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Review

Potential flow of viscous fluids: Historical notes

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Abstract

In this essay I will attempt to identify the main events in the history of thought about irrotational flow of viscous fluids. I am of the opinion that when considering irrotational solutions of the Navier–Stokes equations it is never necessary and typically not useful to put the viscosity to zero. This observation runs counter to the idea frequently expressed that potential flow is a topic which is useful only for inviscid fluids; many people think that the notion of a viscous potential flow is an oxymoron. Incorrect statements like "... irrotational flow implies inviscid flow but not the other way around" can be found in popular textbooks.

Though convenient, phrases like "inviscid potential flow" or "viscous potential flow" confuse properties of the flow (potential or irrotational) with properties of the material (inviscid or viscous); it is better and more accurate to speak of the irrotational flow of an inviscid or viscous fluid.

> Every theorem about potential flow of perfect fluids with conservative body forces applies equally to viscous fluids in regions of irrotational flow.

Keywords: Perfect fluid; Inviscid fluid; Viscous fluid; Potential flow; Irrotational flow; Vorticity; Viscous decay; Dissipation method; Boundary layer; Drag; Lift

Contents

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1. Navier–Stokes equations

The history of Navier–Stokes equations begins with the 1822 memoir of Navier who derived equations for homogeneous incompressible fluids from a molecular argument. Using similar arguments, Poisson (1829) derived the equations for a compressible fluid. The continuum derivation of the Navier–Stokes equation is due to Saint-Venant (1843) and Stokes (1845). In his 1851 paper (Section 49), Stokes wrote that

> Let P_1, P_2, P_3 be the three normal, and T_1, T_2, T_3 be the three tangential pressures in the direction of three rectangular planes parallel to the co-ordinate planes, and let D be the symbol of differentiation with respect to t when the particle and not the point of space remains the same. Then the general equations applicable to a heterogeneous fluid (Eqs. (10) of my former (1845) paper) are
>
> $$\rho\left(\frac{\mathrm{D}u}{\mathrm{D}t}-X\right)+\frac{\mathrm{d}P_1}{\mathrm{d}x}+\frac{\mathrm{d}T_3}{\mathrm{d}y}+\frac{\mathrm{d}T_2}{\mathrm{d}z}=0, \tag{132}$$
>
> with the two other equations which may be written down from symmetry. The pressures P_1, etc. are given by the equations
>
> $$P_1=p-2\mu\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\delta\right),\quad T_1=-\mu\left(\frac{\mathrm{d}v}{\mathrm{d}z}+\frac{\mathrm{d}w}{\mathrm{d}y}\right), \tag{133}$$
>
> and four other similar equations. In these equations
>
> $$3\delta=\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\mathrm{d}v}{\mathrm{d}y}+\frac{\mathrm{d}w}{\mathrm{d}z}. \tag{134}$$

The equations written by Stokes in his 1845 paper are the same ones we use today:

$$\rho\left(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}-\mathbf{X}\right)=\operatorname{div}\mathbf{T}, \tag{1.1}$$

$$\mathbf{T}=\left(-p-\frac{2}{3}\mu\operatorname{div}\mathbf{u}\right)\mathbf{1}+2\mu\mathbf{D}[\mathbf{u}], \tag{1.2}$$

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}=\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}, \tag{1.3}$$

$$\mathbf{D}[\mathbf{u}]=\frac{1}{2}\left[\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathrm{T}}\right], \tag{1.4}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\operatorname{div}\mathbf{u}=0. \tag{1.5}$$

Inviscid fluids are fluids with zero viscosity. Viscous effects on the motion of fluids were not understood before the notion of viscosity was introduced by Navier in 1822. Perfect fluids, following the usage of Stokes and other 19th century English mathematicians, are inviscid fluids which are also incompressible. Statements like Truesdell's (1954),

> In 1781 Lagrange presented his celebrated velocity-potential theorem: if a velocity potential exists at one time in a motion of an inviscid incompressible fluid, subject to conservative extraneous force, it exists at all past and future times.

though perfectly correct, could not have been asserted by Lagrange, since the concept of an inviscid fluid was not available in 1781.

2. Stokes theory of potential flow of viscous fluid

The theory of potential flow of a viscous fluid was introduced by Stokes (1851). All of his work on this topic is framed in terms of the effects of viscosity on the attenuation of small amplitude waves on a liquid–gas surface. Everything he said about this problem is cited below. The problem treated by Stokes was solved exactly using the linearized Navier–Stokes equations, without assuming potential flow, was solved exactly by Lamb (1932).

Stokes discussion is divided into three parts discussed in Sections 51–53:

(1) The dissipation method in which the decay of the energy of the wave is computed from the viscous dissipation integral where the dissipation is evaluated on potential flow (Section 51).
(2) The observation that potential flows satisfy the Navier–Stokes equation together with the notion that certain viscous stresses must be applied at the gas–liquid surface to maintain the wave in permanent form (Section 52).
(3) The observation that if the viscous stresses required to maintain the irrotational motion are relaxed, the work of those stresses is supplied at the expense of the energy of the irrotational flow (Section 53).

Lighthill (1978) discussed Stokes' ideas but he did not contribute more to the theory of irrotational motions of a viscous fluid. On page 234 he notes that

> Stokes ingenious idea was to recognize that the average value of the rate of working given by sinusoidal waves of wave number
>
> $2\mu\lfloor(\partial\phi/\partial x)\partial^2\phi/\partial x\,\partial z+(\partial\phi/\partial z)\partial^2\phi/\partial z^2\rfloor_{z=0}$
>
> which is required to maintain the unattenuated irrotational motions of sinusoidal waves must exactly balance the rate at which the same waves when propagating freely would lose energy by internal dissipation.

Lamb (1932) gave an exact solution of the problem considered by Stokes in which vorticity and boundary layers are not neglected. He showed that the value given for the decay constant computed by Stokes is twice the correct value. Joseph and Wang (2004a) computed the decay constant for gravity waves directly as an ordinary stability problem in which the velocity is irrotational, the pressure is given by Bernoulli's equation and the viscous component of the normal stress is evaluated on the irrotational flow. This kind of analysis we call viscous potential flow or VPF. The decay constant computed by VPF is one half the correct values computed by the dissipation method when the waves are longer than critical value for which progressive waves give way to monotonic waves. For waves shorter than the critical value the decay constant is given by $g/2\upsilon k$; the decay constant from Lambs exact solution agrees with the dissipation value for long waves and with the VPF value for short waves.

2.1. The dissipation method

Section 51. By means of the expression given in Art. 49, for the loss of *vis viva* due to internal friction, we may readily obtain a very approximate solution of the problem: To determine the rate at which the motion subsides, in consequence of internal friction, in the case of a series of oscillatory waves propagated along the surface of a liquid. Let the vertical plane of xy be parallel to the plane of motion, and let y be measured vertically downwards from the mean surface; and for simplicity's sake suppose the depth of the fluid very great compared with the length of a wave, and the motion so small that the square of the velocity may be neglected. In the case of motion which we are considering, $u\,\mathrm{d}x + v\,\mathrm{d}y$ is an exact differential $\mathrm{d}\phi$ when friction is neglected, and

$$\phi = c\varepsilon^{-my}\sin(mx - nt), \tag{140}$$

where c, m, n are three constants, of which the last two are connected by a relation which it is not necessary to write down. We may continue to employ this equation as a near approximation when friction is taken into account, provided we suppose c, instead of being constant, to be parameter which varies slowly with the time. Let V be the *vis viva* of a given portion of the fluid at the end of the time t. Then

$$V = \rho c^2 m^2 \int\int\int \varepsilon^{-2my}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{141}$$

But by means of the expression given in Art. 49, we get for the loss of vis viva during the time $\mathrm{d}t$, observing that in the present case μ is constant, $w = 0$, $\delta = 0$, and $u\,\mathrm{d}x + v\,\mathrm{d}y = \mathrm{d}\phi$, where ϕ is independent of z,

$$4\mu\,\mathrm{d}t \int\int\int \left\{ \left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)^2 + \left(\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}\right)^2 + 2\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x\,\mathrm{d}y}\right)^2 \right\} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

which becomes, on substituting for ϕ its value,

$$8\mu c^2 m^4\,\mathrm{d}t \int\int\int \varepsilon^{-2my}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

But we get from (141) for the decrement of *vis viva* of the same mass arising from the variation of the parameter c,

$$-2\rho m^2 c\frac{\mathrm{d}c}{\mathrm{d}t}\mathrm{d}t \int\int\int \varepsilon^{-2my}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Equating the two expressions for the decrement of *vis viva*, putting for m its value $2\pi\lambda^{-1}$, where λ is the length of a wave, replacing μ by $\mu'\rho$, integrating, and supposing c_0 to be the initial value of c, we get

$$c = c_0\varepsilon^{-\frac{16\pi^2\mu' t}{\lambda^2}}.$$

In a footnote on page 624, Lamb notes that "Through an oversight in the original calculation the value $\lambda^2/16\pi^2\nu$ was too small by one half." The value 16 should be 8.

> It will presently appear that the value of $\sqrt{\mu'}$ for water is about 0.0564, an inch and a second being the units of space and time. Suppose first that λ is 2 in., and t is 10 s. Then $16\pi^2\mu' t\lambda^{-2} = 1.256$, and $c{:}c_0{::}1{:}0.2848$, so that the height of the waves, which varies as c, is only about a quarter of what it was. Accordingly, the ripples excited on a small pool by a puff of wind rapidly subside when the exciting cause ceases to act.
> Now suppose that λ is to fathoms or 2880 in., and that t is 86,400 s or a whole day. In this case $16\pi^2\mu' t\lambda^{-2}$ is equal to only 0.005232, so that by the end of an entire day, in which time waves of this length would travel 574 English miles, the height would be diminished by little more than the one two hundredth part in consequence of friction. Accordingly, the long swells of the ocean are but little allayed by friction, and at last break on some shore situated at the distance of perhaps hundreds of miles from the region where they were first excited.

2.2. The distance a wave will travel before it decays by a certain amount

The observations made by Stokes about the distance a wave will travel before its amplitude decays by a given amount, point the way to a useful frame for the analysis of the effects of viscosity on wave propagation. Many studies of nonlinear irrotational waves can be found in the literature but the only study of the effects of viscosity on the decay of these waves known to me is due to Longuet-Higgins (1997) who used the dissipation method to determine the decay due to viscosity of irrotational steep capillary-gravity waves in deep water. He finds that the limiting rate of decay for small amplitude solitary waves are twice those for linear periodic waves computed by the dissipation method. The dissipation of very steep waves can be more than ten times more than linear waves due to the sharply increased curvature in wave troughs. He assumes that the nonlinear wave maintains its steady form while decaying under the action of viscosity. The wave shape could change radically from its steady shape in very steep waves. These changes could be calculated for irrotational flow using VPF as in the work of Miksis et al. (1982) (see Section 11).

