The motion of a spherical gas bubble in viscous potential flow

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Abstract

A spherical gas bubble accelerates to steady motion in an irrotational flow of a viscous liquid induced by a balance of the acceleration of the added mass of the liquid with the Levich drag. The equation of rectilinear motion is linear and may be integrated giving rise to exponential decay with a decay constant $18\nu t/a^2$ where ν is the kinematic viscosity of the liquid and a is the bubble radius. The problem of decay to rest of a bubble moving initially when the forces maintaining motion are inactivated and the acceleration of a bubble initially at rest to terminal velocity are considered. The equation of motion follows from the assumption that the motion of the viscous liquid is irrotational. It is an elementary example of how potential flows can be used to study the unsteady motions of a viscous liquid suitable for the instruction of undergraduate students. Another example, considered here, is the purely radial irrotational motion of a viscous liquid associated with the motions of a spherical gas bubble. This gives rise to an exact potential flow solution of the Navier-Stokes equations in which the jump of the viscous component of the normal stress is evaluated on the potential flow. The equation of motion for the liquid is almost always called the Reyleigh-Plesset equation but the viscous terms were introduced by Poritsky (1951) and not by Plesset (1949). We show that when the normal stress equation is converted into an energy equation in the conventional way used for inviscid fluid, the viscous normal stress term is converted into the viscous dissipation in the liquid evaluated on potential flow.

We consider a body moving with the velocity U in an unbounded viscous potential flow. Let M be the mass of the body and M' be the added mass, then the total kinetic energy of the fluid and body is

$$T = \frac{1}{2}(M + M')U^2. \tag{1}$$

Let D be the drag and F be the external force in the direction of motion, then the power of D and F should be equal to the rate of the total kinetic energy,

$$(F + D)U = \frac{\mathrm{d}T}{\mathrm{d}t} = (M + M')U\frac{\mathrm{d}U}{\mathrm{d}t}. \tag{2}$$

We next consider a spherical gas bubble, for which $M = 0$ and $M' = \frac{2}{3}\pi a^3 \rho_f$. The drag can be obtained by direct integration using the irrotational viscous normal stress and a viscous pressure correction: $D = -12\pi\mu a U$ (see Joseph and Wang 2004). Suppose the external force just balances the drag, then the bubble moves with a constant velocity $U = U_0$. Imagine that the external force suddenly disappears, then (2) gives rise to

$$-12\pi\mu a U = \frac{2}{3}\pi a^3 \rho_f \frac{\mathrm{d}U}{\mathrm{d}t}. \tag{3}$$

The solution is

$$U = U_0 \mathrm{e}^{-\frac{18\nu}{a^2}t}, \tag{4}$$

which shows that the velocity of the bubble approaches zero exponentially.

If gravity is considered, then $F = \dfrac{4}{3}\pi a^3 \rho_f g$. Suppose the bubble is at rest at $t = 0$ and starts to move due to the buoyant force. Equation (2) can be written as

$$\frac{4}{3}\pi a^3 \rho_f g - 12\pi\mu a U = \frac{2}{3}\pi a^3 \rho_f \frac{\mathrm{d}U}{\mathrm{d}t}. \tag{5}$$

The solution is

$$U = \frac{a^2 g}{9\nu}\left(1 - \mathrm{e}^{-\frac{18\nu}{a^2}t}\right), \tag{6}$$

which indicates the bubble velocity approaches the steady state velocity

$$U = \frac{a^2 g}{9\nu} \tag{7}$$

exponentially.

Another way to obtain the equation of motion is to argue following Lamb (1932) and Levich (1949) that the work done by the external force F is equal to the rate of the total kinetic energy and the dissipation:

$$FU = (M + M')U\frac{\mathrm{d}U}{\mathrm{d}t} + \mathcal{D}. \tag{8}$$

Since $\mathcal{D} = -DU$, (8) is the same as (2).

