

# Drag and Moment in Viscous Potential Flow

D. D. Joseph, T. Y. Liao and H. H. Hu

Department of Aerospace Engineering and Mechanics

University of Minnesota

Minneapolis, Minnesota 55455

May 1992

We consider solutions of the Navier-Stokes equations in which the velocity is given by the gradient of a potential. We show that the drag on bodies and bubbles is the same in viscous and inviscid potential flow. The lift on two-dimensional bodies is given by the usual Kutta condition but the moment about the origin of the stresses acting on the body is given by  $M_I + 2\mu\Gamma$  where  $\mu$  is the viscosity,  $\Gamma$  is the circulation and  $M_I$  is the usual moment for an inviscid fluid.

## I. Introduction

It is well known that potential flows  $\mathbf{u} = \nabla\phi$  are solutions of the Navier-Stokes equations for viscous incompressible fluids. The contribution  $\mu\nabla^2\phi$  of the viscous term to the Bernoulli equation vanishes, but the viscous contribution to the stress in an incompressible fluid (Stokes, 1850)

$$T_{ij} = -p \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = -p \delta_{ij} + 2\mu \frac{\partial^2 \phi}{\partial x_i \partial x_j} \quad (1)$$

does not vanish in general. Potential flows will not generally satisfy the boundary conditions which are associated with the requirement that the tangential component of velocity be continuous across the interface separating the fluid from a solid or another fluid. One view is that potential flow can be a good approximation to the true flow outside boundary layers and wakes where vorticity is important. However, viscosity may act strongly in regions in which vorticity is effectively zero.

The idea that viscous forces in regions of potential flow may come even to dominate the dissipation of energy seems to have been advanced first by Lamb [1924] who showed that in some cases of wave motion the rate of dissipation can be calculated with sufficient accuracy by regarding the motion as irrotational with slipping at solid-fluid boundaries. The computation of the drag  $D$  on a sphere in potential flow using the dissipation method seems to have been given first by Bateman [1932] and repeated Ackeret [1952]. They find that  $D = 12\pi a\mu U$  where  $\mu$  is the viscosity,  $a$  the radius of the sphere and  $U$  its velocity. This drag is twice the Stokes drag and is in better agreement with the measured drag for Reynolds numbers in excess of about 8.

The same calculation for a rising spherical bubble of gas was given by Levich [1949]. Measured values of the drag on spherical gas bubbles are close to  $12\pi a\mu U$  for Reynolds numbers larger than about 20. The reasons for the success of the dissipation method in predicting the drag on gas bubbles has to do with the fact that vorticity is confined to thin layers and the contribution of this vorticity to the drag is smaller in the case of gas bubbles, where the shear traction rather than the relative velocity must vanish on the surface of the sphere. A good explanation was given by Levich [1962] and by Moore [1959, 1963]; a convenient reference is Batchelor [1967]. Brabston and Keller [1975] did a direct numerical simulation of the drag on a spherical bubble of gas in steady ascent at terminal velocity  $U$  in a Navier-Stokes fluid and found the same kinds of agreements seen already in experiments.

The idea that viscosity may act strongly in the regions in which vorticity is effectively zero appears to contradict explanations of boundary layers which have appeared repeatedly since Prandtl. For example, Glauert [1943] says (p.142) that "...Prandtl's conception of the problem is that the effect of the viscosity is important only in a narrow boundary layer surrounding the surface of the body and that the viscosity may be ignored in the free fluid outside this layer." Schlichting [1960] says (p.29) that "...This physical picture suggests that the field of flow in the case of fluids of small viscosity can be divided, for the purpose of mathematical analysis, into two regions; the thin boundary layer near the wall, in which friction must be taken into account, and the region outside the boundary layer, where the forces due to friction are small and may be neglected and where, therefore, the perfect fluid theory offers a very good approximation."

The conventional view of boundary layers could however be wrong in special cases where the gradients of velocity at the boundary are somehow suppressed as in the case of bubbles with zero shear traction or in other special cases which are mentioned

below. Typically at high Reynolds numbers the dissipation in the boundary layer near a solid surface will dominate. It may however be no worse to neglect the effects of viscosity of the fluid near a wall than to put it to zero everywhere outside a thin boundary layer.