Stokes (1880) studied the motion of nonlinear irrotational gravity waves in two dimensions which are propagated with a constant velocity, and without change of form. This analysis led to the celebrated maximum wave whose asymptotic form gives rise to a pointed crest of angle 120°. The effects of viscosity on such extreme waves has not been studied but they may be studied by the dissipation method or same potential flow theory used by Stokes (1851) for inviscid fluids with the caveat that the normal stress condition that p vanish on the free surface be replaced by the condition that

$$p + 2\mu\,\partial u_n/\partial n = 0$$

on the free surface with normal $\mathbf{n}$ where the velocity component $u_n = \partial\phi/\partial n$ is given by the potential.

2.3. The stress of a viscous fluid in potential flow

> Section 52. It is worthy of remark, that in the case of a homogeneous incompressible fluid, whenever $u\,dx + v\,dy + w\,dz$ is an exact differential, not only are the ordinary equations of fluid motion satisfied,[1] but the equations obtained when friction is taken into account are satisfied likewise. It is only the equations of condition which belong to the boundaries of the fluid that are violated. Hence any kind of motion which is possible according to the ordinary equations, and which is such that $u\,dx + v\,dy + w\,dz$ is an exact differential, is possible likewise when friction is taken into account, provided we suppose a

[1] Truesdell (1950) discussed Bernoulli's theorem for viscous compressible fluids under some exotic hypothesis for which in general the vorticity is not zero. He notes "... Long ago Craig 1890 noticed that in the degenerate and physically improbable case of steady irrotational flow of a viscous incompressible fluid... the classical Bernoulli theorem of type (A) still holds..." Type (A) is a Bernoulli equation for a compressible fluid which holds throughout the fluid. Craig does not consider the linearized case for which the Bernoulli equation for compressible fluids has an explicit dependence on viscosity which is neither degenerate or improbable.

certain system of normal and tangential pressures to act at the boundaries of the fluid, so as to satisfy Eqs. (133). Since μ disappears from the general equations (1), it follows that p is the same function as before. But in the first case the system of pressures at the surface was $P_1 = P_2 = P_3 = p$, $T_1 = T_2 = T_3 = 0$. Hence if ΔP_1, etc. be the additional pressures arising from friction, we get from (133), observing that $\delta = 0$, and that $u\,\mathrm{d}x + v\,\mathrm{d}y + w\,\mathrm{d}z$ is an exact differential $\mathrm{d}\phi$,

$$\Delta P_1 = -2\mu\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}, \quad \Delta P_2 = -2\mu\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}, \quad \Delta P_3 = -2\mu\frac{\mathrm{d}^2\phi}{\mathrm{d}z^2}, \tag{142}$$

$$\Delta T_1 = -2\mu\frac{\mathrm{d}^2\phi}{\mathrm{d}y\,\mathrm{d}z}, \quad \Delta T_2 = -2\mu\frac{\mathrm{d}^2\phi}{\mathrm{d}z\,\mathrm{d}x}, \quad \Delta T_3 = -2\mu\frac{\mathrm{d}^2\phi}{\mathrm{d}x\,\mathrm{d}y}. \tag{143}$$

Let $\mathrm{d}S$ be an element of the bounding surface, l', m', n' the direction-cosines of the normal drawn outwards, ΔP, ΔQ, ΔR the components in the direction of x, y, z of the additional pressure on a plane in the direction of $\mathrm{d}S$. Then by the formula (9) of my former paper applied to Eqs. (142), (143) we get

$$\Delta P = -2\mu\left\{ l'\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + m'\frac{\mathrm{d}^2\phi}{\mathrm{d}x\,\mathrm{d}y} + n'\frac{\mathrm{d}^2\phi}{\mathrm{d}x\,\mathrm{d}z}\right\}, \tag{144}$$

with similar expressions for ΔQ and ΔR, and ΔP, ΔQ, ΔR are the components of the pressure which must be applied at the surface, in order to preserve the original motion unaltered by friction.

2.4. Viscous stresses needed to maintain an irrotational wave. Viscous decay of the free wave

Section 53. Let us apply this method to the case of oscillatory waves, considered in Art. 51. In this case the bounding surface is nearly horizontal, and its vertical ordinates are very small, and since the squares of small quantities are neglected, we may suppose the surface to coincide with the plane of xz in calculating the system of pressures which must be supplied, in order to keep up the motion. Moreover, since the motion is symmetrical with respect to the plane of xy, there will be no tangential pressure in the direction of z, so that the only pressures we have to calculate are ΔP_2 and ΔT_3. We get from (140), (142) and (143), putting $y = 0$ after differentiation,

$$\Delta P_2 = -2\mu m^2 c \sin(mx - nt), \quad \Delta T_3 = 2\mu m^2 c \cos(mx - nt). \tag{145}$$

If u_1, v_1 be the velocities at the surface, we get from (140), putting $y = 0$ after differentiation,

$$u_1 = mc\cos(mn - nt), \quad v_1 = -mc\sin(mx - nt). \tag{146}$$

It appears from (145) and (146) that the oblique pressure which must be supplied at the surface in order to keep up the motion is constant in magnitude, and always acts in the direction in which the particles are moving.

The work of this pressure during the time $\mathrm{d}t$ corresponding to the element of surface $\mathrm{d}x\,\mathrm{d}z$, is equal to $\mathrm{d}x\,\mathrm{d}z(\Delta T_3 \cdot u_1\,\mathrm{d}t + \Delta P_1 \cdot v_1\,\mathrm{d}t)$. Hence the work exerted over a given portion of the surface is equal to

$$2\mu m^3 c^2\,\mathrm{d}t \int\int \mathrm{d}x\,\mathrm{d}z.$$

In the absence of pressures ΔP_2, ΔT_3 at the surface, this work must be supplied at the expense of *vis viva*. Hence $4\mu m^3 c^2\,\mathrm{d}t \int\int \mathrm{d}x\,\mathrm{d}z$ is the *vis viva* lost by friction, which agrees with the expression obtained in Art. 51, as will be seen on performing in the latter the integration with respect to y, the limits being $y = 0$ to $y = \infty$.

3. Irrotational solutions of the Navier–Stokes equations; irrotational viscous stresses

Consider first the case of incompressible fluids $\operatorname{div}\mathbf{u} = 0$. If $\mathbf{X}$ has a potential ψ and the fluid is homogeneous (ρ and μ are constants independent of position at all times) then it is readily shown that

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u}\wedge\boldsymbol{\omega}\right] = -\nabla(p+\psi) + \mu\nabla^2\mathbf{u}, \tag{3.1}$$

where $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$. It is evident that $\boldsymbol{\omega} = 0$ is a solution of the curl (3.1). In this case

$$\mathbf{u} = \nabla\phi, \quad \nabla^2\phi = 0. \tag{3.2}$$

Since $\mu\nabla^2\mathbf{u} = \mu\nabla\nabla^2\phi = 0$ independent of μ, for large viscosities as well as small viscosities, (3.1) shows that

$$\nabla\left(\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}|\nabla\phi|^2 + p + \psi\right) = 0, \tag{3.3}$$

and $p = p_I$ is determined by Bernoulli's equation

$$\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}|\nabla\phi|^2 + p_I + \psi = F(t). \tag{3.4}$$

Potential flow $\mathbf{u} = \nabla\phi$, $\nabla^2\phi$ is a solution of the homogeneous, incompressible Navier–Stokes with a pressure $p = p_I$ determined by Bernoulli's equation, independent of viscosity. All of this known, maybe even well known, but largely ignored by the fluid mechanics community from the days of Stokes up till now.

Much less well known, and totally ignored, is the formula (1.2) for the viscous stress evaluated on potential flow $\mathbf{u} = \nabla\phi$,

$$\mathbf{T} = -p\mathbf{1} + 2\mu\nabla\otimes\nabla\phi. \tag{3.5}$$

The formula shows directly and with no ambiguity that viscous stresses are associated with irrotational flow. This formula is one of the most important that could be written about potential flows. It is astonishing, that aside from Stokes (1851), this formula which should be in every book on fluid mechanics, cannot be found in any.

The resultants of the irrotational viscous stresses (3.5) do not enter into the Navier–Stokes equations (3.1). Irrotational motions are determined by the condition that the solenoidal velocity is curl free and the evolution of the potential is associated with the irrotational pressure in the Bernoulli equation. However, the dissipation of the energy of potential flows and the power of viscous irrotational stresses do not vanish. Regions of curl free motions of the Navier–Stokes equations are guaranteed by various theorems concerning the persistence of irrotationality in the motions of parcels of fluid emanating from regions of irrotational flow (see Section 9). All flows on unbounded domains which tend asymptotically to rest or uniform motion and all the irrotational flows outside of vorticity boundary layers give rise to an additional irrotational viscous dissipation which deserves consideration.

The effects of viscous irrotational stresses which are balanced internally enter into the dynamics of motion at places where they become unbalanced such as at free surfaces and at the boundary of regions in which vorticity is important such as boundary layers and even at internal points in the liquid at which stress induced tensions exceed the breaking strength of the liquid. Irrotational viscous stresses enter as an important element in a theory of stress induced cavitation. In this theory, the field of principal stresses which determine the places and times at which the tensile stress is greater than the cavitation threshold must be computed (Funada et al., 2006b; Padrino et al., 2005).

Irrotational flows cannot satisfy no-slip conditions at boundaries when $\mu = 0$. No real fluid has $\mu = 0$. Perfect fluids cannot be used to study viscous effects of real fluids in irrotational flow.

Irrotational flows of a viscous fluid scale with the Reynolds number as do rotational solutions of the Navier–Stokes equations generally. The solutions of the Navier–Stokes equations, rotational and irrotational, are thought to become independent of the Reynolds number at large Reynolds numbers. They can be said to converge to a common set of solutions corresponding to irrotational motion of an inviscid fluid. This limit should be thought to correspond a condition of flow, large Reynolds numbers, and not to a weird material without viscosity; the viscosity should not be put to zero.

