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A note on the elliptic integrals and elliptic functions

T.Funada, Y.Yamashita, N.Tashiro, & Y.Sonoda on April 10, 2004 / ellipsoid-cal-apr10.tex →
ellipsoid-cal-apr22-apr29.tex / printed July 16, 2004

This report reviews the mathematical formulas for elliptic integrals and elliptic functions, based on "Mathematical Formulas I" ("Sugaku-koshiki I," Iwanami).

1 Elliptic integral

(a) Normal form For the integral of a rational function as $\int R(x,\sqrt{\varphi(x)})\,\mathrm{d}x$, the basic integrals are given by

$$I[m]=\int\frac{x^m}{\sqrt{\varphi(x)}}\mathrm{d}x,\quad J[m]=\int\frac{\mathrm{d}x}{(x-\lambda)^m\sqrt{\varphi(x)}},\tag{1}$$

where $\varphi(x)$ is the polynomial of the fourth or third order in x

$$\varphi(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4,\tag{2}$$

and λ is a parameter which may be complex.

(b) Recurrence formula Differentiation gives the following equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^m\sqrt{\varphi}\right)&=mx^{m-1}\sqrt{\varphi}+x^m\frac{1}{2}\frac{1}{\sqrt{\varphi}}\frac{\mathrm{d}\varphi}{\mathrm{d}x}=\frac{1}{\sqrt{\varphi}}\left(mx^{m-1}\varphi+\frac{1}{2}x^m\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)\\&=\frac{1}{\sqrt{\varphi}}\left(a_0mx^{m+3}+a_1mx^{m+2}+a_2mx^{m+1}+a_3mx^m+a_4mx^{m-1}\right.\\&\qquad\left.+2a_0x^{m+3}+\frac{3}{2}a_1x^{m+2}+a_2x^{m+1}+\frac{1}{2}a_3x^m\right)\\&=\frac{1}{2\sqrt{\varphi}}\left[a_0\left(2m+4\right)x^{m+3}+a_1\left(2m+3\right)x^{m+2}+a_2\left(2m+2\right)x^{m+1}+a_3\left(2m+1\right)x^m+a_42mx^{m-1}\right].\end{aligned}\tag{3}$$

Thus integration gives the recurrence formula of $I[m]$

$$\begin{aligned}2x^m\sqrt{\varphi}&=a_0\left(2m+4\right)I[m+3]+a_1\left(2m+3\right)I[m+2]+a_2\left(2m+2\right)I[m+1]\\&\quad+a_3\left(2m+1\right)I[m]+a_42mI[m-1].\end{aligned}\tag{4}$$

When $m=0$, we have

$$2\sqrt{\varphi}=4a_0I[3]+3a_1I[2]+2a_2I[1]+a_3I[0]$$

by which $I[3]$ is obtained if $I[2]$, $I[1]$, $I[0]$ are given.

Differentiation gives the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sqrt{\varphi}}{(x-\lambda)^m}\right]=-m\frac{\sqrt{\varphi}}{(x-\lambda)^{m+1}}+\frac{1}{(x-\lambda)^m}\frac{1}{2}\frac{1}{\sqrt{\varphi}}\frac{\mathrm{d}\varphi}{\mathrm{d}x}=\frac{1}{(x-\lambda)^m\sqrt{\varphi}}\left(\frac{-m\varphi}{x-\lambda}+\frac{1}{2}\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right).\tag{5}$$

Since $\varphi(x)$ is expressed by Taylor series as

$$\begin{aligned}\varphi(x)&=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4\\&=\varphi(\lambda)+\varphi'(\lambda)\left(x-\lambda\right)+\frac{1}{2}\varphi''(\lambda)\left(x-\lambda\right)^2+\frac{1}{6}\varphi(\lambda)'''\left(x-\lambda\right)^3+\frac{1}{24}\varphi(\lambda)''''\left(x-\lambda\right)^4,\end{aligned}\tag{6}$$

