Induced L_2-Norm Control for LPV Systems with Bounded Parameter Variation Rates

Fen Wu, Xin Hua Yang, Andy Packard* and Greg Becker
Department of Mechanical Engineering
University of California, Berkeley, CA 94720

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*Address all correspondence to **Mailing address**: Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA. **E-mail**: `pack@erg.me.berkeley.edu`, **Phone**: (510) 643-7959, **Fax**: (510) 642-6163.

Summary

A linear, finite-dimensional plant, with state-space parameter dependence, is controlled using a parameter-dependent controller. The parameters whose values are in a compact set, are known in real time. Their rates of variation are bounded and known in real time also. The goal of control is to stabilize the parameter-dependent closed-loop system, and provide disturbance/error attenuation as measured in induced $\mathbf{L}_2$ norm. Our approach uses a bounding technique based on a parameter-dependent Lyapunov function, and then solves the control synthesis problem by reformulating the existence conditions into an semi-infinite dimensional convex optimization. We propose finite dimensional approximations to get sufficient conditions for successful controller design.

Keywords: Gain scheduling, linear parameter varying systems, linear matrix inequalities, affine matrix inequalities, control system design.

1 Introduction

In this paper, we consider an approach to reduce the conservatism in quadratic stability analysis of systems with known bounds on the parameter's rate of variation. Our technique is loosely based on the ideas in [26] and [32]. We apply the analysis technique to the design of parameter-dependent controllers for parameter-dependent systems and obtain generalizations of the results in [5], [3] and [2]. A similar method for analysis of parameter-dependent systems has been addressed independently by [10]. Closely related synthesis methods have been developed in [36] and [39]. Different approaches to analyze parameter-dependent systems are derived in [13]. It will be useful to see if those methods can also be used to derive parameter-dependent controller synthesis procedures.

The implication that parameter-dependent systems theory has for gain-scheduling is obvious, since gain-scheduling conceptually involves a linear, parameter-dependent plant. The parameter-dependence can arise in a linear model [32], or in a parametrized family of Jacobian linearizations [33], or from exact linearization techniques [29]. This approach allows us to treat gain-scheduled controllers as a single entity, with the gain-scheduling achieved entirely by the parameter-dependent controller.

The main motivation for our work lies in [16], [15], [32], [34], [33] and [29]. These discuss linear, parameter-varying systems (LPVs) and their importance in gain-scheduling design. Specifically, [16] studies factorizations and realizations of systems over rings. This class of systems includes linear, time-invariant, parameter-dependent systems. In [15], control of time-invariant parameter-dependent systems is considered. The theorems are concerned with the parameter-dependent stabilization, observation and control, and concentrate on the parameter-dependence that is necessary in the controller's state-space entries. Gain-scheduling for linear, parameter-dependent systems is treated in [32]. They derive sufficient conditions for the existence of a parameter-dependent controller which guarantees stability for classes of time-varying parameters. Theoretical issues associated with controlling nonlinear systems using a gain-scheduling perspective are studied in [34]. Several heuristic rules-of-thumb about gain-scheduling are given theoretical interpretation and justification. In [33], a special class of nonlinear systems called "quasi-LPV" are introduced. These types of nonlinear systems can be made to look like linear, parameter-dependent systems using a global diffeomorphism. Finally, [29] is an overview of the extended linearization approach to gain-scheduling.

2 Motivating Example

Consider the parameter-dependent system (modified from [22]) $\dot{x}(t) = A(\rho(t))x(t) + Bu(t)$, where

$$A(\rho) := \begin{bmatrix} a_{11} & a_{12} & \cos(\rho) & \sin(\rho) \\ a_{21} & a_{22} & -\sin(\rho) & \cos(\rho) \\ 0 & 0 & -\tau & 0 \\ 0 & 0 & 0 & -\tau \end{bmatrix}, \qquad B := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \tau & 0 \\ 0 & \tau \end{bmatrix} \qquad A_{11} := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{2.1}$$

This represents a 2-input system with identical actuator dynamics, a time-varying coupling matrix, and 2nd order plant dynamics. The parameter-dependent state-feedback, quadratic stabilization problem is: find a continuous function $F(\rho)$ and a matrix $P \in \mathcal{S}^{4\times 4}, P > 0$ such that

$$[A(\rho) + BF(\rho)]^T P + P\,[A(\rho) + BF(\rho)] < 0$$

for all $\rho \in [-\pi, \ \pi]$. If such matrices existed, then the parameter-dependent state-feedback law $u(t) := F(\rho(t))x(t)$ would render the closed-loop system exponentially stable for any trajectory $\rho(\cdot)$. Unfortunately, if the constant matrix A_{11} is unstable, then the system is not quadratically stabilizable by parameter-dependent state feedback-control. To see this, suppose such matrix $F(\rho)$ exists, then the inequality must hold for $\rho = 0$ and $\rho = \pi$. Adding these two inequalities gives

$$\begin{bmatrix} A_{11} & 0 \\ ? & ? \end{bmatrix}^T P + P \begin{bmatrix} A_{11} & 0 \\ ? & ? \end{bmatrix} < 0$$

(? means "dont't care") which implies that A_{11} must be stable. Hence, when A_{11} is unstable, all of the methods in [5], [4], [3] and [2] are not applicable. The problem is that there are $\rho(\cdot)$ trajectories which allow the upper $(1,2)$ block of $A(\rho)$ in equation (2.1) to switch between I_2 and $-I_2$ arbitrarily fast. So, regardless of the bandwidth τ of the actuators, the rapidly varying parameter $\rho(t)$ do not allow for parameter-dependent quadratic stabilization.

However, a simple singular perturbation argument suggests that the state-feedback

$$F(\rho) := \left[\begin{bmatrix} \cos(\rho) & -\sin(\rho) \\ \sin(\rho) & \cos(\rho) \end{bmatrix} (-\gamma I_2 - A_{11}) \quad 0_{2\times 2} \right]$$

should work (i.e., exponentially stabilize) for $\rho(\cdot)$ trajectories satisfying $\max_{t\geq 0} |\dot{\rho}(t)| \leq B(\tau)$, where $B(\cdot)$ is some monotonically increasing function of τ. In other words, if there is a known rate-bound on $\rho(t)$, then exponentially stabilizing, parameter-dependent state-feedbacks do indeed exist. It is possible (see Lemma 4.1) to produce a parameter-dependent Lyapunov function which demonstrates this stability.

The purpose of this paper is to develop a useful, though somewhat ad-hoc and computationally intensive approach to exploiting rate-of-variation information about the parameters, using parameter-dependent Lyapunov function.