Stokes thought that the motion of perfect fluids is an ideal abstraction from the motion of real fluids with small viscosity, like water. He did not mention irrotational flows of very viscous fluids which are associated with normal stresses

$$\tau_n = 2\mu \mathbf{n} \cdot (\nabla \otimes \nabla \phi) \cdot \mathbf{n}$$

in situations in which the dynamical effects of shear stresses in the direction $\mathbf{t}$,

$$\tau_s = 2\mu \mathbf{t} \cdot (\nabla \otimes \nabla \phi) \cdot \mathbf{n}$$

are negligible. The irrotational purely radial motion of a gas bubble in a liquid (the Rayleigh–Poritsky bubble (Poritsky, 1951), usually incorrectly attributed to Rayleigh–Plesset (Plesset, 1949)) is a potential flow. The shear stresses are zero everywhere but the irrotational normal stresses scale with the viscosity for any viscosity, large or small.

Another exact irrotational solution of the Navier–Stokes equations is the flow between rotating cylinders in which the angular velocities of the cylinders are adjusted to fit the potential solution in circles with

$$\mathbf{u} = \mathbf{e}_\theta u,$$
$$u = a^2\omega_a/r = b^2\omega_b/r.$$

The torques necessary to drive the cylinders are proportional to the viscosity of the liquid for any viscosity, large or small. This motion may be realized approximately in a cylinder of large height with a free surface on top anchored in a bath of mercury below.

A less special example is embedded in almost every complex flow of a viscous fluid at each and every stagnation point. The flow at a point of stagnation is a purely extensional flow, a potential flow with extensional stresses proportional to the product of viscosity times the rate extension there. The irrotational viscous extensional stresses at points of extension can be huge even when the viscosity is small.

A somewhat more complex set of flows of viscous fluids which are very nearly irrotational are generated by waves on free surfaces. The shear stresses on the free surfaces vanish but the normal stresses generated by the up and down motion of the waves do not vanish; gravity waves on highly viscous fluids are greatly retarded by viscosity. It is not immediately obvious that the effects of vorticity on such waves are so well approximated by purely irrotational motions (see Lamb, 1932; Wang and Joseph, 2006c). Many theories of irrotational flows of a viscous fluid which update and greatly improve conventional studies of perfect fluids are assembled and can be downloaded from PDF files at http://www.aem.umn.edu/people/faculty/joseph/ViscousPotentialFlow/.

4. Irrotational solutions of the compressible Navier–Stokes equations and the equations of motion for certain viscoelastic fluids

The velocity may be obtained from a potential provided that the vorticity $\boldsymbol{\omega} = \text{curl}\,\mathbf{u} = 0$ at all points in a simply connected region. This is a kinematic condition which may or may not be compatible with the equations of motion. For example, if the viscosity varies with position or the body forces are not potential, then extra terms, not containing the vorticity will appear in the vorticity equation and $\boldsymbol{\omega} = 0$ will not be a solution in general. Joseph and Liao (1994) formulated a compatibility condition for irrotational solutions $\mathbf{u} = \nabla\phi$ of (1.1) in the form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \text{grad}\,\chi = \frac{1}{\rho}\,\text{div}\,\mathbf{T}[\mathbf{u}]. \tag{4.1}$$

If

$$\frac{1}{\rho}\text{div}\,\mathbf{T}[\nabla\phi] = -\nabla\psi, \tag{4.2}$$

then $\boldsymbol{\omega} = 0$ is a solution of (4.1) and

$$\rho\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+\chi\right)+\psi=f(t) \tag{4.3}$$

is the Bernoulli equation.

Consider first (Joseph, 2003a) the case of viscous compressible flow for which the stress is given by (1.2). The gradient of density and viscosity which may generate vorticity do not enter the equations which perturb uniform states of pressure p_0, density ρ_0 and velocity U_0.

To study acoustic propagation, the equations are linearized; putting $\chi=0$ and

$$[\mathbf{u},p,p]=[\mathbf{u}',p_0+p',\rho_0+\rho'], \tag{4.4}$$

where $\mathbf{u}'$, p' and ρ' are small quantities, we obtain

$$T_{ij}=-\left(p_0+p'+\frac{2}{3}\mu_0\operatorname{div}\mathbf{u}'\right)\delta_{ij}+\mu_0\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right), \tag{4.5}$$

$$\rho_0\frac{\partial\mathbf{u}'}{\partial t}=-\nabla p'+\mu_0\left(\nabla^2\mathbf{u}'+\frac{1}{3}\nabla\operatorname{div}\mathbf{u}'\right), \tag{4.6}$$

$$\frac{\partial\rho'}{\partial t}+\rho_0\operatorname{div}\mathbf{u}'=0, \tag{4.7}$$

where p_0, ρ_0 and μ_0 are constants. For acoustic problems, we assume that a small change in ρ induces small changes in p by fast adiabatic processes; hence

$$p'=C_0^2\rho', \tag{4.8}$$

where C_0 is the speed of sound.

Forming now the curl of (4.6) we find that $\operatorname{curl}\mathbf{u}'=0$ is a solution and $\mathbf{u}'=\nabla\phi$. This gives rise to a viscosity dependent Bernoulli equation

$$\frac{\partial\phi}{\partial t}+\frac{p'}{\rho_0}-\frac{4}{3}v_0\nabla^2\phi=0. \tag{4.9}$$

After eliminating p' in (4.5), using (4.9), we get

$$T_{ij}=-\left(p_0-\rho_0\frac{\partial\phi}{\partial t}+2\mu_0\nabla^2\phi\right)\delta_{ij}+2\mu_0\frac{\partial^2\phi}{\partial x_i\,\partial x_j}.$$

To obtain the equation satisfied by the potential ϕ, we eliminate ρ' in (4.7) with p' using (4.8), then eliminate $\mathbf{u}'=\nabla\phi$ and p' in terms of ϕ using $\rho_0\frac{\partial\phi}{\partial t}+p'-\frac{4}{3}\mu_0\nabla^2\phi=0$ to find

$$\frac{\partial^2\phi}{\partial t^2}=\left(C_0^2+\frac{4}{3}v_0\frac{\partial}{\partial t}\right)\nabla^2\phi, \tag{4.10}$$

where the potential ϕ depends on the speed of sound and the kinematic viscosity $v_0=\mu_0/\rho_0$.

The theory of irrotational motion of a viscous compressible fluid was applied by Funada et al. (2006a) to the study of the stability of a liquid jet in a high Mach number air stream.

Joseph and Liao (1994) showed that most models of a viscoelastic fluid do not satisfy the compatibility condition (4.2) in general but it may be satisfied for particular irrotational flows like stagnation point flow of any fluid. The equations of motion satisfy the compatibility equation (4.2) in the case of inviscid, viscous and linear viscoelastic fluids for which $\psi=0$ is the usual Bernoulli pressure and the second order fluid model (Joseph, 1992, extending results of Pipkin, 1970) for which

$$\psi=p-\hat{\beta}(\nabla\otimes\nabla\phi)^2,$$

where $\beta=n_2-n_1/2$ and n_1 and n_2 are the coefficients of the first and second normal stress difference.

5. Irrotational solutions of the Navier–Stokes equations: viscous contributions to the pressure

A viscous contribution to the pressure in irrotational flow is a new idea which is required to resolve the discrepancy between the direct VPF calculation of the decay of an irrotational wave and the calculation based on the dissipation method. The calculation by VPF differs from the calculation based on potential flow of an inviscid fluid because the viscous component of the normal stress at the free surface is included in the normal stress balance. The viscous component of the normal stress is evaluated on potential flow. The dissipation calculation starts from the evolution of energy equation in which the dissipation integral is evaluated on the irrotational flow; the pressure does not enter into this evaluation. Why does the decay rate computed by these two methods give rise to different values? The answer to this question is associated with a viscous correction of the irrotational pressure which is induced by the uncompensated irrotational shear stress at the free surface; the shear stress should be zero there but the irrotational shear stress, proportional to viscosity, is not zero. The irrotational shear stress cannot be made to vanish in potential flow but the explicit appearance of this shear stress in the traction integral in the energy balance can be eliminated in the mean by the selection of an irrotational pressure which depends on viscosity.

The idea of a viscous contribution to the pressure seems to have been first suggested to Moore (1963) by Batchelor as a method of reconciling the discrepancy in the values of the drag on a spherical gas bubble calculated on irrotational flow by the dissipation method and directly by VPF (Section 11). The first successful calculation of this extra pressure was carried out for the spherical bubble by Kang and Leal (1988a,b). Their work suggested that this extra viscous pressure could be calculated from irrotational flow without reference to boundary layers or vorticity.

Lamb (1932, Sections 348 and 349) performed an analysis of the effect of viscosity on free gravity waves. He computed the decay rate by a dissipation method using the irrotational flow only. He also constructed an exact solution for this problem, which satisfies both the normal and shear stress conditions at the interface.

Joseph and Wang (2004a) studied Lamb's problem using the theory of viscous potential flow (VPF) and obtained a dispersion relation which gives rise to both the decay rate and wave-velocity. They also computed a viscous correction for the irrotational pressure and used this pressure correction in the normal stress balance to obtain another dispersion relation. This method is called a viscous correction of the viscous potential flow (VCVPF). Since VCVPF is an irrotational theory the shear stress cannot be made to vanish. However, the corrected pressure eliminates this uncompensated shear stress from the power of traction integral arising in an energy analysis of the irrotational flow.

Wang and Joseph (2006c) find that the viscous pressure correction of the irrotational motion gives rise to a higher order irrotational correction to the irrotational velocity which is proportional to the viscosity and does not have a boundary layer structure.

The effect of viscosity on the decay of a free gravity wave can be approximated by a purely irrotational theory in which the explicit dependence of the power of traction of the irrotational shear stress is eliminated by a viscous contribution p_v to irrotational pressure. The kinetic energy, potential energy and dissipation of the flow can be computed using the potential flow solution $\mathbf{u} = \nabla\phi$ where $\phi = A\mathrm{e}^{nt+ky+\mathrm{i}kx}$. The potential flow solution is inserted into the mechanical energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_V \rho\,|\mathbf{u}|^2/2\,\mathrm{d}V + \int_0^\lambda \rho g\eta^2/2\,\mathrm{d}x\right) = \int_0^\lambda [v(-p+\tau_{yy}) + u\tau_{xy}]\,\mathrm{d}x + \int_V 2\mu\mathbf{D}:\mathbf{D}\,\mathrm{d}V, \tag{5.1}$$

where η is the elevation of the surface, $p = p_I + p_v$ and p_v is the pressure correction, satisfying $\nabla^2 p_v = 0$ and the correction formula

$$\int_0^\lambda v(-p_v)\,\mathrm{d}x = \int_0^\lambda u\tau_{xy}\,\mathrm{d}x. \tag{5.2}$$

The pressure correction

$$p_v = -2\mu k^2 A\mathrm{e}^{nt+ky+\mathrm{i}kx}$$

can be balanced by a purely irrotational velocity.

