Pendulum Waves

A Lesson in Aliasing

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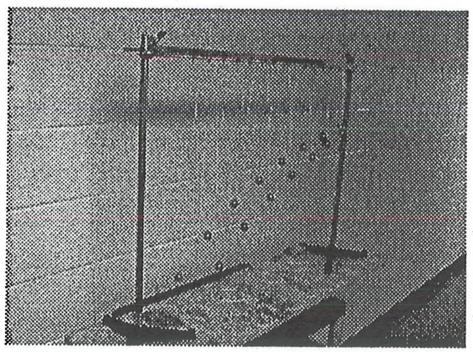
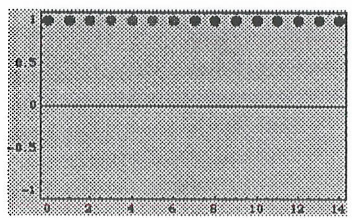


Outline

- "Pendulum Waves" apparatus
- "Mach's Wave Machine" -- analysis of standard sinusoidal traveling waves
- Extending the analysis -- pendulum waves are traveling waves, but with a twist!
- A lesson in aliasing (AKA sampling error)

Pendulum Waves apparatus

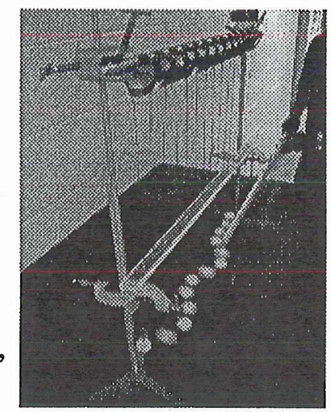
- Richard Berg, Univ. of Maryland -- origins unknown (but this is <u>not</u> Mach's "Wave Machine")
- What continuous function might that be?
- Thanks to Kevin Parendo



Mach's Wave Machine

Standard Sinusoidal Traveling Waves

- Discussed in W. Weiler's Physikbuch #3 entitled "Schwingungen und Wellen," published ~1910
- A set of identical uncoupled pendula, launched one at a time, show traveling waves
- Thanks to Ronnie Cooper, Alex Nugent, Andrew Foltz, and Timo (Lego) Mechler



Motion of a single oscillator

$$y[t] = A\cos\left[\left(\frac{2\pi\ rad}{T}\right)t + \phi_{init}\right] = A\cos[\omega\, t + \phi_{init}]$$

- Displacement from center described by variable y which varies with time t.
- Object moves back and forth sinusoidally with amplitude A.
- Object goes through 2π rad of angle in period T. Call $2\pi\ rad/T$ the angular frequency ω.
- Angle ϕ_{init} is used to get location right at $t = 0$.

Describing a series of oscillators supporting a traveling wave

- Each oscillator gets its own ϕ_{init} which grows linearly with position down the line x.
- To ensure that ϕ_{init} changes by 2π rad as x grows by 1 wavelength λ, try

$$\phi_{init}[x] = \left(\frac{2\pi\ rad}{\lambda}\right)x = k\,x$$

- Thus a traveling wave (moving left) is described by

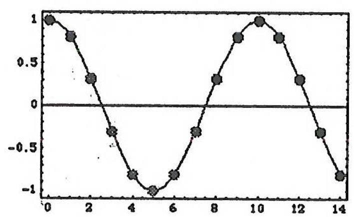
$$y[x,t] = A\cos[\omega\, t + \phi_{init}[x]] = A\cos[\omega\, t + k\, x]$$

Animating this traveling wave functional description

- At $t = 0$ that describes a cosine shape where kx gives each oscillator a unique ϕ_{init}.

$$y[x, t=0] = A\cos[k\,x]$$

- When animated, each oscillator cycles in the same time T and a wave pattern appears to move down the set of oscillators.



Now try to use similar ideas to describe "pendulum waves"

- Begin with a normal traveling wave description

$$y[x, t] = A\cos[\omega\,t + k\,x]$$

- In this case all oscillators are displaced by A at time $t = 0$ so

$y[x, t=0] = A\cos[k\,x] = A$ so apparently $k = 0$!

- The key difference here is that the pendula do not have the same length, so ω is itself a function of x.

$$y[x, t] = A\cos[\omega\,t] \Rightarrow A\cos[\omega[x]\,t]$$

Figuring out the function $\omega[x]$

- Call the overall cycling time Γ (is about 20 sec).
- Next, number the pendula $n = 20, 21, 22, \ldots$ where the n^{th} pendulum has a period $T_n = \Gamma/n$.
- Thus $\omega_n = \dfrac{2\pi\, rad}{T_n} = \dfrac{2\pi\, rad}{\Gamma/n} = n\left(\dfrac{2\pi\, rad}{\Gamma}\right)$.