3 Induced $\mathbf{L}_2$-Norm Analysis

Let $\mathcal{P} \subset \mathbf{R}^s$ be a compact set and $\{\nu_i\}_{i=1}^s$ are nonnegative numbers, then we can define **parameter ν-variation set** as

$$\mathcal{F}_{\mathcal{P}}^{\nu} := \left\{ \rho \in C^1(\mathbf{R}, \mathbf{R}^s) : \ \rho(t) \in \mathcal{P}, \ |\dot{\rho}_i(t)| \leq \nu_i, \ i = 1, 2, \cdots, s, \ \forall\, t \in \mathbf{R}_+ \right\}$$

where $\nu = [\nu_1 \quad \cdots \quad \nu_s]^T$. Consider the behaviour of the LTV system governed by

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t), \dot{\rho}(t)) & B(\rho(t), \dot{\rho}(t)) \\ C(\rho(t), \dot{\rho}(t)) & D(\rho(t), \dot{\rho}(t)) \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} \tag{3.1}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$, $x, \dot{x} \in \mathbf{R}^n$, $d \in \mathbf{R}^{n_d}$ and $e \in \mathbf{R}^{n_e}$. $A : \mathbf{R}^s \times \mathbf{R}^s \to \mathbf{R}^{n \times n}$ is a continuous function, and similarly for B, C and D. It is possible to bound the induced $\mathbf{L}_2$-norm of this system using a parameter-dependent Lyapunov function.

Lemma 3.1 *Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, nonnegative numbers $\{\nu_i\}_{i=1}^s$. If there exists a continuously differentiable function $W : \mathbf{R}^s \to \mathcal{S}^{n \times n}$ such that $W(\rho) > 0$ and*

$$\begin{bmatrix} A^T(\rho,\beta)W(\rho) + W(\rho)A(\rho,\beta) + \sum_{i=1}^s \left(\beta_i \frac{\partial W}{\partial \rho_i}\right) & W(\rho)B(\rho,\beta) & C^T(\rho,\beta) \\ B^T(\rho,\beta)W(\rho) & -I_{n_d} & D^T(\rho,\beta) \\ C(\rho,\beta) & D(\rho,\beta) & -I_{n_e} \end{bmatrix} < 0 \qquad (3.2)$$

for all $\rho \in \mathcal{P}$ and $|\beta_i| \leq \nu_i$, $i = 1, 2, \cdots, s$, then there exists a scalar $\alpha < 1$ such that for any $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$, the LTV system governed by (3.1) is exponentially stable, and if $d \in \mathbf{L}_2$ and $x(0) = 0$, then $\|e\|_2 \leq \alpha \|d\|_2$.

Remark 3.1 *The inequalities in (3.2) represent convex constraints on the set of continuously differentiable functions mapping $\mathbf{R}^s \to \mathcal{S}^{n \times n}$. In section 5, we discuss some computational approaches to determine the feasability of the infinite dimensional AMI's the will arise our controller synthesis. Those inequalities are similar (though simpler) to the inequality in (3.2).*

Proof: Consider any trajectory of $\rho(\cdot)$, which satisfies $\rho(t) \in \mathcal{P}$ and $|\dot{\rho}_i(t)| \leq \nu_i$, $i = 1, 2, \cdots, s$ for all t. The inequality (3.2) gives

$$\begin{bmatrix} A^T(\rho,\dot{\rho})W(\rho) + W(\rho)A(\rho,\dot{\rho}) + \sum_{i=1}^s \left(\dot{\rho}_i \frac{\partial W}{\partial \rho_i}\right) & W(\rho)B(\rho,\dot{\rho}) & C^T(\rho,\dot{\rho}) \\ B^T(\rho,\dot{\rho})W(\rho) & -I_{n_d} & D^T(\rho,\dot{\rho}) \\ C(\rho,\dot{\rho}) & D(\rho,\dot{\rho}) & -I_{n_e} \end{bmatrix} < 0$$

for all t. By Schur complement, this means for such trajectories, the inequality

$$\frac{dW}{dt} + A^T W + WA + C^T C + \left(WB + C^T D\right)\left(I - D^T D\right)^{-1}\left(WB + C^T D\right)^T < 0$$

holds for all $t > 0$. Using the results in [35], [28] and [18], for trajectories $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$, the system (3.1) is exponentially stable, and the induced $\mathbf{L}_2$-norm from d to e is less than 1. ∎

As is customary, [31], [3], [4], [25], [1], [2], [9], [17], we now use this analysis test to derive conditions for control design.

4 Output-feedback Synthesis

Given a parameter-dependent plant, the **LPV Synthesis γ-Performance/ν-Variation Problem** is to determine if there is a parameter-dependent controller and a parameter-dependent Lyapunov function such that the analysis test described in Lemma 3.1 holds for the closed-loop system. In this section we derive necessary and sufficient conditions for this to be possible.

The generalized plant takes on the usual structure, with some regularity assumptions

$$\begin{bmatrix} \dot{x}(t) \\ e_1(t) \\ e_2(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B_{11}(\rho(t)) & B_{12}(\rho(t)) & B_2(\rho(t)) \\ C_{11}(\rho(t)) & 0 & 0 & 0 \\ C_{12}(\rho(t)) & 0 & 0 & I_{n_u} \\ C_2(\rho(t)) & 0 & I_{n_y} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d_1(t) \\ d_2(t) \\ u(t) \end{bmatrix}, \qquad (4.1)$$

where $\rho \in \mathcal{F}_{\mathcal{P}}^{\nu}$, and $x, \dot{x} \in \mathbf{R}^n$, $\begin{bmatrix} d_1^T & d_2^T \end{bmatrix}^T \in \mathbf{R}^{n_d}$, $\begin{bmatrix} e_1^T & e_2^T \end{bmatrix}^T \in \mathbf{R}^{n_e}$, $u \in \mathbf{R}^{n_u}$ and $y \in \mathbf{R}^{n_y}$. The matrix valued functions are of appropriate dimensions. Note that for simplicity of derivation, we assume $D_{11}(\rho) = 0, D_{22}(\rho) = 0$ and $D_{12}(\rho), D_{21}(\rho)$ have been scaled to the standard form. These assumptions can be relaxed at the expense of more complicated formulae (see [38]). Suppose a linear, parameter-dependent controller is used in feedback from y to u,

$$\begin{bmatrix} \dot{x}_k(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_K(\rho(t), \dot{\rho}(t)) & B_K(\rho(t), \dot{\rho}(t)) \\ C_K(\rho(t), \dot{\rho}(t)) & D_K(\rho(t), \dot{\rho}(t)) \end{bmatrix} \begin{bmatrix} x_k(t) \\ y(t) \end{bmatrix}. \tag{4.2}$$