The corrected velocity depends strongly on viscosity and is not related to vorticity; the whole package is purely irrotational. The corrected irrotational flow gives rise to a dispersion relation which is in splendid agreement with Lamb's exact solution, which has no explicit viscous pressure. The agreement with the exact solution holds for fluids even 10^7 times more viscous than water and for small and large wave numbers where the cutoff wave number k_c marks the place where progressive waves give rise to monotonic decay. They find that VCVPF gives rise to the same decay rate as in Lamb's exact solution and in his dissipation calculation when $k < k_c$. The exact solution agrees with VPF when $k > k_c$. The effects of vorticity are sensible only in a small interval centered on the cutoff wave number.

6. Irrotational solutions of the Navier–Stokes equations: classical theorems

An authorative and readable exposition of irrotational flow theory and its applications can be found in Chapter 6 of the book on fluid dynamics by Batchelor (1967). He speaks of the role of the theory of flow of an inviscid fluid. He says

> In this and the following chapter, various aspects of the flow of a fluid regarded as entirely inviscid (and incompressible) will be considered. The results presented are significant only inasmuch as they represent an approximation to the flow of a real fluid at large Reynolds number, and the limitations of each result must be regarded as information as the result itself.

Most of the classical theorems reviewed in Chapter 6 do not require that the fluid be inviscid. These theorems are as true for viscous potential flow as they are for inviscid potential flow. Kelvin's minimum energy theorem holds for the irrotational flow of a viscous fluid. The theory of the acceleration reaction leads to the concept of added mass; it follows from the analysis of unsteady irrotational flow. The theory applies to viscous and inviscid fluids alike.

Jeffreys (1928) derived an equation (his (20)) which replaces the circulation theorem of classical (inviscid) hydrodynamics. When the fluid is homogeneous, Jeffrey's equation may be written as

$$\frac{\mathrm{d}C}{\mathrm{d}t} = -\frac{\mu}{\rho}\oint \operatorname{curl}\boldsymbol{\omega}\cdot \mathrm{d}\mathbf{l}, \tag{6.1}$$

where

$$C(t) = \oint \mathbf{u}\cdot \mathrm{d}\mathbf{l}$$

is the circulation round a closed material curve drawn in the fluid. This equation shows that

> ... the initial value of $\mathrm{d}C/\mathrm{d}t$ around a contour in a fluid originally moving irrotationally is zero, whether or not there is a moving solid within the contour. This at once provides an explanation of the equality of the circulation about an aeroplane and that about the vortex left behind when it starts; for the circulation about a large contour that has never been cut by the moving solid or its wake remains zero, and therefore the circulations about contours obtained by subdividing it must also add up to zero. It also indicates why the motion is in general nearly irrotational except close to a solid or to fluid that has passed near one.

Saint-Venant (1869) interpreted the result of Lagrange about the invariance of circulation $\mathrm{d}C/\mathrm{d}t = 0$ to mean that

> vorticity cannot be generated in the interior of a viscous incompressible fluid, subject to conservative extraneous force, but is necessarily diffused inward from the boundaries.

The circulation formula (6.1) is an important result in the theory of irrotational flows of a viscous fluid. A particle which is initially irrotational will remain irrotational in motions which do not enter into the vortical layers at the boundary.

7. Critical remarks about the "The impossibility of irrotational motions in general"

This topic is treated in Section 37 of the monograph by Truesdell (1954). The basic idea is that, in general, irrotational motions of incompressible fluids satisfy Laplace's equation and the normal and tangential velocities at the bounding surfaces cannot be simultaneously prescribed. The words "in general" allow for rather special cases in which the motion of the bounding surfaces just happens to coincide with the velocities given by the derivatives of the potential. Such special motions were studied for viscous incompressible fluids by Hamel (1941). A bounding surface must always contact the fluid so the normal component of the velocity of the fluid must be exactly the same as the normal component of the velocity of the boundary. The no-slip condition cannot then "in general" be prescribed. Truesdell uses "adherence condition" meaning "sticks fast" rather than the usual no-slip condition of Stokes. The no slip condition is even now a topic of discussion and the mechanisms by which fluids stick fast are not clear. Truesdell does not consider liquid–gas surfaces or, more exactly, liquid–vacuum surfaces on which slip is allowed.

Truesdell's conclusion

> ... that the boundary condition customarily employed in the theory of viscous fluids makes irrotational motion is a virtual impossibility.

is hard to reconcile with the idea that flows outside boundary layers, are asymptotically irrotational. Many examples of irrotational motions of viscous fluids which approximate exact solutions of the Navier–Stokes equations and even agree with experiments at low Reynolds numbers are listed on Joseph's web based archive.

8. The drag on a spherical gas bubble

As in the case of irrotational waves, the problem of the drag on gas bubbles in a viscous liquid may be studied using viscous potential flow directly and by the dissipation method and the two calculations do not agree.

The idea that viscous forces in regions of potential flow may actually dominate the dissipation of energy was first expressed by Stokes (1851), and then, with more details, by Lamb (1932) who studied the viscous decay of free oscillatory waves on deep water (Section 348) and small oscillations of a mass of liquid about the spherical form (Section 355) using the dissipation method. Lamb showed that in these cases the rate of dissipation can be calculated with sufficient accuracy by regarding the motion as irrotational.

8.1. Dissipation calculation

The computation of the drag D on a sphere in potential flow using the dissipation method seems to have been given first by Bateman (1932) (see Dryden et al., 1956) and repeated by Ackeret (1952). They found that $D = 12\pi a\mu U$ where μ is the viscosity, a is the radius of the sphere and U its velocity. This drag is twice the Stokes drag and is in better agreement with the measured drag for Reynolds numbers in excess of about 8.

The same calculation for a rising spherical gas bubble was given by Levich (1949). Measured values of the drag on spherical gas bubbles are close to $12\pi a\mu U$ for Reynolds numbers larger than about 20. The reasons for the success of the dissipation method in predicting the drag on gas bubbles have to do with the fact that vorticity is confined to thin layers and the contribution of this vorticity to the drag is smaller in the case of gas bubbles, where the shear traction rather than the relative velocity must vanish on the surface of the sphere. A good explanation was given by Levich (1949) and by Moore (1959, 1963); a convenient reference is Batchelor (1967). Brabston and Keller (1975) did a direct numerical simulation of the drag on a gas spherical bubble in steady ascent at terminal velocity U in a Newtonian fluid and found the same kind of agreement with experiments. In fact, the agreement between experiments and potential flow calculations using the dissipation method are fairly good for Reynolds numbers as small as 5 and improves (rather than deteriorates) as the Reynolds number increases.

The idea that viscosity may act strongly in the regions in which vorticity is effectively zero appears to contradict explanations of boundary layers which have appeared repeatedly since Prandtl. For example, Glauert (1943) says (p. 142) that

> ...Prandtl's conception of the problem is that the effect of the viscosity is important only in a narrow boundary layer surrounding the surface of the body and that the viscosity may be ignored in the free fluid outside this layer.

According to Harper (1972), this view of boundary layers is correct for solid spheres but not for spherical bubbles. He says that

> ...for $R \gg 1$, the theories of motion past solid spheres and tangentially stress-free bubbles are quite different. It is easy to see why this must be so. In either case vorticity must be generated at the surface because irrotational flow does not satisfy all the boundary conditions. The vorticity remains within a boundary layer of thickness $\delta = O(aR^{-1/2})$, for it is convected around the surface in a time t of order a/U, during which viscosity can diffuse it away to a distance δ if $\delta^2 = O(\nu t) = O(a^2/R)$. But for a solid sphere the fluid velocity must change by $O(U)$ across the layer, because it vanishes on the sphere, whereas for a gas bubble the normal derivative of velocity must change by $O(U/a)$ in order that the shear stress be zero. That implies that the velocity itself changes by $O(U\delta/a) = O(UR^{-1/2}) = o(U)$...
> In the boundary layer on the bubble, therefore, the fluid velocity is only slightly perturbed from that of the irrotational flow, and velocity derivatives are of the same order as in the irrotational flow. Then the viscous dissipation integral has the same value as in the irrotational flow, to the first order, because the total volume of the boundary layer, of order $a^2\delta$, is much less than the volume, of order a^3, of the region in which the velocity derivatives are of order U/a. The volume of the wake is not small, but the velocity derivatives in it are, and it contributes to the dissipation only in higher order terms...

The drag on a spherical gas bubble in steady flow at modestly high Reynolds numbers (say, $Re > 50$) can be calculated using the dissipation method assuming irrotational flow without any reference to boundary layers or vorticity. The dissipation calculation gives $D = 12\pi a\mu U$ or $C_D = 48/R$ where $R = 2aU\rho/\mu$.

8.2. Direct calculation of the drag using viscous potential flow (VPF)

Moore (1959) calculated the drag directly by integrating the pressure and viscous normal stress of the potential flow. The irrotational shear stress is not zero but is not used in the drag calculation. The shear stress which is zero in the real flow was put to zero in the direct calculation. The pressure is computed from Bernoulli's equation and it has no drag resultant. Moore's direct calculation gave $D = 8\pi a\mu U$ or $C_D = 32/R$ instead of $C_D = 48/R$.

8.3. Pressure correction (VCVPF)

The discrepancy between the dissipation calculation leading to $C_D = 48/R$ and the direct VPF calculation leading to $C_D = 32/R$ led Batchelor, as reported in Moore (1963), to suggest the idea of a pressure correction to the irrotational pressure. In that paper, Moore performed a boundary layer analysis and his pressure correction is readily obtained by setting $y = 0$ in his Eq. (2.37):

$$p_v = \frac{4}{R\sin^2\theta}(1-\cos\theta)^2(2+\cos\theta), \tag{8.1}$$

which is singular at the separation point where $\theta = \pi$. The presence of separation is a problem for the application of boundary layers to the calculation of drag on solid bodies. To find the drag coefficient Moore calculated the momentum defect, and obtained the Levich value $48/R$ plus contributions of order $R^{-3/2}$ or lower.