The flow of a viscous fluid, which is at rest at infinity, outside a long cylinder of radius  $a$  rotating with a steady angular velocity  $\omega$  is an exact realization of viscous potential flow valid even when the viscosity  $\mu$  is very large. The exact solution of this problem is given by

$$\mathbf{u} = \frac{\omega a^2}{r} \mathbf{e}_\theta \quad (2)$$

and it is a potential flow solution of the Navier-Stokes equations with a circulation

$$\Gamma = -2\pi a^2 \omega \quad (3)$$

which satisfies the no slip condition. The viscosity enters this problem through the couple

$$M = 2\mu\Gamma \quad (4)$$

required to turn the cylinder.

The chances of getting good approximations for the true drag per unit length on a cylinder from the potential flow calculation using the dissipation method are best when the dissipation in vorticity layers is small and separation is suppressed. Two cases in which separation can be suppressed by rotation come to mind. The potential flow drag for both cases can be obtained from the formula for the dissipation

$$8\pi\mu U^2 \left[ 1 + \frac{\Gamma^2}{8\pi^2 U^2 a^2} \right] \quad (5)$$

of a uniform flow past a cylinder of radius  $a$  with circulation  $\Gamma$ , which was calculated by Ackeret [1952]. Then, according to (18), the drag per unit length is

$$8\pi\mu U \left[ 1 + \frac{\Gamma^2}{8\pi^2 U^2 a^2} \right]. \quad (6)$$

The first case involves the use of the Kutta-Joukowski condition. In general this potential flow has two stagnation points on the surface of the cylinder. The exact viscous flow corresponding to this circulation is a separated flow (see Batchelor [1967], Fig 6.6.2)

except in the case in which the circulation is adjusted so that the two stagnation points coalesce on the cylinder,  $\Gamma = 4\pi aU$ . This is the Kutta-Joukowski circulation. In this case the drag per unit length given by (6) is  $24\pi\mu U$ . The second case concerns the symmetric problem,  $\Gamma = 0$ , of flow past a cylinder. This flow is strongly separated, but Taneda [1977] has shown that the potential flow solution with  $\Gamma = 0$  may be obtained by oscillatory rotation of the cylinder around its axis ( see his Fig 54) and by (6) the drag per unit length is  $8\pi\mu U$ . This same drag,  $8\pi\mu U$  per unit length, on an infinitely long cylindrical bubble is predicted by the dissipation method. It would be of interest to do experiments to test the prediction that this drag, is roughly independent of cylinder radius  $a$ , for some range of  $a$ .

## II. Formula relating drag to dissipation

Consider the motion of a solid body or bubble in a viscous liquid in three dimensions. Suppose that the body  $B$  moves forward with a velocity  $U \mathbf{e}_x$  and that it neither rotates nor changes shape or volume. The absolute velocity  $\mathbf{u}$  and the relative velocity  $\mathbf{v}$  of the fluid are then related by

$$\mathbf{u} = U \mathbf{e}_x + \mathbf{v} \quad (7)$$

with

$$\mathbf{v} \cdot \hat{\mathbf{n}} \big|_{\partial B} = 0 \quad (8)$$

where  $\hat{\mathbf{n}}$  is the outward normal on the boundary  $\partial B$  of  $B$ . The fluid outside  $B$  is unbounded. The volume  $V$  of the fluid outside  $B$  is a material volume because no mass crosses  $\partial B$  and therefore satisfies the requirements for the validity of Reynolds transport theorem which, when applied to the kinetic energy  $E$  of the fluid in  $V$ , becomes

$$\frac{dE}{dt} = \frac{d}{dt} \int_{V(t)} \rho \frac{|\mathbf{u}|^2}{2} dV = \int_{V(t)} \rho \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} dV. \quad (9)$$

Since

$$\rho \frac{d\mathbf{u}}{dt} = \text{div } \mathbf{T} \quad (11)$$

with  $\mathbf{T}$  given by (1), we find, using (11) in (9) and applying the divergence theorem using  $\mathbf{u} \sim O(\frac{1}{R})$  as  $R \rightarrow \infty$ , that

$$\begin{aligned}
\frac{dE}{dt} &= \int_{V(t)} \mathbf{u} \cdot \operatorname{div} \mathbf{T} dV \\
&= \int_{\partial B} \mathbf{u} \cdot (\mathbf{T} \cdot \mathbf{n}) dS - \int_{V(t)} 2\mu \mathbf{D}[\mathbf{u}] : \mathbf{D}[\mathbf{u}] dV
\end{aligned} \tag{12}$$

where  $\mathbf{n} = -\hat{\mathbf{n}}$ ,  $\mathbf{D}[\mathbf{u}]$  is the symmetric part of  $\nabla \mathbf{u}$  and  $\mathbf{T} \cdot \mathbf{n}$  is the negative of the traction vector expressing the force exerted by the fluid on the body. We may rewrite this, using (7), as