the above expression is then arranged as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{2\sqrt{\varphi(x)}}{(x-\lambda)^m}\right]=\frac{-2m}{(x-\lambda)^{m+1}\sqrt{\varphi}}\left[\varphi(\lambda)+\varphi'(\lambda)\left(x-\lambda\right)+\frac{1}{2}\varphi''(\lambda)\left(x-\lambda\right)^2+\frac{1}{6}\varphi'''(\lambda)\left(x-\lambda\right)^3\right.$$

$$+\frac{1}{24}\varphi''''(\lambda)\,(x-\lambda)^4\Big] + \frac{1}{(x-\lambda)^m\sqrt{\varphi}}\left[\varphi'(\lambda)+\varphi''(\lambda)\,(x-\lambda)+\frac{1}{2}\varphi'''(\lambda)\,(x-\lambda)^2\ +\frac{1}{6}\varphi''''(\lambda)\,(x-\lambda)^3\right]. \quad (7)$$

Thus integration gives the recurrence formula of $J[m]$

$$\frac{2\sqrt{\varphi(x)}}{(x-\lambda)^m} = -2m\left[\varphi(\lambda)J[m+1]+\varphi'(\lambda)J[m]+\frac{1}{2}\varphi''(\lambda)J[m-1]+\frac{1}{6}\varphi'''(\lambda)J[m-2]+\frac{1}{24}\varphi''''(\lambda)J[m-3]\right]$$

$$+\varphi'(\lambda)J[m]+\varphi''(\lambda)J[m-1]+\frac{1}{2}\varphi'''(\lambda)J[m-2]+\frac{1}{6}\varphi''''(\lambda)J[m-3],$$

$$\frac{-2\sqrt{\varphi(x)}}{(x-\lambda)^m} = 2m\varphi(\lambda)J[m+1]+(2m-1)\,\varphi'(\lambda)J[m]+(m-1)\,\varphi''(\lambda)J[m-1]$$

$$+\frac{2m-3}{6}\varphi'''(\lambda)J[m-2]+\frac{m-2}{12}\varphi''''(\lambda)J[m-3]. \quad (8)$$

When $m=2$, we have

$$\frac{-2\sqrt{\varphi(x)}}{(x-\lambda)^2} = 4\varphi(\lambda)J[3]+3\varphi'(\lambda)J[2]+\varphi''(\lambda)J[1]+\frac{1}{6}\varphi'''(\lambda)J[0],$$

by which $J[3]$ is obtained provided $J[2]$, $J[1]$, $J[0]$ are given.

When $m=1$, we have

$$\frac{-2\sqrt{\varphi(x)}}{(x-\lambda)} = 2\varphi(\lambda)J[2]+\varphi'(\lambda)J[1]-\frac{1}{6}\varphi'''(\lambda)J[-1]-\frac{1}{12}\varphi''''(\lambda)J[-2],$$

by which $J[2]$ is obtained if $J[1]$, $J[-1]$, $J[-2]$ are given. Since $J[0]=I[0]$, $J[-1]=I[1]-\lambda I[0]$, $J[-2]=I[2]-2\lambda I[1]+\lambda^2 I[0]$, we need to seek $I[0]$, $I[1]$, $I[2]$, $J[1]$ in the following.

(c) Transformation of $\varphi(x)$ The polynomial ($\varphi(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4$) is assumed to be of the fourth or third order, thus it allows $a_0=0$, for which two cases are shown to be transformed into the fourth one:

$$a_0=0 \quad \& \quad a_4\neq 0 \to x=1/t,$$
$$a_0=0 \quad \& \quad a_4=0 \to x=t^2.$$

For the former, we have

$$\frac{\mathrm{d}x}{\sqrt{\varphi(x)}} = \frac{-t^{-2}\mathrm{d}t}{\sqrt{a_1t^{-3}+a_2t^{-2}+a_3t^{-1}+a_4}} = \frac{-\mathrm{d}t}{\sqrt{a_4t^4+a_3t^3+a_2t^2+a_1t}}.$$