The first successful calculation of a viscous pressure correction was carried out by Kang and Leal (1988a). They calculated a viscous correction of the irrotational pressure by solving the Navier–Stokes equations under the condition that the shear stress on the bubble surface is zero. Their calculation could not be carried out to very high Reynolds numbers, and it was not verified that the dissipation in the liquid is close to the value given by potential flow. They find indications of a boundary layer structure but they do not establish the existence of properties of a layer in which the vorticity is important. They obtain the drag coefficient $48/R$ by direct integration of the normal stress and viscous pressure over the boundary. This shows that the force resultant of the pressure correction does indeed contribute exactly the $16/R$ which is needed to reconcile the difference between the dissipation calculation and the direct calculation of drag.

Kang and Leal (1988a) obtain their drag result by expanding the pressure correction as a spherical harmonic series and noting that only one term of this series contributes to the drag, no appeal to the boundary layer approximation being necessary. Kang and Leal (1988b) remark that

> In the present analysis, we therefore use an alternative method which is equivalent to Lamb's dissipation method, in which we ignore the boundary layer and use the potential flow solution right up to the boundary, with the effect of viscosity included by adding a viscous pressure correction and the viscous stress term to the normal stress balance, using the inviscid flow solution to estimate their values.

The VCVPF approach to problems of gas–liquid flows taken by Joseph and Wang (2004a,b) and Wang and Joseph (2006c), in which the viscous contribution to the pressure is selected to remove the uncompensated irrotational shear stress τ_s from the traction integral as was done in (5.2), is different than that used by Kang and Leal (1988a,b).

For the case of a gas bubble rising with the velocity U in a viscous fluid, it is possible to prove that the drag D_1 computed indirectly by the dissipation method is equal to the drag D_2 computed directly by our formulation of VCVPF. Suppose that the drag on the bubble is given as $D_1 = \mathscr{D}/U$, where $\mathscr{D}$ is the dissipation. Then

$$D_1 = \mathscr{D}/U = \int_V 2\mu\mathbf{D} : \mathbf{D}\,\mathrm{d}V/U = \int_A n \cdot 2\mu\mathbf{D} \cdot u\,\mathrm{d}A/U = \int_A (\tau_n u_n + \tau_s u_s)\,\mathrm{d}A/U = \int_A (-p^v + \tau_n)u_n\,\mathrm{d}A/U$$
$$= \int_A e_x \cdot e_n(-p^v - p^i + \tau_n)\,\mathrm{d}A = \int_A e_x \cdot \mathbf{T} \cdot e_n\,\mathrm{d}A = D_2,$$

where we have used the normal velocity continuity $u_n = Ue_x \cdot e_n$, the zero-shear-stress condition at the gas–liquid interface and the fact that the Bernoulli pressure does not contribute to the drag.

Dissipation calculations for the drag on a rising oblate ellipsoidal bubble was given by Moore (1965) and for the rise of a spherical liquid drop, approximated by Hill's spherical vortex in another liquid in irrotational motion, by Harper and Moore (1968). The drag results from these dissipation calculations were obtained by Joseph and Wang (2004a), using the irrotational viscous pressure.

8.4. Acceleration of a spherical gas bubble to steady flow

A spherical gas bubble accelerates to steady motion in an irrotational flow of a viscous liquid induced by a balance of the acceleration of the added mass of the liquid with the Levich drag. The equation of rectilinear motion is linear and may be integrated giving rise to exponential decay with decay constant $18\nu t/a^2$ where ν is the kinematic viscosity of the liquid and a is the bubble radius. The problem of decay to rest of a bubble moving initially when the forces maintaining motion are inactivated and the acceleration of a bubble initially at rest to terminal velocity are considered (Joseph and Wang, 2004b). The equation of motion follows from the assumption that the motion of the viscous liquid is irrotational. It is an elementary example of how potential flows can be used to study the unsteady motions of a viscous liquid suitable for the instruction of undergraduate students.

Consider a body moving with the velocity U in an unbounded viscous potential flow. Let M be the mass of the body and M' be the added mass. Then the total kinetic energy of the fluid and body is

$$T = \frac{1}{2}(M + M')U^2. \tag{8.2}$$

Let D be the drag and F be the external force in the direction of motion. Then the power of D and F should be equal to the rate of the total kinetic energy,

$$(F + D)U = \frac{\mathrm{d}T}{\mathrm{d}t} = (M + M')U\frac{\mathrm{d}U}{\mathrm{d}t}. \tag{8.3}$$

We next consider a spherical gas bubble, for which $M = 0$ and $M' = \frac{2}{3}\pi a^3\rho_f$. The drag can be obtained by direct integration using the irrotational viscous normal stress and a viscous pressure correction: $D = -12\pi\mu aU$. Suppose the external force just balances the drag, then the bubble moves with a constant velocity $U = U_0$. Imagine that the external force suddenly disappears, then (8.3) gives rise to

$$-12\pi\mu aU = \frac{2}{3}\pi a^3\rho_f\frac{\mathrm{d}U}{\mathrm{d}t}. \tag{8.4}$$

The solution is

$$U = U_0 e^{-\frac{18\nu}{a^2}t}, \tag{8.5}$$

which shows that the velocity of the bubble approaches zero exponentially.

If gravity is considered, then $F = \frac{4}{3}\pi a^3 \rho_f g$. Suppose the bubble is at rest at $t = 0$ and starts to move due to the buoyant force. Eq. (8.3) can be written as

$$\frac{4}{3}\pi a^3 \rho_f g - 12\pi\mu a U = \frac{2}{3}\pi a^3 \rho_f \frac{dU}{dt}. \tag{8.6}$$

The solution is

$$U = \frac{a^2 g}{9\nu}\left(1 - e^{-\frac{18\nu}{a^2}t}\right), \tag{8.7}$$

which indicates the bubble velocity approaches the steady state velocity

$$U = \frac{a^2 g}{9\nu}. \tag{8.8}$$

8.5. The rise velocity and deformation of a gas bubble computed using VPF

The shape of a rising bubble, or of a falling drop, in an incompressible viscous liquid was computed numerically by Miksis et al. (1982), omitting the condition on the tangential traction at the bubble or drop surface. The shape is found, together with the flow of the surrounding fluid, by assuming that both are steady and axially symmetric, with the Reynolds number being large. The flow is taken to be a potential flow and the viscous normal stress, evaluated on the irrotational flow, is included in the normal stress balance. This study is exactly what we have called VPF; it follows the earlier study of Moore (1959), but it differs markedly from Moore's study because the bubble shape is computed.

When the bubble is sufficiently distorted, its top is found to be spherical and its bottom is found to be rather flat. Then the radius of its upper surface is in fair agreement with the formula of Davies and Taylor (1950). This distortion occurs when the effect of gravity is large while that of surface tension is small. When the effect of surface tension is large, the bubble is nearly a sphere. The difference in these two cases is associated with large and small Morton numbers.

8.6. The rise velocity of a spherical cap bubble computed using VPF

Davies and Taylor (1950) studied the rise velocity of a lenticular or spherical cap bubble assuming that motion was irrotational and the liquid inviscid. The spherical cap is round at the top and rather flat at the bottom. These are the shapes of large volume bubbles of gas rising in the liquid. They measured the bubble shape and showed that it indeed had a spherical cap when rising in water. Brown (1965) did experiments which shows the cap is very nearly spherical even when the liquid in which the gas bubble rises is very viscous.

Joseph (2003b) and Funada et al. (2004a) applied the theory of viscous potential flow VPF to the problem of finding the rise velocity U of a spherical cap bubble. The rise velocity is given by

$$\frac{U}{\sqrt{gD}} = -\frac{8}{3}\frac{\nu}{\sqrt{gD^3}} + \frac{\sqrt{2}}{3}\left[1 + \frac{32\nu^2}{gD^3}\right]^{1/2}, \tag{8.9}$$

where $R = D/2$ is the radius of the cap and ν is the kinematic viscosity of the liquid. Davies and Taylor's (1950) result follows from (8.9) when the viscosity is zero. Eq. (8.9) may be expressed as a drag law

$$C_D = 6 + 32/R_e. \tag{8.10}$$

This drag law is in excellent agreement with experiments at large Morton numbers reported by Bhaga and Weber (1981) after the drag law is scaled so that the effective diameter used in the experiments and the spherical cap radius of Davies and Taylor (1950) are the same (see Fig. 1).

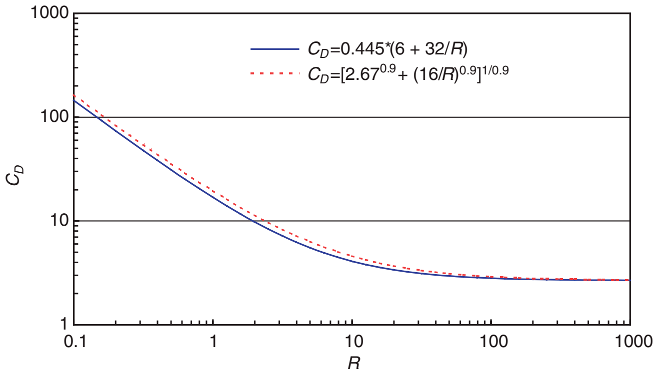


Fig. 1. Comparison of the empirical drag law with the theoretical drag law (8.10) scaled by the factor 0.445 required to match the experimental data reported by Bhaga and Weber (1981) with the experiments of Davies and Taylor (1950) at large *Re*. The agreement between theory and experiment based on the irrotational flow of a viscous fluid is excellent even for Reynolds numbers as small as 0.1.

Viscous potential flow VPF works well in the case of spherical cap bubbles, but a viscous pressure correction VCVPF is required to bring theory and experiment together in the case of small spherical gas bubbles. This is like the problem of the effect of viscosity on the decay of irrotational gravity waves studied by Lamb (1932) and Wang and Joseph (2006c) in which the decay of long waves is correctly predicted by VCVPF and the decay of short waves is correctly predicted by VPF.

9. Dissipation and drag in irrotational motions over solid bodies

Contrived examples of irrotational motions of viscous fluids over solid bodies are those in which it is imagined that the boundary of the solid moves with exactly same velocity as the potential flow. These irrotational motions are exactly the same as those in which the viscous fluid is allowed to slip at the boundary of the solid. Some authors claim that the drag on a solid body exerted by a viscous liquid in potential flow can be obtained by equating the product DU of drag time's velocity to the total dissipation in the liquid evaluated on the potential; other authors say the drag is zero. The relation of drag to dissipation in irrotational flow is subtle and its utility as theoretical tool depends critically on understanding what to do with unphysical irrotational shear stress at the boundary of the body.