$$\frac{dE}{dt} = UD + \int_{\partial B} \mathbf{v} \cdot (\mathbf{T} \cdot \mathbf{n}) dS - \int_{V(t)} 2\mu \mathbf{D}[\mathbf{u}] : \mathbf{D}[\mathbf{u}] dV \tag{13}$$

where

$$D = \int_{\partial B} \mathbf{e}_x \cdot (\mathbf{T} \cdot \mathbf{n}) dS \tag{14}$$

is the drag exerted by the fluid on the body and

$$\int_{\partial B} \mathbf{v} \cdot (\mathbf{T} \cdot \mathbf{n}) dS = 2\mu \int_{\partial B} \mathbf{v} \cdot (\mathbf{D}[\mathbf{u}] \cdot \mathbf{n}) dS \tag{15}$$

because  $p \mathbf{v} \cdot \mathbf{n} = 0$  at each point of  $\partial B$ , according to (8).

There are two standard cases in which

$$\mathbf{v} \cdot (\mathbf{T} \cdot \mathbf{n})|_{\partial B} = 0. \tag{16}$$

If  $B$  is a rigid solid, then  $\mathbf{v}|_{\partial B} = 0$ . If  $B$  is a bubble and the tangential component of the traction vector vanishes,

$$\boldsymbol{\tau} \cdot (\mathbf{T} \cdot \mathbf{n})|_{\partial B} = 0, \text{ for all } \boldsymbol{\tau} \perp \mathbf{n}. \tag{17}$$

Since  $\mathbf{v} \perp \mathbf{n}$  by (8), we obtain (16). In the steady case,  $\frac{dU}{dt} = \frac{dE}{dt} = 0$  and

$$UD = \int_{V(t)} 2\mu \mathbf{D}[\mathbf{u}] : \mathbf{D}[\mathbf{u}] dV. \tag{18}$$

Levich [1949] used the right side of (18), evaluating  $\mathbf{D}[\mathbf{u}]$  in potential flow, to compute the drag in the steady ascent of a bubble in a viscous fluid. The formula (18) appears in the book by Batchelor, but not (13).

Landau and Lifshitz have given the Levich [1949] result on page 172, §45 of their [1959] book. But  $\frac{dE}{dt} = \dot{E}_{kin} = 0$  in steady flow and their discussion is confused. They prove that  $\dot{E}_{kin} < 0$  on page 54 in the case when there is **no** body. Their result on that page should be framed as a stability result,  $\dot{E}_{kin} \rightarrow 0$  as  $t \rightarrow \infty$ . The dissipation is an energy rate; it gives rise to the generation of heat from friction and not to the decay of the kinetic energy in steady flow.

We must maintain a distinction between potential flow approximations outside of vorticity layers and strict potential flow which in general satisfies neither (16) nor (17). For strict potential flow we have

$$\begin{aligned} \int_{\partial B} \mathbf{v} \cdot (\mathbf{D}[\mathbf{u}] \cdot \mathbf{n}) dS &= \int_{\partial B} \mathbf{v} \cdot (\mathbf{D}[\mathbf{v}] \cdot \mathbf{n}) dS \\ &= \int_{\partial B} v_i \frac{\partial v_i}{\partial x_j} n_j dS = \int_V \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dV = \int_V \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} dV = \int_V D_{ij} D_{ij} dV. \end{aligned} \quad (19)$$

Hence, for strict potential flow,

$$\int_{\partial B} \mathbf{v} \cdot (\mathbf{T} \cdot \mathbf{n}) dS = \int_{V(t)} 2\mu \mathbf{D}[\mathbf{u}] : \mathbf{D}[\mathbf{u}] dV \quad (20)$$

and (13) reduces to

$$M' \frac{dU}{dt} = D \quad (21)$$

where  $M'$  is the apparent mass of the fluid. This equation proves the theorem enunciated in the summary at the beginning of the paper. For flows in which the vorticity is confined to narrow layers the kinetic energy  $E$  should be well approximated by potential flow (even if the dissipation is not).

We have already noted that the boundary term on the left of (20) is zero when no-slip (16) or no-traction (17) conditions are prescribed. In this case we may approximate the true drag by evaluating (18) on potential flow. There is also an additional drag due to the vorticity layer neglected in the dissipation calculation. In the case of a rising bubble, by using the control volume method, one would find that the additional drag manifests itself in the momentum balance in the form of a thin wake of lower order trailing the bubble.