Note that we allow the controller to depend explicitly on ρ and $\dot{\rho}$. Define $x_{\text{clp}}^T := \begin{bmatrix} x^T & x_k^T \end{bmatrix}$. The closed-loop system can be written as

$$\begin{bmatrix} \dot{x}_{\text{clp}}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A_{\text{clp}}(\rho(t), \dot{\rho}(t)) & B_{\text{clp}}(\rho(t), \dot{\rho}(t)) \\ C_{\text{clp}}(\rho(t), \dot{\rho}(t)) & D_{\text{clp}}(\rho(t), \dot{\rho}(t)) \end{bmatrix} \begin{bmatrix} x_{\text{clp}}(t) \\ d(t) \end{bmatrix}, \tag{4.3}$$

where

$$A_{\text{clp}}(\rho, \dot{\rho}) := \begin{bmatrix} A(\rho) + B_2(\rho) D_K(\rho, \dot{\rho}) C_2(\rho) & B_2(\rho) C_K(\rho, \dot{\rho}) \\ B_K(\rho, \dot{\rho}) C_2(\rho) & A_K(\rho, \dot{\rho}) \end{bmatrix}, \tag{4.4.a}$$

$$B_{\text{clp}}(\rho, \dot{\rho}) := \begin{bmatrix} B_{11}(\rho) & B_{12}(\rho) + B_2(\rho) D_K(\rho, \dot{\rho}) \\ 0 & B_K(\rho, \dot{\rho}) \end{bmatrix}, \tag{4.4.b}$$

$$C_{\text{clp}}(\rho, \dot{\rho}) := \begin{bmatrix} C_{11}(\rho) & 0 \\ C_{12}(\rho) + D_K(\rho, \dot{\rho}) C_2(\rho) & C_K(\rho, \dot{\rho}) \end{bmatrix}, \tag{4.4.c}$$

$$D_{\text{clp}}(\rho, \dot{\rho}) := \begin{bmatrix} 0 & 0 \\ 0 & D_K(\rho, \dot{\rho}) \end{bmatrix}. \tag{4.4.d}$$

Definition 4.1 *Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, nonnegative numbers $\{\nu_i\}_{i=1}^s$, performance level $\gamma > 0$, the open-loop LPV system in (4.1). The* **LPV Synthesis γ-Performance/ν-Variation Problem** *is solvable if there exist an $m \geq 0$, a continuously differentiable matrix function $W : \mathbf{R}^s \to \mathcal{S}^{(n+m)\times(n+m)}$, and continuous matrix functions $(A_K, B_K, C_K, D_K) : \mathbf{R}^s \times \mathbf{R}^s \to (\mathbf{R}^{m\times m}, \mathbf{R}^{m\times n_y}, \mathbf{R}^{n_u\times m}, \mathbf{R}^{n_u\times n_y})$ such that $W(\rho) > 0$ and*

$$\begin{bmatrix} A_{\text{clp}}^T(\rho, \beta) W(\rho) + W(\rho) A_{\text{clp}}(\rho, \beta) + \sum_{i=1}^s \left(\beta_i \frac{\partial W}{\partial \rho_i}\right) & W(\rho) B_{\text{clp}}(\rho, \beta) & C_{\text{clp}}^T(\rho, \beta) \\ B_{\text{clp}}^T(\rho, \beta) W(\rho) & -\gamma I_{n_d} & D_{\text{clp}}^T(\rho, \beta) \\ C_{\text{clp}}(\rho, \beta) & D_{\text{clp}}(\rho, \beta) & -\gamma I_{n_e} \end{bmatrix} < 0 \tag{4.5}$$

for all $\rho \in \mathcal{P}$ and $|\beta_i| \leq \nu_i$, $i = 1, 2, \cdots, s$. Here the matrices $A_{\text{clp}}, B_{\text{clp}}, C_{\text{clp}}$ and D_{clp} are defined in (4.4.).

Note that if **LPV Synthesis γ-Performance/ν-Variation Problem** is solvable, then induced $\mathbf{L}_2$-norm of the closed-loop system in (4.3) is less than γ. This problem represents a generalization of the standard sub-optimal $\mathcal{H}_\infty$ optimal control problem (no parameter-dependence in plant, no parameter-dependence in controller, constant W) and conceptually expands the applicability and usefulness of the $\mathcal{H}_\infty$ methodology. Additionally, the solution can be put inside a larger design iteration, such as a $D-K$ iteration, to achieve robustness to other perturbations, such as unmodeled dynamics.

We begin with a lemma that could be used to derive state-feedback synthesis result. In this paper, this lemma is only used to prove the sufficient condition in Theorem 4.1.

Lemma 4.1 *Given a compact set $\mathcal{P} \subset \mathbf{R}^s$, nonnegative numbers $\{\nu_i\}_{i=1}^s$, a scalar $\gamma > 0$, and define $A_F(\rho) := A(\rho) + B_2(\rho)F(\rho)$, $C_F^T(\rho) := \left[C_{11}^T(\rho) \;\; C_{12}^T(\rho) + F^T(\rho)\right]$. There exist a continuous function $F(\rho)$ and a continuously differentiable function $S(\rho)$ such that $S(\rho) > 0$ and*

$$\begin{aligned} &A_F^T(\rho)S(\rho) + S(\rho)A_F(\rho) + \sum_{i=1}^{s}\left(\beta_i \frac{\partial S}{\partial \rho_i}\right) + \gamma^{-1}C_F^T(\rho)C_F(\rho) \\ &+ \gamma^{-1}S(\rho)\left[B_{11}(\rho)B_{11}^T(\rho) + B_{12}(\rho)B_{12}^T(\rho)\right]S(\rho) < 0 \end{aligned} \tag{4.6}$$

for all $\rho \in \mathcal{P}$ and $|\beta_i| \leq \nu_i$, $i = 1, 2, \cdots, s$ if and only if there exists a continuously differentiable matrix function $Y(\rho)$ such that for all $\rho \in \mathcal{P}$, $Y(\rho) > 0$ and

$$\begin{bmatrix} Y(\rho)\hat{A}^T(\rho) + \hat{A}(\rho)Y(\rho) - \sum_{i=1}^{s} \pm\left(\nu_i \frac{\partial Y}{\partial \rho_i}\right) - \gamma B_2(\rho)B_2^T(\rho) & Y(\rho)C_{11}^T(\rho) & [B_{11}(\rho) \;\; B_{12}(\rho)] \\ C_{11}(\rho)Y(\rho) & -\gamma I & 0 \\ \begin{bmatrix} B_{11}^T(\rho) \\ B_{12}^T(\rho) \end{bmatrix} & 0 & -\gamma I \end{bmatrix} < 0, \tag{4.7}$$

where $\hat{A}(\rho) := A(\rho) - B_2(\rho)C_{12}(\rho)$.

Remark 4.1 *The notation $\sum_{i=1}^{s} \pm(\cdot)$ in (4.7) indicates that every combination of $+(\cdot)$ and $-(\cdot)$ should be included in the inequality. This means that the 3×3 "inequality" in (4.7) actually represents 2^s different inequalities which must be checked.*