9.1. Energy equation

Consider the motion of a body in a fluid at rest at infinity. If there are no body forces and the fluid is homogeneous, the evolution of the mechanical energy is given by

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho|\mathbf{u}|^2\,\mathrm{d}V = \int_S [-p\mathbf{u}\cdot\mathbf{n} + 2\mu\mathbf{u}\cdot\mathbf{D}\cdot\mathbf{n}]\,\mathrm{d}S - \mathscr{D}, \tag{9.1}$$

where $\mathbf{D} = \mathbf{D}[\mathbf{u}]$,

$$\mathscr{D} = 2\mu\int_V \mathbf{D} : \mathbf{D}\,\mathrm{d}V \tag{9.2}$$

is the dissipation, V is the region exterior to the body with surface S and normal $\mathbf{n}$ drawn into the body. The stress vector on S is given by

$$\boldsymbol{\tau} = 2\mu\mathbf{D}\cdot\mathbf{n} = \tau_n\mathbf{n} + \tau_s\mathbf{t}, \tag{9.3}$$

where $\tau_n = 2\mu\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n}$ is the component of the component of the stress vector normal to S and $\tau_s = 2\mu\mathbf{t}\cdot\mathbf{D}\cdot\mathbf{n}$ is the component of stress vector tangent to S. We may write (9.1) as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_S [(-p + \tau_n)\mathbf{n} \cdot \mathbf{u} + \tau_s \mathbf{t} \cdot \mathbf{u}]\mathrm{d}S - \mathscr{D}, \tag{9.4}$$

where $E = \frac{1}{2}\int_V \rho|\mathbf{u}|^2 \mathrm{d}V$.

If the body a solid moving with a velocity $\mathbf{u} = U\mathbf{e}_x$ and the no-slip condition applies; it can be shown using $\operatorname{div}\mathbf{u} = 0$, that $\tau_n = 0$ and

$$\frac{\mathrm{d}E}{\mathrm{d}t} = DU - \mathscr{D}, \tag{9.5}$$

where

$$D = \int_S [-pn_x + \tau_s t_x]\,\mathrm{d}S \tag{9.6}$$

is the drag. For steady flow

$$DU = \mathscr{D} \tag{9.7}$$

the rate of working of the drag force is balanced by dissipation.

9.2. d'Alembert paradox

If $\mu = 0$, then $\mathscr{D} = 0$ and $D = 0$. The drag of a body in an inviscid flow vanishes. However, for irrotational flow of a viscous fluid

$$\mathscr{D} = 2\mu \int_V \frac{\partial^2 \phi}{\partial x_i \partial x_j}\frac{\partial^2 \phi}{\partial x_i \partial x_j}\,\mathrm{d}V = \mu \int_S \mathbf{n} \cdot \nabla|\nabla\phi|^2\,\mathrm{d}S \tag{9.8}$$

is not zero. However, if the fluid is viscous and the flow is irrotational, then, using $\mathbf{u} = \nabla\phi$ and (9.8) we find that

$$\int_S u_i D_{ij} n_j\,\mathrm{d}S - \int_V D_{ij} D_{ij}\,\mathrm{d}V = 0. \tag{9.9}$$

Hence, for steady flow, using (9.1) and $\mathbf{u} = U\mathbf{e}_x + \mathbf{v}$, where $\mathbf{v} \cdot \mathbf{n} = 0$, we get

$$U \int_S pn_x\,\mathrm{d}S = UD = 0.$$

It follows that in the irrotational flow of a viscous fluid over a body

$$D = 0, \tag{9.10}$$

even though $\mathscr{D} \neq 0$. Now we have two paradoxes instead of one.

9.3. Different interpretations of the boundary conditions for irrotational flows over solids

Finzi (1925) calculated the dissipation due to the irrotational motion of a viscous fluid when the fluid is allowed to slip at the boundary of a solid of unspecified shape. His goal was to compute the drag associated with the dissipation of the irrotational flow. He computes the dissipation as the sum of the usual volume integral of the square of the rate of strain over the fluid times the viscosity plus the power of the irrotational viscous stresses on the boundary. He shows that the dissipation so defined vanishes when the velocity is harmonic as is the case for irrotational flow. Thus in Finzi's theory there is no dissipation of energy in irrotational flow (cf. Bateman in Dryden et al., 1956, p. 158).

Bateman (1932) computed $D = 12\pi\mu a U$ from (9.7) for a solid sphere of radius a moving forward with velocity U in a potential flow motionless at infinity. Exactly the same value for drag on a spherical gas bubble was computed in exactly the same way by Levich (1949). Levich justified his result by arguing that the drag due to the weak vorticity boundary layer at the gas–liquid interface is much smaller than the drag due to the dissipation of the irrotational flow. There are two ways to interpret Bateman's result: (1) the fluid slips at the solid and the dissipation is balanced by the power of the traction leading to zero drag (9.10) or (2) the fluid does not slip, there is a region at the boundary which may be small or large in which vorticity is important and $12\pi\mu aU$ approximates the additional drag due to the viscous dissipation in the irrotational flow outside the boundary layer.

Some German researchers (Hamel, 1941; Ackeret, 1952; Romberg, 1967; Zierep, 1984) considered the problem of dissipation and drag on solid bodies in contrived irrotational motions of a viscous fluid. Hamel (1941) noted that though the resultants of the viscous irrotational stresses vanish, the work done by these stresses do not vanish. This observation motivated his discussion of dissipation and drag. The papers of Hamel and Ackeret are very similar; they both use the formula (9.7) relating dissipation and drag for solid bodies under the assumption that the boundary of the solid moves with the velocity of the irrotational flow. Ackeret gives drag results for circular cylinders in a uniform stream with circulation, for elliptic cylinders, for spheres and other bodies. Zierep (1984) in a paper on viscous potential flow discusses the dissipation and drag associated with the moving wall calculations of Hamel and Ackeret; he calls this a pseudo drag which "originates from friction." He says that "... A real drag does not appear in potential flows including those with friction." Zierep's claim that the drag on a solid body in irrotational flow vanishes is a consequence of d'Alembert's principle for flows with a non-zero dissipation embodied in (9.10). The assumption that the boundary of the solid moves with the velocity of the irrotational flow could be interpreted to mean that the edge of the boundary layer on the body moves with the velocity of the irrotational flow because the velocity is continuous there. The difference between the dissipation in the irrotational flow outside the boundary layer on a rigid body and the dissipation outside the same body whose velocity is contrived to move at the velocity of the irrotational flow could be close if the boundary layer is not too thick and the dissipation in regions of strong vorticity arising from boundary layer separation is ignored. The dissipation in the irrotational flow outside a moving body is exactly the same when the fluid slips and when the motion of the boundary of the body is imagined to move with the slip velocity.

Zierep's discussion of viscous potential flow is confused: he says that viscous potential flow is independent of the Reynolds number; he did not discuss the irrotational viscous stresses which scale with the Reynolds number. He notes that the drag on a sphere with a moving wall computed from (9.7) is the same as the drag on a spherical gas bubble computed by Levich (1949). He notes that drag on the bubble is in good agreement with experiment and he attributes this difference to the fact that the shear stress on the bubble surface vanishes and the gas liquid is like moving wall. He does not confront the discrepancy posed by the non-zero viscous and irrotational shear stress in the theory and the zero shear stress condition required in practice.

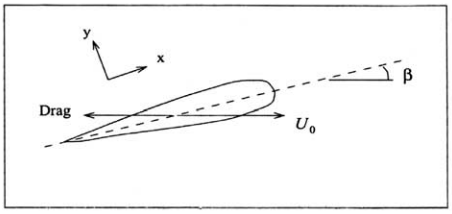
9.4. Viscous dissipation in the irrotational flow outside the boundary layer and wake

The paper on dissipation and drag by Romberg (1967) is important and deserves to be better known. In this paper

> The connection between drag and dissipation in incompressible flows is derived. Two cases are considered: Firstly the surface of the body is at rest and secondly the surface elements can move.In the further case the flow outside the boundary layer and the wake contributes a term to the drag coefficient which is proportional to Reynolds number to the minus one (high Reynolds number assumed). The coefficient of this term is completely fixed by the frictionless flow.

The dissipation in the irrotational flow of a viscous fluid outside the boundary layer had not been considered before Romberg. I learned about this paper from Zierep (1984) that discussed the first two topics but ignored the flow outside the boundary layer. Much later and independently, Wang and Joseph (2006a) prepared a work on pressure corrections for the effects of viscosity on the irrotational flow outside Prandtl's boundary layer; in one chapter they calculated the additional drag on a Joukowski airfoil at an angle of attack by the dissipation method.

The existence of an added drag due to viscosity in the irrotational flow outside the very small vorticity layer at the bubble surface is widely accepted by fluid mechanics researchers. The situation there and in other gas–liquid free surface problems is complicated by the uncompensated irrotational shear stress at the interface which is intimately connected to the viscous action in the irrotational flow. This complication does not appear in the equivalent formulation in which the irrotational effects of viscosity are computed using the dissipation



β \ ε	0.3	0.2	0.1	0.05	0.01
0	4.34	3.48	2.53	2.05	1.67
$\pi/20$	6.24	6.30	9.39	22.7	410
$\pi/10$	11.8	14.5	29.3	82.8	1.59×10^3
$\pi/6$	23.7	32.3	72.6	213	4.17×10^3
$\pi/4$	43.2	61.1	142	424	8.34×10^3

Fig. 2. Added drag due to the viscous dissipation in the irrotational flow outside the boundary layer. This drag is negligible at very high Reynolds numbers.

method. It is hard to envisage a situation in which the viscous effects associated with irrotational gas–liquid flows do not occur in other irrotational flows, like those outside Prandtl's boundary layer. The effects of the shear stress at the surface of a rigid body are well described with traditional boundary layer theory but in flows in which the irrotational shear stresses at the edge of the boundary layer are not the same as the ones obtained from boundary layer analysis there is mismatch which begs for resolution. Indeed we can think that the body plus the boundary moves through the irrotational flow like a bubble in which the resolution of discontinuous velocity derivatives rather than the resolution of a discontinuous velocity is at issue. The simplest kind of analysis to try for the boundary layer problem is the dissipation method where the dissipation would be computed in the region outside the boundary layer. This approach is difficult to implement because there is no definite end to the boundary layer and for other reasons. To avoid these difficulties, the viscous dissipation of the irrotational flow could be computed everywhere outside and inside the boundary layer, right up to the boundary as was done by Romberg (1967) and Wang and Joseph (2006a). If this approach has merit, all the calculations of the drag and dissipation in irrotational motions around solids given by Hamel (1941) and Ackeret (1952) which I called contrived might actually give an approximation to the added drag due to the irrotational flow outside the boundary layer.

