \mathcal{A}_{F_i}(M_{\text{pre},i}(k-1)) \geq 0 \; \forall i \in \mathcal{N}$. The lemma statement therefore holds for k. ■

Lemma 3 (Ordering of $\mathcal{A}$-iteration) Given any set of corrector gains, $\mathcal{F}$, let $\{\tilde{M}_i(k)\}$ and $\{M_i(k)\}$ denote solutions of the $\mathcal{A}$-iteration. Assume their initial conditions satisfy: $0 \leq \tilde{M}_i(0) \leq M_i(0) \; \forall i \in \mathcal{N}$. Then $\tilde{M}_i(k) \leq M_i(k) \; \forall i \in \mathcal{N}$ and $\forall k$.

Proof: The proof is by induction. $\tilde{M}_i(k) \leq M_i(k) \; \forall i \in \mathcal{N}$ holds for $k = 0$ by the choice of initial conditions. Assume that it holds for some $k \geq 0$. Then $\tilde{M}_{\text{pre},i}(k) \leq M_{\text{pre},i}(k) \; \forall i \in \mathcal{N}$. It then follows from the definition of $\mathcal{A}_{F_i}(\cdot)$ that $\mathcal{A}_{F_i}\left(\tilde{M}_{\text{pre},i}(k)\right) \leq \mathcal{A}_{F_i}(M_{\text{pre},i}(k)) \;\; \forall i \in \mathcal{N}$. Thus $\tilde{M}_i(k+1) \leq M_i(k+1) \; \forall i \in \mathcal{N}$ and the lemma is true by induction. ■

Lemma 4 (Monotonicity of $\mathcal{R}$-iteration) The $\mathcal{R}$-iteration (Definition 18) started with initial conditions $M_i(0) = 0 \; \forall \; i \in \mathcal{N}$ satisfies $M_i(k+1) \geq M_i(k) \geq 0 \; \forall i \in \mathcal{N}$.

Proof: The proof relies on the following direct application of Lemma 3.1 in [16]:

$$M_1 \geq M_2 \geq 0 \;\Rightarrow\; \mathcal{R}_i(M_1) \geq \mathcal{R}_i(M_2) \geq 0 \;\; \forall i \in \mathcal{N} \tag{24}$$

The lemma is proved by induction. If $M_i(0) = 0\ \forall i \in \mathcal{N}$, then $M_{\text{pre},i}(0) = 0\ \forall i \in \mathcal{N}$. Hence $M_i(1) = BWB^T \geq 0\ \forall i \in \mathcal{N}$ and the lemma statement holds for $k = 0$. Assume the lemma statement holds for $k-1$: $M_i(k) \geq M_i(k-1) \geq 0\ \forall i \in \mathcal{N}$. It follows that $M_{\text{pre},i}(k) \geq M_{\text{pre},i}(k-1) \geq 0\ \forall i \in \mathcal{N}$. Using Eq (24) $\mathcal{R}_i(M_{\text{pre},i}(k)) \geq \mathcal{R}_i(M_{\text{pre},i}(k-1)) \geq 0\ \forall i \in \mathcal{N}$. The lemma statement therefore holds for k. ∎

Lemma 5 (Minimum Property) Given any set of corrector gains, $\mathcal{F}$, let $\{M_i(k)\}$ denote the solution of the $\mathcal{A}$-iteration with initial conditions given by $M_i(0) \geq 0\ \forall i \in \mathcal{N}$. Let $\{\tilde{M}_i(k)\}$ denote the solution of the $\mathcal{R}$-iteration starting at $\tilde{M}_i(0) = 0\ \forall i \in \mathcal{N}$. Then $\tilde{M}_i(k) \leq M_i(k)\ \forall i \in \mathcal{N}$ and $\forall k$.

Proof: This is a generalization of a result by Caines and Mayne [17] for the standard Riccati equation. For each $i \in \mathcal{N}$, $\mathcal{A}_{F_i}(M)$ can be rewritten as follows:

$$\mathcal{A}_{F_i}(M) = \mathcal{R}_i(M) + A(\tilde{F}_i - F_i)\left(C_{g(i)} M C_{g(i)}^T + V\right)(\tilde{F}_i - F_i)^T A^T$$

where $\tilde{F}_i$ is defined as: $\tilde{F}_i = M C_{g(i)}^T \left(C_{g(i)} M C_{g(i)}^T + V\right)^{-1}$.