Proof: $\Rightarrow$ Let $Y(\rho) := S^{-1}(\rho)$, pre and post-multiply the left hand side of equation (4.6) by $Y(\rho)$, then

$$\begin{aligned} &Y(\rho)\left[A(\rho) + B_2(\rho)F(\rho)\right]^T + \left[A(\rho) + B_2(\rho)F(\rho)\right]Y(\rho) - \sum_{i=1}^{s}\beta_i \frac{\partial Y}{\partial \rho_i} \\ &+ \gamma^{-1}Y(\rho)\left[C_{11}^T(\rho) \;\; C_{12}^T(\rho) + F^T(\rho)\right]\begin{bmatrix} C_{11}(\rho) \\ C_{12}(\rho) + F(\rho) \end{bmatrix} Y(\rho) \\ &+ \gamma^{-1}\left[B_{11}(\rho)B_{11}^T(\rho) + B_{12}(\rho)B_{12}^T(\rho)\right] < 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &Y(\rho)\hat{A}^T(\rho) + \hat{A}(\rho)Y(\rho) - \sum_{i=1}^{s}\beta_i \frac{\partial Y}{\partial \rho_i} - \gamma B_2(\rho)B_2^T(\rho) + \gamma^{-1}Y(\rho)C_{11}^T(\rho)C_{11}(\rho)Y(\rho) \\ &+ \gamma^{-1}\left[B_{11}(\rho)B_{11}^T(\rho) + B_{12}(\rho)B_{12}^T(\rho)\right] \\ &+ \left[\gamma^{-\frac{1}{2}}(C_{12}(\rho) + F(\rho))Y(\rho) + \gamma^{\frac{1}{2}}B_2^T(\rho)\right]^T\left[\gamma^{-\frac{1}{2}}(C_{12}(\rho) + F(\rho))Y(\rho) + \gamma^{\frac{1}{2}}B_2^T(\rho)\right] < 0. \end{aligned}$$

Note that the left hand side is affine function of β, so that

$$Y(\rho)\hat{A}^T(\rho)+\hat{A}(\rho)Y(\rho)-\sum_{i=1}^{s}\pm(\nu_i\frac{\partial Y}{\partial\rho_i})-\gamma B_2(\rho)B_2^T(\rho)+\gamma^{-1}Y(\rho)C_{11}^T(\rho)C_{11}(\rho)Y(\rho)$$
$$+\gamma^{-1}\left[B_{11}(\rho)B_{11}^T(\rho)+B_{12}(\rho)B_{12}^T(\rho)\right]<0$$

The above inequality is exactly the Schur complement of equation (4.7).

$\Leftarrow$ Choose $F(\rho):=-\left[\gamma B_2^T(\rho)Y^{-1}(\rho)+C_{12}(\rho)\right]$ and $S(\rho):=Y^{-1}(\rho)$, then equation (4.6) follows from inequality (4.7) by simple algebraic manipulations. ∎

We now state the main theorem for the **LPV Synthesis γ-Performance/ν-Variation Problem.**

Theorem 4.1 *Given a compact set $\mathcal{P}\subset\mathbf{R}^s$, nonnegative numbers $\{\nu_i\}_{i=1}^s$, performance level $\gamma>0$, and the open-loop LPV system in (4.1), the* **LPV Synthesis γ-Performance/ν-Variation Problem** *is solvable if and only if there exist continuously differentiable matrix functions $X:\mathbf{R}^s\to\mathcal{S}^{n\times n}$ and $Y:\mathbf{R}^s\to\mathcal{S}^{n\times n}$, such that for all $\rho\in\mathcal{P}$, $X(\rho),Y(\rho)>0$, and*

$$\begin{bmatrix} Y(\rho)\hat{A}^T(\rho)+\hat{A}(\rho)Y(\rho)-\sum_{i=1}^{s}\pm\left(\nu_i\frac{\partial Y}{\partial\rho_i}\right)-\gamma B_2(\rho)B_2^T(\rho) & Y(\rho)C_{11}^T(\rho) & B_1(\rho)\\ C_{11}(\rho)Y(\rho) & -\gamma I_{n_{e1}} & 0\\ B_1^T(\rho) & 0 & -\gamma I_{n_d}\end{bmatrix}<0 \tag{4.8.a}$$

$$\begin{bmatrix} \tilde{A}^T(\rho)X(\rho)+X(\rho)\tilde{A}(\rho)+\sum_{i=1}^{s}\pm\left(\nu_i\frac{\partial X}{\partial\rho_i}\right)-\gamma C_2^T(\rho)C_2(\rho) & X(\rho)B_{11}(\rho) & C_1^T(\rho)\\ B_{11}^T(\rho)X(\rho) & -\gamma I_{n_{d1}} & 0\\ C_1(\rho) & 0 & -\gamma I_{n_e}\end{bmatrix}<0 \tag{4.8.b}$$

$$\begin{bmatrix} X(\rho) & I_n\\ I_n & Y(\rho)\end{bmatrix}\geq 0, \tag{4.8.c}$$

where

$$\hat{A}(\rho):=A(\rho)-B_2(\rho)C_{12}(\rho),\qquad B_1(\rho)=\left[B_{11}(\rho)\ \ B_{12}(\rho)\right],$$
$$\tilde{A}(\rho):=A(\rho)-B_{12}(\rho)C_2(\rho),\qquad C_1^T(\rho)=\left[C_{11}^T(\rho)\ \ C_{12}^T(\rho)\right].$$

If the conditions are satisfied, then by continuity and compactness, it is possible to perturb X such that the three LMIs (4.8.a)–(4.8.b) still hold and $(X-Y^{-1})>0$ uniformly on $\mathcal{P}$. Define $Q(\rho):=X(\rho)-Y^{-1}(\rho)$, $F(\rho):=-\left[\gamma B_2^T(\rho)Y^{-1}(\rho)+C_{12}(\rho)\right]$, $L(\rho):=-\left[\gamma X^{-1}(\rho)C_2^T(\rho)+B_{12}(\rho)\right]$. Let

$$\begin{aligned}H(\rho,\dot{\rho})\ :=\ -&\Bigg[A_F^T(\rho)Y^{-1}(\rho)+Y^{-1}(\rho)A_F(\rho)+\sum_{i=1}^{s}\left(\dot{\rho}_i\frac{\partial Y^{-1}}{\partial\rho_i}\right)+\gamma^{-1}C_F^T(\rho)C_F(\rho)\\ &+\gamma^{-1}Y^{-1}(\rho)B_1(\rho)B_1^T(\rho)Y^{-1}(\rho)\Bigg],\end{aligned}$$

with $A_F(\rho):=A(\rho)+B_2(\rho)F(\rho)$ and $C_F^T(\rho):=\left[C_{11}^T(\rho)\ \ C_{12}^T(\rho)+F^T(\rho)\right]$. One n-dimensional, strictly proper controller that solves the feedback problem is defined as:

$$A_K(\rho,\dot{\rho})\ :=\ A(\rho)+\gamma^{-1}\left[Q^{-1}(\rho)X(\rho)L(\rho)B_{12}^T(\rho)+B_1(\rho)B_1^T(\rho)\right]Y^{-1}(\rho)$$

$$\begin{aligned} & + B_2(\rho)F(\rho) + Q^{-1}(\rho)X(\rho)L(\rho)C_2(\rho) - Q^{-1}(\rho)H(\rho,\dot\rho) && \text{(4.9.a)}\\ B_K(\rho) &:= -Q^{-1}(\rho)X(\rho)L(\rho) && \text{(4.9.b)}\\ C_K(\rho) &:= F(\rho) && \text{(4.9.c)}\\ D_K(\rho) &:= 0. && \text{(4.9.d)} \end{aligned}$$

Proof: $\Rightarrow$ Let $W: \mathbf{R}^s \to \mathcal{S}^{(n+m)\times(n+m)}$ be the continuously differentiable, parameter-dependent Lyapunov function that satisfies the analysis test in Lemma 3.1 for the closed-loop system. Hence, W is bounded and uniformly positive definite over $\mathcal{P}$. For each $\rho \in \mathcal{P}$, define $Z(\rho) := W^{-1}(\rho)$. Clearly, Z is also continuously differentiable, bounded and uniformly positive definite over $\mathcal{P}$.