- Now if the spacing between pendula is d, then they are located at $x = 0, d, 2d, 3d, \ldots$ which means

$$x_n = (n - 20)\, d = n\, d - 20d.$$

- Solving that for n gives x for the n^{th} pendulum.

$$n = \frac{x_n + 20d}{d} = \frac{x_n}{d} + 20$$

- Thus $\omega_n = n\left(\dfrac{2\pi\, rad}{\Gamma}\right) = \left(\dfrac{x_n}{d} + 20\right)\left(\dfrac{2\pi\, rad}{\Gamma}\right)$

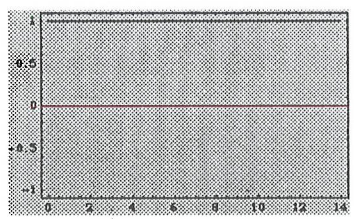
and hence, continuously, $\omega[x] = \left(\dfrac{x}{d} + 20\right)\left(\dfrac{2\pi\, rad}{\Gamma}\right)$

Proposed math description for a "pendulum waves" function

- Putting it all together, we have

$$y[x,t] = A\cos[\omega[x]\,t] = A\cos\left[\left(\frac{x}{d}+20\right)\left(\frac{2\pi\,rad}{\Gamma}\right)t\right]$$

- Try animating that.



Rewriting that expression

- This may be easier to understand when rewritten as

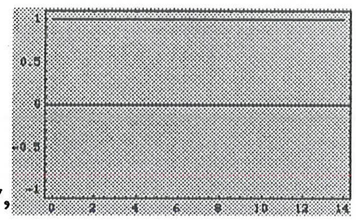
$$y[x,t] = A\cos\left[\left(\frac{2\pi\,rad}{\Gamma\,d}\,t\right)x + \left(\frac{40\pi\,rad}{\Gamma}\right)t\right]$$

which means $y[x,t] = A\cos[k[t]\,x + \omega_{20}\,t]$

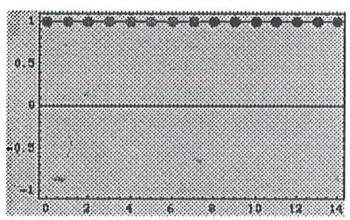
where $k[t] = \left(\frac{2\pi\,rad}{\Gamma\,d}\right)t$ so $\lambda[t] = \frac{2\pi\,rad}{k[t]} = \frac{\Gamma\,d}{t}$.

- Thus $y[x,t]$ describes a traveling wave that begins with $\lambda[t=0] = \infty$ and then the wavelength λ shrinks as t^{-1}.

- That makes the animation easier to understand, but can $y[x, t]$ actually explain <u>all</u> the patterns the pendula show, including those after $t = \Gamma/2$ when the pendulum waves appear to reverse direction and get longer?



- Try superimposing the pendulum movie and the proposed function $y[x, t]$.



Wow! Aliasing!

- Thus the proposed function $y[x, t]$ <u>does</u> match the pendula at all times, even though $y[x, t]$ continuously shrinks and never reverses direction!
- Notice that $\lambda[t = \Gamma/2] = \frac{\Gamma d}{\Gamma/2} = 2d$, exactly what we expect for the out-of-phase pattern.
- Similarly $\lambda[t = \Gamma] = \frac{\Gamma d}{\Gamma} = d$. Thus when the pendula "come back in phase again" there is actually a full cycle of $y[x, t]$ between adjacent pendula!

Aliasing (AKA sampling error)

- Notice that for times $t > \Gamma/2$ there are more peaks and valleys in the function $y[x, t]$ than there are pendula, so the pendula cannot possibly capture the true complexity of $y[x, t]$.

- Indeed, at certain times the pendula appear to show a pattern much broader than $y[x, t]$ that appears to travel to the right while $y[x, t]$ consists of shrinking waves that travel to the left.

- Periodic (under)sampling of a periodic function can lead to patterns in the data that are very misleading. This is called "aliasing" or "sampling error."

- Aliasing is due to a (usually-unintended) coupling between the frequency of the signal and the frequency at which the signal is being sampled.

- For example, aliasing can occur if data is collected only at certain <u>times</u>. e.g. strobe effects

- In the case of pendulum waves, information about the functional shape is available at all times, but only at certain <u>locations</u> (the positions of the pendula).

References

• J. A. Flaten and K. A. Parendo, American Journal of Physics, 69 (7), July 2001, pp. 778 - 782.

This paper also explores questions like "Does the function $y[x, t]$, examined only at locations where there are pendula, look identical at $t = (\Gamma/2) + \varepsilon$ and $t = (\Gamma/2) - \varepsilon$ (times equally spaced before then after the out-of-phase time)? Answer : Yes!

• www.mrs.umn.edu/~flatenja/pendulumwaves.shtml

• www.physics.umd.edu/lecdem/services/demos/demosg1/g1-82.htm

Addendum

• One could extending the apparatus to longer and longer pendula, building $n = 19, 18, 17, \ldots, 1, 0$!!

• If x is now measured from the $n = 0$ pendulum, we realize that pendulum waves are just a limited view of a collapsing accordion function.