The only papers known to us which address the d'Alembert paradox for viscous flow are by Cisotti [1917] and Finzi [1925]. Cisotti asserts that the drag on a body in steady translation vanishes without using potential flow or the vanishing of vorticity. This is wrong. He says this "...conclusion seems to be in contradiction with the classic solution of Stokes relative to the slow uniform translation of a sphere in an unbounded viscous liquid."

Finzi's [1925] paper is written in old notation which is hard to follow. He seems to have proved (20) without explicitly introducing the drag as in (13). A more accessible exposition of Finzi's mathematical work can be found in the review by Bateman[1932], who says that "...in Finzi's theory there is no dissipation of energy in irrotational flow."

### **III. Force on a two dimensional body in steady incompressible viscous potential flow**

After integrating the Navier-Stokes equations for steady motion over the volume  $V$  between the body  $B$  and the control surface  $C$  in Figure 1, using  $\mathbf{u} \cdot \mathbf{n} = 0$  on  $\partial B$ , we find that

$$\mathbf{F} = \int_C [\mathbf{T} \cdot \mathbf{n} - \rho(\mathbf{n} \cdot \mathbf{u})\mathbf{u}] dS \quad (22)$$

where  $\mathbf{F}$  is the force of the fluid on the body. The  $x$  and  $y$  components of the force are then given by

$$X = \int_C [n_x T_{xx} + n_y T_{yx} - \rho u(\mathbf{n} \cdot \mathbf{u})] dS \quad (23)$$

$$Y = \int_C [n_x T_{xy} + n_y T_{yy} - \rho v(\mathbf{n} \cdot \mathbf{u})] dS \quad (24)$$

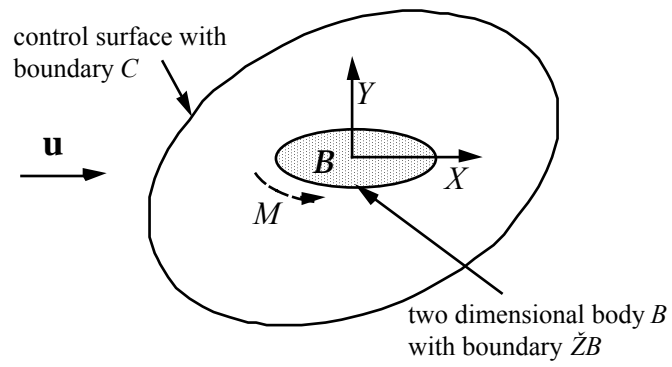


Figure 1: An arbitrary body  $B$  in two dimensions is enclosed by a control surface. Here  $X$  and  $Y$  are the components of the force exerted by the fluid on the body and  $M$  is the hydrodynamic couple.



where the velocity  $\mathbf{u} = (u, v)$ ,

$$\mathbf{n}dS = (n_x dS, n_y dS) = (dy, -dx), \quad (25)$$

and  $\mathbf{T}$  is given by (1). Since the Bernoulli equation for steady potential flow

$$p = -\frac{\rho}{2} |\mathbf{u}|^2 + \text{constant}, \quad (26)$$

is the same with and without viscosity, we may decompose the force expressions (23) and (24) into viscous and inviscid parts. Thus

$$X = \int_C 2\mu \left[ \frac{\partial^2 \phi}{\partial x^2} dy - \frac{\partial^2 \phi}{\partial x \partial y} dx \right] + X_I \quad (27)$$

$$Y = \int_C 2\mu \left[ \frac{\partial^2 \phi}{\partial x \partial y} dy - \frac{\partial^2 \phi}{\partial y^2} dx \right] + Y_I \quad (28)$$

where  $X_I$  and  $Y_I$  are the forces that arise in the classical case of inviscid potential flow, and are given by the Blasius integral formula

$$X_I - i Y_I = i \frac{\rho}{2} \int_C W^2 dz \quad (29)$$

where  $W = u - i v$  is the complex velocity, an analytic function of the complex variable  $z = x + iy$ . A straightforward calculation, using the fact that  $\phi$  satisfies Laplace's equation, leads to the expression

$$X - i Y = i \int_C \left[ \frac{\rho}{2} W^2 - 2\mu \frac{dW}{dz} \right] dz \quad (30)$$

for the forces on a body in viscous potential flow. Using the residue theorem and the expression (38) for complex velocity  $W$ , we find that the viscous force on the body in steady potential flow vanishes.