Partition W and Z as

$$W = \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}, \quad Z = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix},$$

where $X: \mathbf{R}^s \to \mathcal{S}^{n\times n}, Y: \mathbf{R}^s \to \mathcal{S}^{n\times n}$, and $X_3: \mathbf{R}^s \to \mathcal{S}^{m\times m}$ and $Y_3: \mathbf{R}^s \to \mathcal{S}^{m\times m}$.

By the matrix inversion lemma, it follows that for all $\rho \in \mathcal{P}$

$$\begin{bmatrix} X(\rho) & I \\ I & Y(\rho) \end{bmatrix} \geq 0.$$

To show necessity of inequalities (4.8.a) and (4.8.b), write the left hand side of the closed-loop LMI in (4.5) as

$$G(\rho,\beta) := R(\rho,\beta) + U(\rho)K(\rho,\beta)V^T(\rho) + V(\rho)K(\rho,\beta)U^T(\rho),$$

where

$$R := \begin{bmatrix} \begin{bmatrix} A^T & 0 \\ 0 & 0 \end{bmatrix} W + W \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \sum_{i=1}^{s} \left(\beta_i \frac{\partial W}{\partial \rho_i}\right) & W \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix} & \begin{matrix} C_{11}^T & C_{12}^T \\ 0 & 0 \end{matrix} \\ \begin{bmatrix} B_{11}^T & 0 \\ B_{12}^T & 0 \end{bmatrix} W & \begin{matrix} -\gamma I & 0 \\ 0 & -\gamma I \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} C_{11} & 0 \\ C_{12} & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} -\gamma I & 0 \\ 0 & -\gamma I \end{matrix} \end{bmatrix},$$

$$U := \begin{bmatrix} W \begin{bmatrix} B_2 & 0 \\ 0 & I \end{bmatrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \end{matrix} \end{bmatrix}, \quad V := \begin{bmatrix} C_2^T & 0 \\ 0 & I \\ 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad K := \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix}.$$

Define

$$U_\perp := \begin{bmatrix} Y & 0 & 0 & 0 \\ Y_2^T & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -B_2^T & 0 & 0 & 0 \end{bmatrix}, \quad V_\perp := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -C_2 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Note that for all $\rho \in \mathcal{P}$, $U_\perp^T U = 0$, $V_\perp^T V = 0$, and $[U \;\; U_\perp]$, $[V \;\; V_\perp]$ are full rank. Since both $U_\perp$ and $V_\perp$ are full column rank for all $\rho \in \mathcal{P}$, it is clear that if $G < 0$ for all $\rho \in \mathcal{P}$ and $|\beta_i| \leq \nu_i$, $i = 1, 2, \cdots, s$ then

$$U_\perp^T(\rho)G(\rho,\beta)U_\perp(\rho) < 0 \quad \text{and} \quad V_\perp^T(\rho)G(\rho,\beta)V_\perp(\rho) < 0$$

for all $\rho \in \mathcal{P}$ and $|\beta_i| \le \nu_i$, $i = 1, 2, \cdots, s$. This leads to

$$U_\perp^T(\rho)R(\rho,\beta)U_\perp(\rho) < 0 \quad \text{and} \quad V_\perp^T(\rho)R(\rho,\beta)V_\perp(\rho) < 0$$

for all $\rho \in \mathcal{P}$ and $|\beta_i| \le \nu_i$, $i = 1, 2, \cdots, s$. Carrying out the algebraic manipulations for the first inequality $U_\perp^T(\rho)R(\rho,\beta)U_\perp(\rho) < 0$, it is equivalent to

$$\begin{bmatrix} \Omega(\rho,\beta) & B_1(\rho) & Y(\rho)C_{11}^T(\rho) \\ B_1^T(\rho) & -\gamma I & 0 \\ C_{11}(\rho)Y(\rho) & 0 & -\gamma I \end{bmatrix} < 0 \tag{4.10}$$

where

$$\begin{aligned} \Omega &:= YXAY + YA^TXY + YA^TX_2Y_2^T + Y_2X_2^TAY - B_2C_{12}Y - YC_{12}^TB_2^T \\ &\quad + \sum_{i=1}^{s} \beta_i \left(Y\frac{\partial X}{\partial \rho_i}Y + Y\frac{\partial X_2}{\partial \rho_i}Y_2^T + Y_2\frac{\partial X_2^T}{\partial \rho_i}Y + Y_2\frac{\partial X_3}{\partial \rho_i}Y_2^T \right) - \gamma B_2B_2^T. \end{aligned}$$

Differentiating W^{-1} gives $\frac{\partial Z}{\partial \rho_i} = -Z\frac{\partial W}{\partial \rho_i}Z$, which has as its $(1,1)$ entry

$$Y\frac{\partial X}{\partial \rho_i}Y + Y\frac{\partial X_2}{\partial \rho_i}Y_2^T + Y_2\frac{\partial X_2^T}{\partial \rho_i}Y + Y_2\frac{\partial X_3}{\partial \rho_i}Y_2^T = -\frac{\partial Y}{\partial \rho_i}.$$

Furthermore $ZW = I$, it follows that $YX + Y_2X_2^T = XY + X_2Y_2^T = I$. This simplifies Ω to

$$\Omega(\rho,\beta) = Y(\rho)\hat{A}^T(\rho) + \hat{A}(\rho)Y(\rho) - \sum_{i=1}^{s}\left(\beta_i\frac{\partial Y}{\partial \rho_i}\right) - \gamma B_2(\rho)B_2^T(\rho).$$

Hence the condition in (4.10) is

$$\begin{bmatrix} Y(\rho)\hat{A}^T(\rho) + \hat{A}(\rho)Y(\rho) - \sum_{i=1}^{s}\left(\beta_i\frac{\partial Y}{\partial \rho_i}\right) - \gamma B_2(\rho)B_2^T(\rho) & B_1(\rho) & Y(\rho)C_{11}^T(\rho) \\ B_1^T(\rho) & -\gamma I & 0 \\ C_{11}(\rho)Y(\rho) & 0 & -\gamma I \end{bmatrix} < 0. \tag{4.11}$$

As β enters the left hand side of (4.11) affinely and $|\beta_i| \le \nu_i$, $i = 1, 2, \cdots, s$, (4.11) is equivalent to (4.8.a). Simpler manipulations show that $V_\perp^T(\rho)R(\rho,\beta)V_\perp(\rho) < 0$ is equivalent to (4.8.b).