For the latter, we have

$$\frac{\mathrm{d}x}{\sqrt{\varphi(x)}} = \frac{2t\mathrm{d}t}{\sqrt{a_1t^6+a_2t^4+a_3t^2}} = \frac{2\mathrm{d}t}{\sqrt{a_1t^4+a_2t^2+a_3}}.$$

Therefore we may consider the polynomial of the fourth order without any loss of generality:

$$\varphi(x)=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4 \quad (a_0\neq 0)$$

where a_0, a_1, a_2, a_3, a_4 are real. On account of Ferrari's formula, this $\varphi(x)$ can be factorized as

$$\varphi(x)=\left(ax^2+bx+c\right)\left(\alpha x^2+\beta x+\gamma\right) \quad (9)$$

where a, b, c, α, β, γ are real. Arranging Eq(9) with $X=x+b/(2a)$, we consider two cases:

$$\varphi(x)=\left(ax^2+bx+c\right)\left(\alpha x^2+\beta x+\gamma\right)=\left[a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a}\right]\left[\alpha\left(x+\frac{\beta}{2\alpha}\right)^2+\gamma-\frac{\beta^2}{4\alpha}\right]$$

$$= \left[aX^2 + c - \frac{b^2}{4a}\right]\left[\alpha\left(X + \frac{\beta}{2\alpha} - \frac{b}{2a}\right)^2 + \gamma - \frac{\beta^2}{4\alpha}\right],$$

$$\text{if} \quad \frac{\beta}{\alpha} = \frac{b}{a}, \ \varphi(x) = \left(aX^2 + c - \frac{b^2}{4a}\right)\left(\alpha X^2 + \gamma - \frac{\beta^2}{4\alpha}\right), \tag{10}$$

$$\text{if} \ \frac{\beta}{\alpha} \neq \frac{b}{a}, \ \text{the transformation from } x \text{ to } t \text{ is made using } x = \frac{At + B}{t + 1}, \tag{11}$$

by which

$$\mathrm{d}x = \frac{A\mathrm{d}t}{t+1} - \frac{At + B}{(t+1)^2}\mathrm{d}t = \frac{A - B}{(t+1)^2}\mathrm{d}t,$$

$$\frac{\mathrm{d}x}{\sqrt{\varphi(x)}} = \frac{1}{\sqrt{\varphi(x)}}\frac{A - B}{(t+1)^2}\mathrm{d}t, \quad (t+1)^4\varphi(x) = \left[a(At+B)^2 + b(At+B)(t+1) + c(t+1)^2\right]$$

$$\times\left[\alpha(At+B)^2 + \beta(At+B)(t+1) + \gamma(t+1)^2\right]$$

$$= \left[\left(aA^2 + bA + c\right)t^2 + \left(2aAB + b(A+B) + 2c\right)t + aB^2 + bB + c\right]$$

$$\times\left[\left(\alpha A^2 + \beta A + \gamma\right)t^2 + \left(2\alpha AB + \beta(A+B) + 2\gamma\right)t + \alpha B^2 + \beta B + \gamma\right],$$

which will be simplified if A and B are chosen so as to satisfy the equations:

$$\left.\begin{array}{l} 2aAB + b(A+B) + 2c = 0 \rightarrow A = -\dfrac{bB + 2c}{2aB + b}, \\ 2\alpha AB + \beta(A+B) + 2\gamma = 0. \end{array}\right\} \tag{12}$$

Thus we have the following equation of B

$$-2\alpha B\frac{bB + 2c}{2aB + b} + \beta\left(-\frac{bB + 2c}{2aB + b} + B\right) + 2\gamma = 0, \tag{13}$$

which then leads to the quadratic equation of B

$$-2b\alpha B^2 - 4\alpha cB + \beta\left(2aB^2 - 2c\right) + 2\gamma(2aB + b) = 2(a\beta - b\alpha)B^2 + 4(\gamma a - \alpha c)B - 2\beta c + 2\gamma b = 0.$$