The lift  $L$  and drag  $D$  on a body in a uniform stream at attack angle  $\alpha$

$$U + iV = \sqrt{U^2 + V^2} e^{i\alpha} \quad (31)$$

is given by

$$D - iL = (X - iY) e^{i\alpha}. \quad (32)$$

**IV. Moment on a two dimensional body  
in steady incompressible viscous potential flow**

A circulation  $\Gamma$  around the body will induce a viscous contribution  $2\mu\Gamma$  to the moment of forces on the body. The moment  $M$  of hydrodynamic forces about the origin is given by

$$M = M_I + \int_C 2\mu \left[ (xdy+ydxdx) \frac{\partial^2 \phi}{\partial x \partial y} - y \frac{\partial^2 \phi}{\partial x^2} dy - x \frac{\partial^2 \phi}{\partial y^2} dx \right] \quad (33)$$

where

$$M_I = \int_C p (xdx+ydxdy) + \rho(udy - vdx) (uy - vx) \quad (34)$$

is the moment in ideal potential flow (see, for example, Currie [1974]). It is well known that (34) may be expressed in terms of complex variables as

$$M_I = - \operatorname{Re} \int_C z \frac{\rho}{2} W^2 dz . \quad (35)$$

A straightforward calculation shows that the viscous moment may also be expressed in terms of the complex velocity, so that

$$M = \operatorname{Re} \int_C z \left[ -\frac{\rho}{2} W^2 + 2\mu \frac{dW}{dz} \right] dz. \quad (36)$$

In the unsteady case, the viscous terms in the force (30) and moment (36) stay unchanged, while the Blasius formulae for  $X_I - i Y_I$  and  $M_I$  have to be replaced by the extended version for the unsteady flow which can be found in Milne-Thomson [1968].

Now we show that the viscous moment does not vanish when  $\Gamma \neq 0$ . The far field form of the potential  $\phi(z)$  of the flow past a cylinder of arbitrary cross section in an infinite domain is given by

$$\phi(z) = z e^{-i\alpha} \sqrt{U^2+V^2} + \frac{m+i\Gamma}{2\pi} \ln z + \sum_{k=1}^{\infty} \frac{a_k+ib_k}{z^k} \quad (37)$$

where the  $\Gamma$  is the circulation, which is positive if clockwise,  $m$  is the volume flux across the boundary of the cylinder, which vanishes for a solid body, and  $a_k, b_k$  are real constants determined by the shape of the cylinder. The complex velocity at far field is then given by

$$W = \frac{d\phi}{dz} = e^{-i\alpha\sqrt{U^2+V^2}} + \frac{m+i\Gamma}{2\pi z} - \sum_{k=1}^{\infty} k \frac{a_k+ib_k}{z^{k+1}}. \quad (38)$$

The force (30) and moment (36) may now be evaluated in the usual way, using residues:

$$X - iY = -\rho (m + i\Gamma) e^{-i\alpha\sqrt{U^2+V^2}}, \quad (39)$$

$$M = \frac{\rho m \Gamma}{2\pi} + 2\rho\pi (a_1 V - b_1 U) + 2\mu\Gamma. \quad (40)$$

Equations (39) and (40) hold for bodies of arbitrary shapes when the singularities are all in the interior of the body. The lift and drag may be obtained from (32). The lift and drag reduce to the classical result for inviscid fluids but a moment may be associated with viscosity.

In the case of an elliptic cylinder of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (41)$$

we have  $m = 0$ ,  $a_1 = Ub \frac{a+b}{2}$  and  $b_1 = Va \frac{a+b}{2}$ . And (39), (32) and (40) give rise to

$$D = 0, \quad (42)$$

$$L = \rho\Gamma\sqrt{U^2+V^2}, \quad (43)$$

$$M = 2\mu\Gamma - \rho\pi (a^2 - b^2) UV. \quad (44)$$

Batchelor [1967] has considered this case for an inviscid fluid and he notes “... that the pressure distribution over the surface of the ellipse tends to turn it about the origin to a position broadside-on to the stream.” In the viscous case there is moment  $M = 2\mu\Gamma$  even without a stream. Perhaps the circulation which best represents the action of the true fluid is the  $\Gamma$  given by (44) where  $M$  is the measured value of torque.

## Acknowledgments

This work was supported by the NSF, fluid, particulate and hydraulic systems; by US Army, Mathematics; AHPCRC and DOE, Department of Basic Energy Sciences. We gratefully acknowledge the help of Geraldo Ribeiro with several aspects of this paper.

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