The operator $\mathcal{R}_i(\cdot)$ is a minimum in the following sense:

$$\mathcal{R}_i(M) \leq \mathcal{A}_{F_i}(M) \;\; \forall i \in \mathcal{N} \tag{25}$$

This fact and induction are used to prove the lemma. For the given initial conditions, $\tilde{M}_i(0) \leq M_i(0)\ \forall i \in \mathcal{N}$. Assume the proposed inequality holds for k. Referring to the definition of the 'pre'-covariances in Eq (16), $\tilde{M}_{\text{pre},i}(k) \leq M_{\text{pre},i}(k)\ \forall i \in \mathcal{N}$. The following inequalities hold $\forall i \in \mathcal{N}$:

$$\tilde{M}_i(k+1) = \mathcal{R}_i(\tilde{M}_{\text{pre},i}(k)) \overset{(a)}{\leq} \mathcal{A}_{F_i}\left(\tilde{M}_{\text{pre},i}(k)\right) \overset{(b)}{\leq} \mathcal{A}_{F_i}\left(M_{\text{pre},i}(k)\right) = M_i(k+1) \tag{26}$$

Inequality (a) follows from Eq (25). Inequality (b) is a consequence of an easily verified fact: $M_1 \geq M_2 \Rightarrow \mathcal{A}_{F_i}(M_1) \geq \mathcal{A}_{F_i}(M_2) \quad \forall i \in \mathcal{N}$. The lemma follows by induction. ■

Lemma 6: Assume (A,B) controllable. Let $\{\overline{M}_i\}$ be a set of positive semi-definite matrices satisfying:

$$\overline{M}_{\mathrm{pre},i} = \sum_{j=1}^{N_n} p^*_{ij}\overline{M}_j \quad \text{and} \quad \overline{M}_i = \mathcal{A}_{F_i}\left(\overline{M}_{\mathrm{pre},i}\right) \quad \forall i \in \mathcal{N} \tag{27}$$

Then $\overline{M}_i > 0$ for all $i \in \mathcal{N}$.

Proof: This is a non-trivial proof. The critical part of the proof is a generalization of Theorem 4.1 in [18]. The covariances are bounded below by controllability grammians of a synthetic time varying system. It is shown that controllability of the synthetic system can be identified with controllability of the pair (A, B), proving the lemma.

Let $\{M_i(k)\}$ denote the solution of the $\mathcal{A}$-iteration starting with the initial conditions $\{\overline{M}_i\}$. By Eq (27), $M_i(k) = \overline{M}_i$ $\forall i \in \mathcal{N}$ and $\forall k$. Using the definition of the $\mathcal{A}$-iteration, this can be written as an equality for time $k+1$ as:

$$\overline{M}_i = M_i(k+1) = A_i M_{\mathrm{pre},i}(k) A_i^T + L_i L_i^T \tag{28}$$

where A_i and L_i have been defined as follows:

$$A_i \doteq A(I - F_i C_{g(i)}) \qquad\qquad L_i L_i^T \doteq A F_i V F_i^T A^T + BWB^T$$

Given any $i_k \in \mathcal{N}$, there exists a set of indices, $\{i_0, \ldots, i_{k-1}\}$, such that $\{p^*_{i_k i_{k-1}}, \ldots, p^*_{i_2 i_1}, p^*_{i_1 i_0}\}$ are all strictly positive. We can now use these indices and induction to lower bound $\overline{M}_{i_k}$:

$$\overline{M}_{i_k} = M_{i_k}(k+1) \geq \left(p^*_{i_k i_{k-1}} \cdots p^*_{i_2 i_1} \cdot p^*_{i_1 i_0}\right) \mathcal{C}(k)\mathcal{C}(k)^T \tag{29}$$

where $\mathcal{C}(k) \doteq \begin{bmatrix} L_{i_k} & A_{i_k} L_{i_{k-1}} & \ldots & \left(A_{i_k} A_{i_{k-1}} \cdots A_{i_1}\right) L_{i_0} \end{bmatrix}$. The proof that Eq (29) holds for $k = 0$ follows simply from Eq (28). Assume that Eq (29) holds for k. As shown below, it must also hold for $k+1$:

$$\begin{aligned} M_{i_k}(k+1) &= A_{i_k} M_{\text{pre},i_k}(k) A_{i_k}^T + L_{i_k} L_{i_k}^T \\ &\overset{(a)}{\geq} p^*_{i_k i_{k-1}} A_{i_k} M_{i_{k-1}}(k) A_{i_k}^T + L_{i_k} L_{i_k}^T \\ &\overset{(b)}{\geq} \left(p^*_{i_k i_{k-1}} \cdots p^*_{i_1 i_0}\right) A_{i_k} \mathcal{C}(k-1)\mathcal{C}(k-1)^T A_{i_k}^T + L_{i_k} L_{i_k}^T \\ &\geq \left(p^*_{i_k i_{k-1}} \cdots p^*_{i_1 i_0}\right) \mathcal{C}(k)\mathcal{C}(k)^T \end{aligned}$$

Inequality (a) uses a lower bound for $M_{\text{pre},i_k}(k)$ and inequality (b) uses the induction assumption. This string of inequalities shows that Eq (29) holds for all $k > 0$.

Now, let n denote the dimension of the plant matrix, A. To complete the proof, it is sufficient to show that $\mathcal{C}(n)$ is full rank for any set of indices $\{i_0, \ldots, i_{n-1}\}$, further implying that $\mathcal{C}(n)\mathcal{C}(n)^T > 0$. That $\overline{M}_{i_n} > 0 \ \forall i_n \in \mathcal{N}$ follows, using appropriate indices and the lower bound of Eq (29).