$\Leftarrow$ For sufficiency, the approach of [30] is adopted. We verify the controller given in (4.9.) satisfies the **LPV Synthesis γ-Performance/ν-Variation Problem** using

$$W(\rho) := \begin{bmatrix} X(\rho) & -(X(\rho) - Y^{-1}(\rho)) \\ -(X(\rho) - Y^{-1}(\rho)) & X(\rho) - Y^{-1}(\rho) \end{bmatrix}.$$

First, note that by Schur complements, $W > 0$ for all $\rho \in \mathcal{P}$. Define

$$\begin{aligned} \Gamma(\rho,\beta) &:= A_{\rm clp}^T(\rho,\beta)W(\rho) + W(\rho)A_{\rm clp}(\rho,\beta) + \sum_{i=1}^{s}\left(\beta_i\frac{\partial W}{\partial \rho_i}\right) + \gamma^{-1}C_{\rm clp}^T(\rho)C_{\rm clp}(\rho) \\ &\quad + \gamma^{-1}W(\rho)B_{\rm clp}(\rho)B_{\rm clp}^T(\rho)W(\rho) \end{aligned} \tag{4.12}$$

where the closed loop matrices $A_{\rm clp}$, $B_{\rm clp}$ and $C_{\rm clp}$ are defined in (4.4.). We perform the following matrix manipulations to verify $\Gamma(\rho,\beta) < 0$. Partition Γ into $n \times n$ blocks $\Gamma_{11}, \Gamma_{12}, \Gamma_{22}$. Define a transformation $T := \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$, note $\Gamma(\rho,\beta) < 0$ if and only if $T^T\Gamma(\rho,\beta)T < 0$. Define transformed state-space data $\tilde{A}_{\rm clp}(\rho,\beta) := T^{-1}A_{\rm clp}(\rho,\beta)T$, $\tilde{B}_{\rm clp}(\rho) := T^{-1}B_{\rm clp}(\rho)$, and $\tilde{C}_{\rm clp}(\rho) := C_{\rm clp}(\rho)T$. Let $\tilde{W}(\rho) := T^TW(\rho)T$. Then $T^T\Gamma(\rho,\beta)T < 0$ can be rewritten as

$$\begin{aligned}&\tilde{A}^T_{\rm clp}(\rho,\beta)\tilde{W}(\rho) + \tilde{W}(\rho)\tilde{A}_{\rm clp}(\rho,\beta) + \sum_{i=1}^{s}\left(\beta_i\frac{\partial \tilde{W}}{\partial \rho_i}\right) + \gamma^{-1}\tilde{C}^T_{\rm clp}(\rho)\tilde{C}_{\rm clp}(\rho)\\ &+ \gamma^{-1}\tilde{W}(\rho)\tilde{B}_{\rm clp}(\rho)\tilde{B}^T_{\rm clp}(\rho)\tilde{W}(\rho) < 0 \end{aligned} \tag{4.13}$$

Denote the left hand side of (4.13) as $\tilde{\Gamma}$ and partition it into blocks $\tilde{\Gamma}_{11}, \tilde{\Gamma}_{12}, \tilde{\Gamma}_{22} \in \mathbf{R}^{n\times n}$. Verify that

$$\tilde{\Gamma}(\rho,\beta) = \begin{bmatrix} -H(\rho,\beta) & -H(\rho,\beta) \\ -H(\rho,\beta) & \Gamma_{11} - H(\rho,\beta) \end{bmatrix}.$$

Using Schur complements, $\tilde{\Gamma} < 0$ if and only if $\tilde{\Gamma}_{22}(\rho,\beta) - \tilde{\Gamma}^T_{12}(\rho,\beta)\tilde{\Gamma}^{-1}_{11}(\rho,\beta)\tilde{\Gamma}_{12}(\rho,\beta) < 0$ and $-H(\rho,\beta) < 0$. But,

$$\begin{aligned}-H(\rho,\beta) \;=\;& A^T_F(\rho)Y^{-1}(\rho) + Y^{-1}(\rho)A_F(\rho) + \sum_{i=1}^{s}\left(\beta_i\frac{\partial Y^{-1}}{\partial \rho_i}\right) + \gamma^{-1}C^T_F(\rho)C_F(\rho)\\ &+ \gamma^{-1}Y^{-1}(\rho)B_1(\rho)B_1^T(\rho)Y^{-1}(\rho)\end{aligned}$$

with $A_F = A + B_2F, C^T_F = \begin{bmatrix} C^T_{11} & C^T_{12} + F^T\end{bmatrix}$. So $\tilde{\Gamma}_{11}$ is negative definite by Lemma 4.1. Also

$$\begin{aligned}&\tilde{\Gamma}_{22}(\rho,\beta) - \tilde{\Gamma}^T_{12}(\rho,\beta)\tilde{\Gamma}^{-1}_{11}(\rho,\beta)\tilde{\Gamma}_{12}(\rho,\beta)\\ &= \Gamma_{11}(\rho,\beta) - H(\rho,\beta) + H(\rho,\beta)H^{-1}(\rho,\beta)H(\rho,\beta)\\ &= \Gamma_{11}(\rho,\beta)\\ &= A^T_L(\rho)X(\rho) + X(\rho)A_L(\rho) + \sum_{i=1}^{s}\left(\beta_i\frac{\partial X}{\partial \rho_i}\right) + \gamma^{-1}C^T_1(\rho)C_1(\rho) + \gamma^{-1}X(\rho)B_L(\rho)B^T_L(\rho)X(\rho),\end{aligned}$$

with $A_L = A + LC_2, B_L = [B_{11} \;\; B_{12} + L]$, which is negative definite by the dual of Lemma 4.1. ∎

5 Computational Considerations

Several ad-hoc approaches to solve the LMIs in Theorem 4.1 can be proposed. For instance, let $\{f_i\}_{i=1}^N$ and $\{g_i\}_{i=1}^N$ be user-defined sets of continuously differentiable functions from $\mathbf{R}^s$ to $\mathbf{R}$. For any matrices $\{X_i\}_{i=1}^N$, $X_i \in \mathcal{S}^{n\times n}$ and $\{Y_i\}_{i=1}^N$, $Y_i \in \mathcal{S}^{n\times n}$, the functions

$$X(\rho) := \sum_{i=1}^{N} f_i(\rho)X_i, \qquad Y(\rho) := \sum_{i=1}^{N} g_i(\rho)Y_i$$

are continuously differentiable on $\mathbf{R}^s \to \mathcal{S}^{n\times n}$. So, once the basis functions f_i and g_i are chosen, we can attempt to solve the synthesis equations (4.8.) by optimizing over the matrices $X_i, Y_i \in \mathcal{S}^{n\times n}$. The three conditions become (for space reasons, we have omitted the condition for (4.8.b) involving $R_X(X_1,\cdots,X_N,\rho)$ – it is similar to that for (4.8.a) and R_Y defined below)

$$R_Y(Y_1,\cdots,Y_N,\rho) :=$$

$$\left[\begin{array}{ccc} \left\{\begin{array}{c} \sum_{i=1}^{N} g_i(\rho)\left(Y_i\hat{A}^T(\rho)+\hat{A}(\rho)Y_i\right)-\sum_{j=1}^{s}\pm\left(\nu_j\sum_{i=1}^{N}\frac{\partial g_i}{\partial\rho_j}Y_i\right) \\ -\gamma B_2(\rho)B_2^T(\rho) \end{array}\right\} & \sum_{i=1}^{N} g_i(\rho)Y_iC_{11}^T(\rho) & B_1(\rho) \\ C_{11}(\rho)\sum_{i=1}^{N} g_i(\rho)Y_i & -\gamma I_{n_{e1}} & 0 \\ B_1^T(\rho) & 0 & -\gamma I_{n_d} \end{array}\right] < 0 \tag{5.1.a}$$

$$S_{XY}(X_1,\cdots,X_N,Y_1,\cdots,Y_N,\rho) := \left[\begin{array}{cc} \sum_{i=1}^{N} f_i(\rho)X_i & I_n \\ I_n & \sum_{i=1}^{N} g_i(\rho)Y_i \end{array}\right] \geq 0. \tag{5.1.c}$$

with $\hat{A}, \tilde{A}, B_1, C_1$ defined as in Theorem 4.1. This represents $2^{s+1}+1$ Affine Matrix Inequalities in the variables X_i, Y_i that must be satisfied for all $\rho \in \mathcal{P}$.