Romberg calculated the additional drag due to viscosity in the irrotational flow over an ellipse at a zero angle of attack. He finds that the coefficient of the drag is given by

$$C_d = 4\pi(1+\tau)^2/Re,$$

where τ is the aspect ratio of the ellipse. The drag coefficient tends to $4\pi/Re$ as the ellipse collapses onto its major axis. This is the flow over a flat plate. Romberg notes that the dissipation integral (9.8) remains finite because the integrand is singular at the front at the front stagnation point.

Wang and Joseph (2006a) computed the additional viscous drag on a Joukowski airfoil in the well-known irrotational streaming flow obtained from the conformal transformation $z = \zeta + c^2/\zeta$ from a circle of radius $a = c(1+\varepsilon)$ with circulation $\Gamma = 4\pi U_0 a \sin\beta$ where β is the attack angle and ε is the sharpness parameter; the smaller ε the sharper the nose. They calculated this drag from the dissipation integral (12.8) using the complex potential $f(z) = \varphi + \mathrm{i}\psi$. The drag coefficient $C_d = -2I(\varepsilon, \beta)/Re$ where $Re = 4cU_0/\nu$ is given in the table in Fig. 2.

10. Major effects of viscosity in irrotational motions can be large; they are not perturbations of potential flows of inviscid fluids

At the risk of repeating myself, I feel that it is necessary to write this section forcefully because the contrary opinion is so widespread. The best way to establish this point is to list many examples in which viscous effects computed from purely irrotational theories are both large and in good agreement with exact theoretical results and experiments.

10.1. Exact solutions

By exact solutions I mean irrotational solutions which also satisfy commonly accepted boundary conditions for viscous fluids. Irrotational flows of viscous fluids cannot in general satisfy no-slip conditions at solid–fluid surfaces or continuity conditions on the tangential components of velocity and stress at the interface between liquids. At a gas–liquid surface the tangential component velocity in the liquid is essentially unrestricted as it would be at a vacuum–liquid surface but the continuity of the shear stress leads to the condition that the shear stress must be essentially zero in the liquid as it is in the vacuum or nearly in the gas. I know only of two non-contrived examples of potential flows which satisfy these strict conditions. The first is the purely rotary flow between rotating cylinders adjusted so that the velocity of the fluid in circles is proportional to c/r. The second solution arises in purely radial gas–liquid flows of spherical bubbles or drops (Poritsky, 1951). In these purely radial flows, shear stresses cannot develop but irrotational viscous normal stresses proportional to the viscosity are not zero. The irrotational solutions of these problems work for water and pitch, independent of viscosity.

10.2. Gas–liquid flows: bubbles, drops and waves

Irrotational studies of gas–liquid flows cannot be made to satisfy the zero shear stress condition on the free surface. This mismatch leads to the generation of vorticity. Very often the contribution of vorticity is confined to a boundary layer and the viscous effects in these layers are much smaller than the viscous effects in the purely irrotational flow. The purely irrotational analysis of the rise velocity of a spherical cap bubble described in Fig. 1 is in good agreement with experiment at high Morton numbers even at Reynolds numbers of 0.1. Classical studies of interfacial stability for potential flow of inviscid fluids are as easily done for the potential flow of viscous and even viscoelastic fluids.

10.3. Rayleigh–Taylor instability

Joseph et al. (1999) studied the breakup of drops in a high speed air stream behind a shock wave. At first the drop flattens due to high pressure at the front and back of drop. The accelerations can be 10^5 times gravity; this huge acceleration is main dynamical feature for generating the RT instability which give rise to the corrugations seen on the front face of the drop before it is set into motion by acceleration (see Fig. 3). The analysis of RT instability can be done exactly from the linearized Navier–Stokes equations and from viscous potential flow VPF. The growth rates and the wave length for maximum growth depend strongly on viscosity; the difference between VPF and the exact solution is about 2% at most for small and large viscosities. Similar agreement between the irrotational theory, exact solutions and experiments for RT instability were given by Joseph et al. (2002).

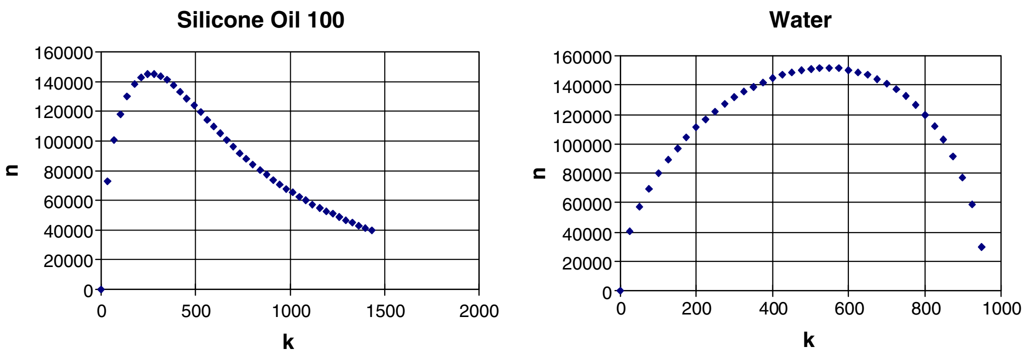


Fig. 3. Growth rate curves for Rayleigh–Taylor instability for water (1 cp) silicon oil (100 cp). The critical cut off wave number is given by $k_c = (\rho a/\gamma)^{1/2}$, where a is acceleration γ is surface tension, is independent of viscosity. The maximum value of the growth rate decreases strongly with viscosity and the maximum value of the growth rate shifts strongly to longer waves. The growth rate curves here computed from viscous potential flow are less than 2% different than the curves from the exact theory. The irrotational theory agrees with exact theory and with experiments even when the viscosity is very large.

10.4. Capillary instability

Wang et al. (2005a) studied capillary instability using VPF and VCVPF; they compared their results to the exact solution of Tomotika (1935). The differences between the exact solution and VCVPF are at most a few percent even for viscosities a million times larger than water. A vorticity layer exists at the interface but for gas–liquid flows, its effect on the linearized dynamics is negligible. It is unexpected that the purely irrotational analysis of capillary instability of two very viscous liquids like Golden syrup and paraffin should give results close to the exact solution for the shorter waves in the region of cutoff and maximum growth rates. The effects of vorticity are not uniform in k, they are more important for long waves (small k) (Fig. 4).

Similar agreements between purely irrotational analysis of capillary instability of viscoelastic fluids and exact solutions have been obtained by Wang et al. (2005c).

10.5. Kelvin–Helmholtz instability

This is an instability which arises from a discontinuity in the velocity of uniform parallel streams. It is usually studied using Euler's equations because the continuity of the tangential components of velocity and stress required in the Navier–Stokes theory will not allow discontinuities (Basset, 1888, Section 518, *Vortex sheets cannot exist in a viscous liquid*, p. 308). However, these discontinuities are compatible with viscous potential flow. Funada and Joseph (2001) studied KH instability in a channel. Funada et al. (2004b) studied KH instability of a liquid jet into incompressible gases and liquid using VPF, of the same problem in Funada et al. (2005) using VCVPF and of a liquid cylinder in a supersonic compressible gas in Funada et al. (2006a). The instabilities in all these studies depend strongly on viscosity but there are no exact solutions to which they may be compared.

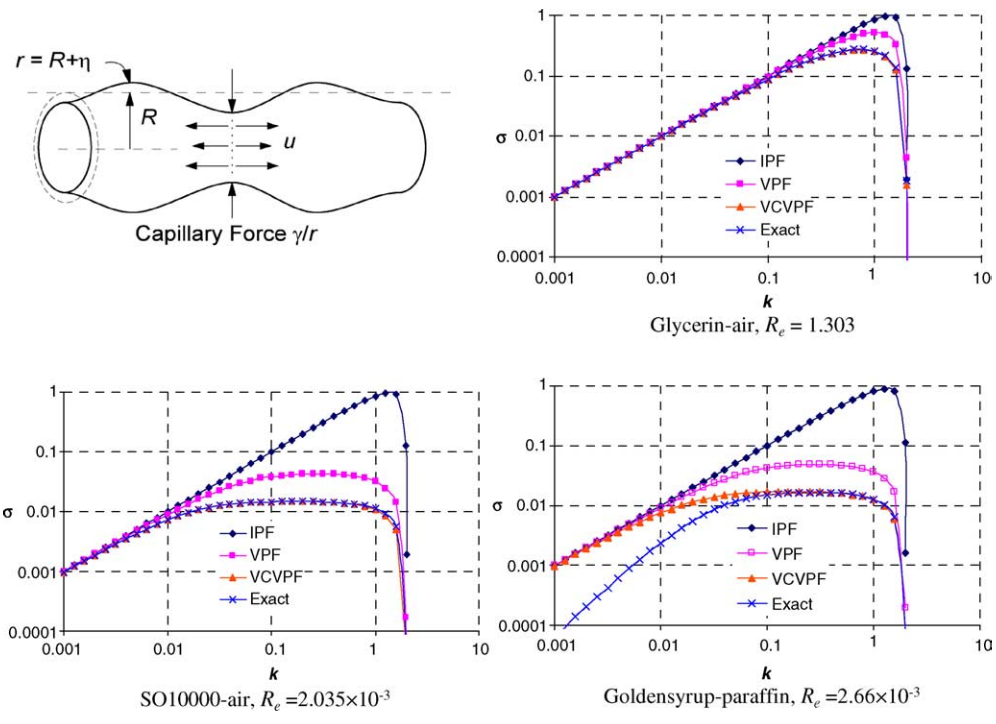


Fig. 4. Growth rate curves for capillary instability of a liquid cylinder. The growth rates given by VCVPF are in good agreement with the exact solution for all k in the case of liquids in air (Wang et al., 2005a). The agreement in the two-liquid case (Wang et al., 2005b) is good for the maximum growth rates but poor for long waves (small k).

10.6. Free waves on highly viscous liquids

Here we return to the discussion in Section 2 of purely irrotational effects of the viscosity on the decay of free gravity waves. In Fig. 5 we compare Lamb's exact solution with the irrotational solutions VPF and VCVPF in the case when the viscosity of the liquid is 10^8 larger than water. The exact solution and the irrotational solutions are in good agreement uniformly in k except near the critical value k_c where progressive waves change to monotonic decay. The exact solutions agree with VCVPF for long waves and with VPF for short waves.