Note that $\mathcal{C}(n)$ is the controllability matrix for the following time-varying system:

$$x(k+1) = A_{i_k} x(k) + L_{i_k} u(k)$$

If $x(0) = 0$, induction can be used to show:

$$x(n+1) = \mathcal{C}(n) \begin{bmatrix} u(n) \\ \vdots \\ u(0) \end{bmatrix}$$

The state of this system can be driven from $x(0) = 0$ to $x(n+1) = x_{n+1}$ for any $x_{n+1} \in \mathbb{R}^n$ if and only if rank $[\mathcal{C}(n)] = n$.

The next piece of the proof generalizes Theorem 4.1 of [18]. Let $\mathbf{N}\{X\}$ and $\mathbf{R}\{X\}$ denote the null and range spaces of a matrix X. For any vector z, $z^T L_{i_k} L_{i_k}^T z = 0$ implies that both $z^T B W B^T z = 0$ and $z^T A F_{i_k} V F_{i_k}^T A^T z = 0$. Thus $\mathbf{N}\{L_{i_k}\} \subset \left(\mathbf{N}\{W^{1/2} B^T\} \cap \mathbf{N}\{V^{1/2} F_{i_k}^T A^T\}\right)$. Taking orthogonal complements gives set containment (a):

$$\mathbf{R}\{L_{i_k}\} \overset{(a)}{\supset} \left(\mathbf{R}\{B W^{1/2}\} \cup \mathbf{R}\{A F_{i_k} V^{1/2}\}\right) \supset \mathbf{R}\{A F_{i_k} V^{1/2}\}$$

This implies that $\mathbf{R}\{L_{i_k}\} \supset \mathbf{R}\big\{AF_{i_k}V^{1/2}G\big\}$ for any matrix G. In particular, the choice of $G = V^{-1/2}C_{g(i_k)}$ leads to the conclusion that $\mathbf{R}\{L_{i_k}\} \supset \mathbf{R}\big\{AF_{i_k}C_{g(i_k)}\big\}$. Hence, there must exist some matrix K_{i_k} such that $AF_{i_k}C_{g(i_k)} = L_{i_k}K_{i_k}$. We now use this linear algebra result to show that $\mathcal{C}(n)$ is full rank. Plugging in for A_{i_k}, we can rewrite the system dynamics as:

$$x(k+1) = A(I - F_{i_k}C_{g(i_k)})x(k) + L_{i_k}u(k)$$

By choosing the control law, $u(k) = K_{i_k}x(k) + Bv(k)$, these system dynamics become:

$$x(k+1) = Ax(k) + Bv(k)$$

Since (A,B) controllable, we can always choose $v(k)$ to transfer the system from $x(0) = 0$ to $x(n+1) = x_{n+1}$ for any $x_{n+1} \in \mathbb{R}^n$. Thus $\mathcal{C}(n)$ is full rank and the proof is complete. ■

B. Powered-up Markov Chain Data

The one-step transition matrices and numbering scheme f (which maps sequences of $\theta(k)$ to a Markov state, $n(k)$), for the Markov chains depicted in Figure 3 are given below. Also given is the auxillary function g used in writing $\mathcal{A}_{F_i}(\cdot)$ and $\mathcal{R}_i(\cdot)$. For Markov chain (a), $r = 1$:

$$P_1 = \begin{bmatrix} 1-\gamma & \gamma \\ 1-\alpha & \alpha \end{bmatrix}, \qquad \begin{bmatrix} R \\ L \end{bmatrix} \xrightarrow{f} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow{g} \begin{bmatrix} R \\ L \end{bmatrix} \tag{30}$$

For Markov chain (b), $r = 2$:

$$P_2 = \begin{bmatrix} 1-\gamma & 0 & \gamma & 0 \\ 1-\gamma & 0 & \gamma & 0 \\ 0 & 1-\alpha & 0 & \alpha \\ 0 & 1-\alpha & 0 & \alpha \end{bmatrix}, \qquad \begin{bmatrix} R\mathbf{R} \\ L\mathbf{R} \\ R\mathbf{L} \\ L\mathbf{L} \end{bmatrix} \xrightarrow{f} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \xrightarrow{g} \begin{bmatrix} R \\ R \\ L \\ L \end{bmatrix} \tag{31}$$

For Markov chain (c), $r = 3$:

$$P_3 = \begin{bmatrix} 1-\gamma & 0 & 0 & 0 & \gamma & 0 & 0 & 0 \\ 1-\gamma & 0 & 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & 1-\gamma & 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 1-\gamma & 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 1-\alpha & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 1-\alpha & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 1-\alpha & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 1-\alpha & 0 & 0 & 0 & \alpha \end{bmatrix}, \qquad \begin{bmatrix} RR\mathbf{R} \\ LR\mathbf{R} \\ RL\mathbf{R} \\ LL\mathbf{R} \\ RR\mathbf{L} \\ LR\mathbf{L} \\ RL\mathbf{L} \\ LL\mathbf{L} \end{bmatrix} \xrightarrow{f} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \xrightarrow{g} \begin{bmatrix} R \\ R \\ R \\ R \\ L \\ L \\ L \\ L \end{bmatrix} \tag{32}$$