In order to solve this infinite dimensional convex optimization, we solve a slightly more stringent problem over a finite set of ρ_k. For simplicity of explanation, assume that $\mathcal{P} \subset \mathbf{R}$, and grid the set $\mathcal{P}$ by L points $\{\rho_k\}_{k=1}^{L}$. Pick a (large) number $T > 0$ and a (small) number $\delta > 0$. Consider the finite dimensional feasibility problem

$$\begin{aligned} R_Y(Y_1,\cdots,Y_N,\rho_k) &\leq -\delta && (5.2.b) \\ R_X(X_1,\cdots,X_N,\rho_k) &\leq -\delta && (5.2.c) \\ S_{XY}(X_1,\cdots,X_N,Y_1,\cdots,Y_N,\rho_k) &\geq \delta && (5.2.d) \\ \|X_i\|_F \leq T,\ \|Y_i\|_F &\leq T && (5.2.e) \\ & && (5.3.e) \end{aligned}$$

for $k = 1,\cdots,L$. We now want to decide that how dense the points $\{\rho_k\}_{k=1}^{L}$ should be to guarantee the global solvability of the LMIs over all $\rho \in \mathcal{P}$.

Lemma 5.1 *Assume that all the state-space data are continuously differentiable and f_i, g_i are twice continuously differentiable. Let*

$$\begin{aligned} h_{min} := \delta\cdot\min\Bigg\{ & \left[2T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left\|\frac{d(g_i\hat{A}^T)}{d\rho}\right\|_F + \nu T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left|\frac{d^2g_i}{d\rho^2}\right| + \gamma\max_{\rho\in\mathcal{P}}\left\|\frac{d(B_2B_2^T)}{d\rho}\right\|_F \right. \\ & \left. + 2T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left\|\frac{d(g_iC_{11}^T)}{d\rho}\right\|_F + 2\max_{\rho\in\mathcal{P}}\left\|\frac{dB_1}{d\rho}\right\|_F\right]^{-1}, \\ & \left[2T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left\|\frac{d(f_i\tilde{A})}{d\rho}\right\|_F + \nu T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left|\frac{d^2f_i}{d\rho^2}\right| + \gamma\max_{\rho\in\mathcal{P}}\left\|\frac{d(C_2^TC_2)}{d\rho}\right\|_F \right. \\ & \left. + 2T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left\|\frac{d(f_iB_{11})}{d\rho}\right\|_F + 2\max_{\rho\in\mathcal{P}}\left\|\frac{dC_1^T}{d\rho}\right\|_F\right]^{-1}, \\ & \left[T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left|\frac{df_i}{d\rho}\right| + T\sum_{i=1}^{N}\max_{\rho\in\mathcal{P}}\left|\frac{dg_i}{d\rho}\right|\right]^{-1}\Bigg\}. \end{aligned}$$

If $|\rho_k - \rho_{k+1}| \leq h_{min}$ for all $k = 1, \ldots, L-1$, and there exist matrices $\bar{X}_i, \bar{Y}_i$ solving equation (5.2.e)–(5.2.d), then for all $\rho \in \mathcal{P}$

$$R_Y(\bar{Y}_1, \cdots, \bar{Y}_N, \rho) < 0 \quad (5.4.a)$$

$$R_X(\bar{X}_1, \cdots, \bar{X}_N, \rho) < 0 \quad (5.4.b)$$

$$S_{XY}(\bar{X}_1, \cdots, \bar{X}_N, \bar{Y}_1, \cdots, \bar{Y}_N, \rho) \geq 0. \quad (5.4.c)$$

Proof: The proof is based on norm triangular inequality and intermediate value theorem. For details, see [38]. ∎

Since $\rho \subset \mathbf{R}^s$, it will require approximately L^s points to grid $\mathcal{P}$ with approximately L points in each dimension. So the optimization problem to determine appropriate X_i and Y_i is approximately $L^s\,(2^{s+1}+1)$ affine matrix inequalities in the matrix variables $(X_1, Y_1, \ldots, X_N, Y_N)$. The feasibility of these finite number of inequalities can then be determined with several techniques [24], [23], [6], [8], [12].

6 Example

As a simple example, we return to the problem which motivated the rate-dependent stabilization. We consider the output-feedback performance of this problem. The control problem involves stabilization, tracking, disturbance rejection and input penalty. A block diagram of the generalized plant is shown in Figure 1. The plant T_ρ is governed by

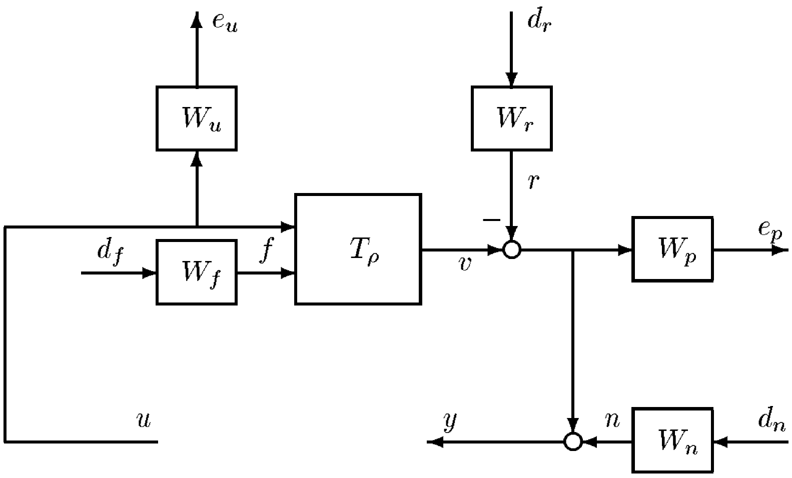

Figure 1: Control system configuration

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} 0.75 & 2 & \cos(\rho(t)) & \sin(\rho(t)) & 0 & 0 & 0 \\ 0 & 0.5 & -\sin(\rho(t)) & \cos(\rho(t)) & 3 & 0 & 0 \\ 0 & 0 & -10 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & -10 & 0 & 0 & 10 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ f(t) \\ u_1(t) \\ u_2(t) \end{bmatrix}.$$

The weighting functions are as follows

$$W_p = I_2, \quad W_n = \frac{10(s+10)}{s+1000} I_2, \quad W_f = 1, \quad W_u = \frac{1}{280} I_2, \quad W_r = \frac{20}{s+0.2} I_2.$$

These roughly imply objectives of decoupled command response and disturbance rejection, with tracking errors less than 1% of command, and control signals bounded by 280 times the size of $y(t)$.