The motion of a single spherical gas bubble in a viscous liquid has been considered by some authors. Typically, these authors assemble terms arising in various situations, like Stokes flow (Hadamard-Rybczynski drag, Basset memory integral) and high Reynolds number flow (Levich drag, boundary layer drag, induced mass) and other terms into a single equation. Such general equations have been presented by Yang and Leal (1991) and by Park, Klausner and Mei (1995) and they have been discussed in the review paper of Magnaudet and Eams (2000, see their section 4). Yang and Leal's equation has Stokes drag and no Levich drag. Our equation is not embedded in their equation. Park *et al.* listed five terms for the force on a gas bubble; our equation may be obtained from theirs if the free stream velocity U is put to zero, the memory term is dropped, and the boundary layer contribution to the drag given by Moore (1963) is neglected. Park *et al.* did not write down the same equation as our equation (1) and did not obtain the exponential decay.

It is generally believed that the added mass contribution, derived for potential flow is independent of viscosity. Magnaudet and Eames say that "... results all indicate that the added mass coefficient is independent of the Reynolds, strength of acceleration and ... boundary conditions." This independence of added mass on viscosity follows from the assumption that the motion of viscous fluids can be irrotational. The results cited by Magnaudet and Eams seem to suggest that induced mass is also independent of vorticity.

Chen (1974) did a boundary layer analysis of the impulsive motion of a spherical gas bubble which shows that the Levich drag $48/R_e$ at short times evolves to the drag $\frac{48}{R_e}\left(1 - \frac{2.21}{\sqrt{R_e}}\right)$ obtained in a boundary layer analysis by Moore (1963). The Moore drag cannot be distinguished from the Levich drag when R_e is large. The boundary layer contribution is vortical and is neglected in our potential flow analysis.

Another problem of irrotational motion of a spherical gas bubble in a viscous liquid is the expanding or contracting gas bubble first studied by Rayleigh (1917). The problem is also framed by Batchelor 1967 (p.479) but, as in Rayleigh's work, with viscosity and surface tension neglected. Vicosity μ and surface tension γ effects can be readily introduced into this problem without approximation because the motion is purely radial and irrotational; shear stresses do not arise. Though Plesset (1949) introduced a variable external driving pressure and surface tension, the effects of surface tension were also introduced and the effects of viscosity were first introduced by Poritsky (1951). His understanding of irrotational viscous stresses is exemplary, unique for his time and not usual even in ours. The equation

$$\frac{2\gamma}{R} = p_b - p_\infty - R\ddot{R} - \frac{3}{2}\dot{R}^2 - 4\mu\frac{\dot{R}}{R} \tag{9}$$

for the bubble radius $R(t)$, is always called the Rayleigh-Plesset equation but Plesset did not present or discuss this equation which is given as (62) in the 1951 paper of Poritsky. It is well known when γ and μ are neglected, that equation (9) can be formulated as an energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}KE = (p_b - p_\infty)\,\dot{V}$$

where

$$KE = \frac{1}{2}\int_R^\infty \rho\left(\frac{\partial\phi}{\partial r}\right)^2 4\pi r^2 \mathrm{d}r$$

and

$$\dot{V} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\pi R^3\right)$$

The equation

$$(p_b - p_\infty)\,\dot{V} = \frac{\mathrm{d}KE}{\mathrm{d}t} + \mathcal{D} + 2\gamma\frac{\dot{V}}{R} \tag{10}$$

where the dissipation

$$\mathcal{D} = 2\mu\int_R^\infty \frac{\partial^2\phi}{\partial x_i \partial x_j}\frac{\partial^2\phi}{\partial x_i \partial x_j} 4\pi r^2 \mathrm{d}r = 16\pi\mu^2 R\dot{R}^2$$

follows from (9) after multiplication by $\dot{V}$. In this problem we demonstrate a direct connection between the irrotational viscous normal stress and the dissipation integral $\mathcal{D}$ computed on potential flow.

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