The solutions are given by

$$B = -\frac{\gamma a - \alpha c}{a\beta - b\alpha} \pm \sqrt{\left(\frac{\gamma a - \alpha c}{a\beta - b\alpha}\right)^2 + \frac{\beta c - \gamma b}{a\beta - b\alpha}} = B_1, \ B_2, \ \text{say}, \tag{14}$$

with

$$B_1 + B_2 = -2\frac{\gamma a - \alpha c}{a\beta - b\alpha}, \quad B_1 B_2 = \frac{\beta c - \gamma b}{a\beta - b\alpha}.$$

Then A is obtained as

$$(b\alpha - a\beta)(A + B) + 2(c\alpha - a\gamma) = 0 \quad \rightarrow \quad A = -B - 2\frac{\gamma a - \alpha c}{a\beta - b\alpha}, \tag{15}$$

thus $A = B_2$ if $B = B_1$ or $A = B_1$ if $B = B_2$. Using these, we have

$$\varphi(A) = \left(aA^2 + bA + c\right)\left(\alpha A^2 + \beta A + \gamma\right), \quad (t+1)^4\varphi(x) = \varphi(A)\left[\pm\left(t^2 \pm \mu^2\right)\left(t^2 \pm \nu^2\right)\right], \tag{16}$$

$$\pm\mu^2 = \frac{aB^2 + bB + c}{aA^2 + bA + c} = \frac{aB_2^2 + bB_2 + c}{aB_1^2 + bB_1 + c}, \quad \pm\nu^2 = \frac{\alpha B^2 + \beta B + \gamma}{\alpha A^2 + \beta A + \gamma} = \frac{\alpha B_2^2 + \beta B_2 + \gamma}{\alpha B_1^2 + \beta B_1 + \gamma} \quad (\mu, \ \nu > 0), \tag{17}$$

and

$$\frac{\mathrm{d}x}{\sqrt{\varphi(x)}} = \frac{A - B}{\sqrt{\varphi(A)}}\frac{\mathrm{d}t}{\sqrt{\pm\left(t^2 \pm \mu^2\right)\left(t^2 \pm \nu^2\right)}}. \tag{18}$$

(d) Further transformation The expression (18) is then transformed into the expression:

$$\frac{\mathrm{d}x}{\sqrt{\varphi(x)}} = \frac{c\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}} \qquad \left[c : \text{constant},\ 0 < k^2 < 1\right] \tag{19}$$

using the transformation in the following table.

denominator in Eq(18)	transformation $(t \to u)$	k^2
$\left(t^2-\mu^2\right)\left(t^2-\nu^2\right)$	$\begin{cases} t^2 = \mu^2 u^2 & \left[t^2 < \mu^2\right] \\ t^2 = \nu^2/u^2 & \left[t^2 > \nu^2\right] \end{cases}$	$\mu^2/\nu^2 \quad \left[\mu^2 < \nu^2\right]$
$\left(t^2-\mu^2\right)\left(\nu^2-t^2\right)$	$t^2 = \nu^2\left(1-k^2u^2\right) \quad \left[\mu^2 < t^2 < \nu^2\right]$	$\left(\nu^2-\mu^2\right)/\nu^2 \quad \left[\mu^2 < \nu^2\right]$
$\left(t^2+\mu^2\right)\left(t^2-\nu^2\right)$	$t^2 = \nu^2/\left(1-u^2\right) \quad \left[t^2 > \nu^2\right]$	$\mu^2/\left(\mu^2+\nu^2\right)$
$\left(t^2+\mu^2\right)\left(\nu^2-t^2\right)$	$t^2 = \nu^2\left(1-u^2\right) \quad \left[t^2 < \mu^2\right]$	$\nu^2/\left(\mu^2+\nu^2\right)$
$\left(t^2+\mu^2\right)\left(t^2+\nu^2\right)$	$t^2 = \mu^2u^2/\left(1-u^2\right)$	$\left(\nu^2-\mu^2\right)/\nu^2 \quad \left[\mu^2 < \nu^2\right]$

The case for which $-\left(t^2+\mu^2\right)\left(t^2+\nu^2\right)$ is taken away since the term in the square root is negative.