For this example, we pick $N = 3$, and basis functions

$$f_1(\rho) = g_1(\rho) := 1, \quad f_2(\rho) = g_2(\rho) := \cos(\rho), \quad f_3(\rho) = g_3(\rho) := \sin(\rho).$$

These seem natural, given the dependence of the plant on ρ. We choose a rather coarse gridding of interval $\mathcal{P} := [-\pi, \pi]$, namely 6 points, uniformly spaced (i.e., every 60°). First, using the optimal output feedback synthesis [7], we determined that for fixed values of ρ, the optimal achievable γ level is approximately 0.80, and is independent of ρ (although the optimal controllers are **strongly dependent** on ρ). Using LMIlab [11] with the 6-point gridding, we solved the **LPV Synthesis γ-Performance/ν-Variation Problem** at various levels of ν. The relationship between the optimally achievable γ and ν are shown in the Figure 2.

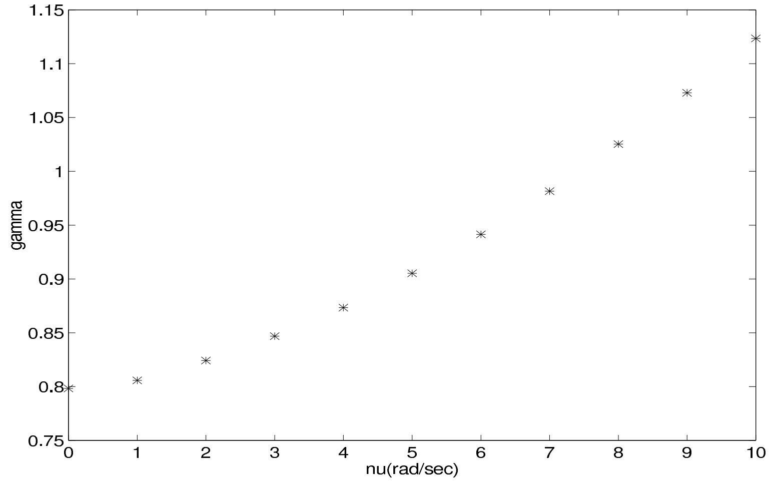

Figure 2: $\mathcal{H}_\infty$ performance γ vs. bound of variation-rate ν

Since the optimization only guarantees that the LMIs hold at the grid points, we further analyzed the resulting solution at 360 points (every 1°). Plotting the eigenvalues of equation (4.8.c) and (4.8.a)–(4.8.b) clearly indicate that the LMIs are indeed solved over the whole interval $\mathcal{P}$.

A time-varying $\rho(\cdot)$ is also simulated. The trajectory considered is $\rho(t) = \pi \sin(5t/\pi)$. Note that for this trajectory, $\rho(t) \in [-\pi, \ \pi]$ and $|\dot{\rho}(t)| \leq 5$. The simulation results for $r(t) = [r_1(t) \ \ 0]^T$ where r_1 is given in Figure 3, $f(t) = (0.01)\mathbf{1}(\mathbf{t})$, and $n(t)$ as uniformly distributed random process in $[-0.005, 0.005]$ are shown in Figure 3.

7 Comments and Conclusions

Many people will complain about the need to grid the $\mathcal{P}$ set. We too feel that this is a disadvantage of the method. However in many gain-scheduling applications, the number of scheduling variables

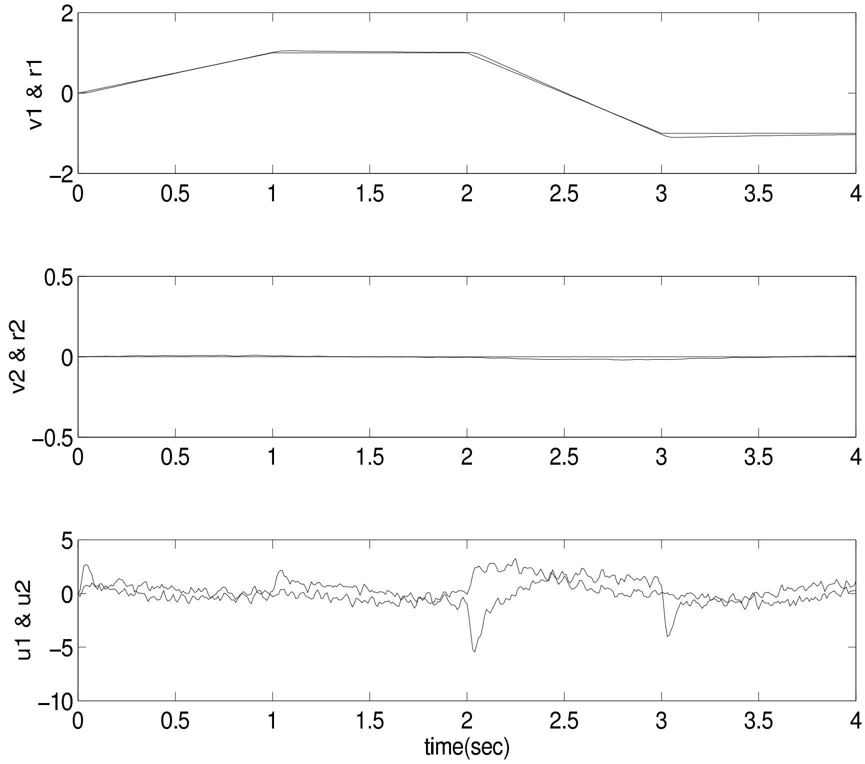

Figure 3: Reference & output channel 1; reference & output channel 2; controller input responses for time-varying $\rho(t) = \pi \sin(\frac{5}{\pi}\, t)$ with output-feedback controller.

is small, usually 3 or less. Hence the dimensionality of the gridding, while extremely cumbersome, is not overwhelming. Of course, for a problem with many parameters, the gridding procedure will become prohibitively expensive. This clearly indicates the drawbacks associated with using Lemma 3.1 as a general robustness analysis tool for systems with time-varying real uncertainty.

Another significant problem is the complete lack of guidance provided by the theory to pick the basis functions, namely, f_i and g_i. Hopefully additional study will yield some results along these lines.

This method has been applied to an example where the results of [1], [2], [3], [25] and [5] will not work. This is due to the bound on the rate-of-variation in the parameter. This is pointed out in [14], where a difficult $\mathcal{H}_\infty$ scheduled control law is designed for vertical short take-off and landing vehicles is performed. It would be interesting to try to apply the results here to that specific example. Also note that in Theorem 4.1, by restricting X, Y be constant, we recover the results of [3], [5] and [2], where the quadratic performance problem is considered.

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