10.7. The effect of viscosity on the small oscillations of a mass of liquid about the spherical form

This problem is like the problem of the effect of viscosity on free oscillatory waves on deep water. Both problems were studied by Lamb (1932). Lamb does not present an exact solution as comprehensive as the one for free waves on deep water; he presents a detailed analysis of oscillations using the dissipation method. An exact solution of this problem is embedded in the study by Miller and Scriven (1968) of the oscillations of a fluid droplet immersed in another fluid. They note that "…the approximation based on inviscid profiles is adequate …when the interface is free and either the interior or exterior fluid is a gas of negligible density and viscosity." The word inviscid in the previous sentence should be replaced by irrotational. Miller and Scriven say that

> Lamb (1932) had previously developed approximate expressions for the rate of damping of oscillations for such a droplet when its viscosity is small and for a cavity or bubble of low density gas oscillating in a liquid of low viscosity.

This statement is not correct.

Prosperetti (1977), following his earlier 1975 work on viscous effects on the initial value problem for small-amplitude surface waves on a deep liquid, studied viscous effects on the perturbed spherical flows. For both problems he identifies two asymptotic results, for small times and large times. He says irrotational results are valid for short times and "…on the surface of a liquid of small viscosity." For large times, the solution tends to the viscous normal mode solution for free waves and spherical oscillations. We have shown that these normal mode solutions are also essentially irrotational.

Lamb gives results for the dissipation method and places no restrictions on the viscosity. Without actually carrying out the analysis for the irrotational effects of viscosity, it is clear the results will closely correspond to those given by Wang and Joseph (2006c) for free waves on deep liquid. For the oscillating sphere, the exact

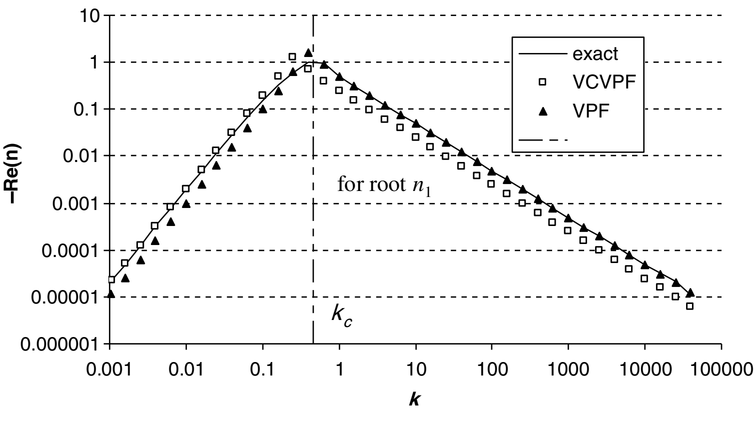


Fig. 5. (Wang and Joseph, 2006c) (a) Decay rate $-Re(n)$ vs. wave number k for $\nu = 10$ m^2/s. $Re(n)$ is computed for the exact solution, for VPF and for VCVPF. When the wave number $k < k_c$, the decay rate $-2\nu k^2$ for VCVPF is in good agreement with the exact solution, whereas the decay rate $-\nu k^2$ for VPF is only half of the exact solution. When $k > k_c$, n has two real solutions in each theory. In this figure, we plot the decay rate n_1 corresponding to the more relevant slower decaying mode.

solution is well approximated by the dissipation method which probably can also be computed from VCVPF. When the viscosity is very great the oscillations stop and the surface perturbations will decay monotonically; the monotonic decay is probably well approximated by VPF.

The following citation from Lamb makes the point about viscosity and irrotationality in oscillations of a liquid globe. He gives the formula for decay constant τ for $\exp(-t/\tau)$ from the dissipation analysis $\tau = a^2/(n-1)(2n+1)v$ and says that

> the most remarkable feature of this result is the excessively minute extent to which the oscillations of a globe of moderate dimensions are affected by such a degree of viscosity as is ordinarily met in nature. For a globe of the size of the earth, and of the same kinematic viscosity as water, we have, on the cgs system, $a = 6.37 \times 10^8$, $v = .0178$ and the value of τ for the gravitational oscillation of longest period ($n = 2$) is therefore
>
> $\tau = 1.44 \times 10^{11}$ years.
>
> Even with the value found by Darwin (1878) for the viscosity of pitch near the freezing temperature, viz. $\mu = 1.3 \times 10^8 \times g$, we find, taking $g = 980$, the value
>
> $\tau = 180$ h
>
> for the modulus of decay of the slowest oscillation of a globe the size of the earth, having the density of water and the viscosity of pitch. Since this is still large compared with the period of 1 h 34 min found in Art. 262, it appears that such a globe would oscillate almost like a perfect fluid.

10.8. Viscosity and vorticity

Vorticity is generated by the no-slip condition at the boundary of solids; it is generated by continuity conditions for the tangential components of velocity and stress at liquid–liquid surfaces and by the zero shear condition at a liquid–gas surface. The effects of viscosity on the formation of regions of vorticity are a topic at the foundation of boundary layer theory. In solid–liquid flows the major effects of viscosity emanate from regions of non-zero vorticity. At high Reynolds numbers, the major part of the drag is due to the shear stress at the boundary; a small contribution to the drag may arise from the viscous contribution to dissipation in the irrotational flow outside the boundary layer. Typically, the effects of viscosity arising from the boundary layer at liquid–gas flows are small because the rates of strain in these layers are no larger than in the irrotational flow and the layer thickness is small. It is possible to study the effect of viscosity emanating from such layers by perturbing the irrotational flow of an inviscid fluid with terms emanating from the viscous terms in the normal stress balance at the free surface for small viscosity as was done by Lundgren and Mansour (1988). This approach captures the viscous effects from the boundary layer when the viscosity is small. These effects of viscosity due to the formation of boundary layers of vorticity to remove the unphysical irrotational shear stress at the bubble interface may be added to the direct effects of viscosity on the irrotational flow. In the direct irrotational theories which we called VPF and VCVPF the effects of viscosity on vorticity generation are ignored but there is there is no restriction on the size of the viscosity. The limits of applicability of irrotational theories are determined by an analysis of the effects of vorticity. This is well known. The new idea is that significant effects of viscosity arise in regions irrotational flow even in problems in which vorticity is important.

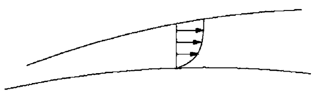
Saffman (1992) in his monograph on vortex dynamics collects results on the classical theory of inviscid incompressible fluids containing finite regions of vorticity. This approach is consistent with the fact that vortex sheets cannot exist in a viscous fluid which is perhaps an intuitively obvious result proved in Section 518 of Basset (1888). In view of the results collected in this essay, a theory of irrotational flow of viscous fluids containing finite regions of vorticity might be considered.

11. Boundary layers when the Reynolds number is not so large

Prandtl's boundary layer theory arises when the Reynolds number is very large. The flow in the boundary layer satisfies the no-slip boundary conditions the solid wall. Vorticity is generated at the wall. At large

Reynolds numbers the boundary layer is thin and outside this thin layer the flow is irrotational. In the limit of very large Reynolds numbers the irrotational flow tends to a limit which is independent of Reynolds number. This flow is sometimes said to be inviscid potential flow which is an unfortunate turn of phrase since the fluid is not viscous in some places and inviscid in others. It is well known that there is no definite end to the boundary layer; the vorticity decays rapidly away from wall and a criterion can be defined for the place beyond which the vorticity is so small that the flow out there is effectively irrotational.

The classical high Reynolds theory of Prandtl does not account for the viscous effects of the outer irrotational flow. Viscous effects on the normal stresses at the boundary of a solid cannot be obtained from Prandtl's theory. It is very well known and easily demonstrated that as a consequence of the continuity equation, the viscous normal stress must vanish on a rigid solid. The only way that viscous effects can act a boundary is through the pressure but the pressure in Prandtl's theory is not viscous. It is determined by Bernoulli's equation in the irrotational flow and is impressed unchanged on the wall through the thin boundary layer. Turning couples due to viscous stresses as well as the all important pressure drag cannot be computed from Prandtl's theory and they have not been computed from the matched asymptotic techniques used in higher order boundary layer theory. The shear stress at the wall is computed by Prandtl's theory but the mismatch between the irrotational shear stress and the zero shear stress at the edge of the boundary layer $y = \delta$ given by Prandtl's theory is not resolved.



Typically, at smaller Reynolds numbers, the flow outside some vortical region which can be a thick layer can become effectively irrotational but viscous. We can think that the solid plus the boundary layer is like a gas bubble because the shear stress vanishes at the edge of the boundary layer but the irrotational shear stress does not.

Some recent papers (Padrino and Joseph, 2006; Wang and Joseph, 2006a,b) attempt to deal with problems of boundary layers at Reynolds numbers which are not so large. They considered the problem of boundary layers on a rotating and translating cylinder of radius a previously studied by Glauert (1957). This is a good test problem for boundary layer analysis because wakes may be suppressed by rotation. The order of magnitude of the terms in the continuity and momentum equations are estimated inside the boundary layer. When terms of order δ/a and higher are dropped, Glauert's equations are covered. This procedure follows Prandtl's recipe for boundary analysis but by dropping those terms he throws out the baby with the bath water in the following sense. If the stream velocity is put to zero a boundary layer does not develop because the flow on circles around the cylinder is a viscous potential flow which requires a torque to maintain the motion. Glauert's high Reynolds boundary layer solution gives a zero torque when the steam velocity is zero. To get a correct result for the torque terms of the order neglected need to be retained. A new boundary layer for this problem was derived by Wang and Joseph (2006a) and compared with the direct numerical simulation of Padrino and Joseph (2006).

There are significant differences between the new theory and Glauert's. In Glauert's study the pressure is assumed to be a constant across the boundary layer and the momentum equation in the radial direction is not used. In the boundary layer solution of Glauert (and Prandtl), the normal stress on a solid is imposed by the irrotational pressure, independent of the Reynolds number. Viscous effects on the normal stress on a solid wall, which always exist at finite-Reynolds number, no matter how large, cannot be obtained from the Glauert theory.

In the new theory, the pressure is an unknown and the momentum equation in the radial direction is needed. Since there is an extra unknown, an extra boundary condition is needed and found in the requirement that the shear stress at the outer edge of the boundary layer is continuous. Solutions in power series, like Glauert's, are obtained. The inertia terms in the momentum equations give rise to the irrotational pressure and the viscous terms lead to a viscous pressure which contributes to both lift and drag. The solution is good to excellent agreement with the numerical solution when the boundary layer thickness δ is properly chosen. The boundary layer is relatively thick and the Reynolds numbers are not so large. The new solution is not self contained because δ is not selected internally but is selected to fit data.

Acknowledgments

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