(e) Normal form of elliptic integrals When $\varphi(x)$ is transformed as in (d), the elliptic integrals are expressed, after the operation (a) and (b), in the following 4 expressions:

$$I[0] = \int \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}}, \quad I[1] = \int \frac{u}{\sqrt{(1-u^2)(1-k^2u^2)}}\mathrm{d}u, \tag{20}$$

$$I[2] = \int \frac{u^2}{\sqrt{(1-u^2)(1-k^2u^2)}}\mathrm{d}u, \quad J[1] = \int \frac{\mathrm{d}u}{(u-\lambda)\sqrt{(1-u^2)(1-k^2u^2)}}. \tag{21}$$

$I[0]$ is called the elliptic integral of the first kind. Using the transformation $v = u^2$, $I[1]$ is reduced to the indefinite integral of the second order irrational function. $I[2]$ is expressed as

$$I[2] - \frac{1}{k^2}I[0] = -\frac{1}{k^2}\int \sqrt{\frac{1-k^2u^2}{1-u^2}}\mathrm{d}u, \tag{22}$$

which is denoted as the elliptic integral of the second kind. $J[1]$ is expressed as

$$J[1] = \int \frac{u}{(u^2-\lambda^2)\sqrt{(1-u^2)(1-k^2u^2)}}\mathrm{d}u + \lambda\int \frac{\mathrm{d}u}{(u^2-\lambda^2)\sqrt{(1-u^2)(1-k^2u^2)}}, \tag{23}$$

in which the first term is reduced to the indefinite integral of the second order irrational function, using the transformation $v = u^2$. The second integral is the elliptic integral of the third kind.

(f) Legendre-Jacobi normal form On account of the above results, the elliptic integrals are denoted as the Legendre-Jacobi elliptic integrals of the following three kinds with $[u = \sin\theta,\ 0 < k^2 < 1]$.

the elliptic integral of the first kind	$\displaystyle\int_0^x \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}} = \int_0^\varphi \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\varphi, k)$
the elliptic integral of the second kind	$\displaystyle\int_0^x \sqrt{\frac{1-k^2u^2}{1-u^2}}\mathrm{d}u = \int_0^\varphi \sqrt{1-k^2\sin^2\theta}\mathrm{d}\theta = E(\varphi, k)$
the elliptic integral of the third kind	$\displaystyle\int_0^x \frac{\mathrm{d}u}{(1+cu^2)\sqrt{(1-u^2)(1-k^2u^2)}} = \int_0^\varphi \frac{\mathrm{d}\theta}{\left(1+c\sin^2\theta\right)\sqrt{1-k^2\sin^2\theta}} = \Pi(\varphi; c, k)$

Here, k is the elliptic modulus, φ is jacobi amplitude, and c in $\Pi(\varphi; c, k)$ is called the parameter. When $k \neq 0$, 1, these integrals cannot be expressed by elementary functions.

We may take normal forms as in the following.

Riemann's form:

$$\int_0^x \frac{\mathrm{d}u}{\sqrt{u(1-u)(1-\lambda u)}}, \quad \int_0^x \frac{u}{\sqrt{u(1-u)(1-\lambda u)}}\mathrm{d}u, \quad \int_0^x \frac{\mathrm{d}u}{(u-a)\sqrt{u(1-u)(1-\lambda u)}}$$

Weierstrass' form:

$$\int_\infty^x \frac{\mathrm{d}u}{\sqrt{4u^3 - g_2 u - g_3}}, \quad \int_\infty^x \frac{u}{\sqrt{4u^3 - g_2 u - g_3}}\mathrm{d}u, \quad \int_\infty^x \frac{\mathrm{d}u}{(u-a)\sqrt{4u^3 - g_2 u - g_3}}$$

The inverse of the elliptic integral of the first kind $z = \int_0^x \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}}$ is the Jacobi elliptic function $x = \mathrm{sn}(z)$, whence we may define the relevant elliptic functions:

$$\mathrm{cn}(z) = \sqrt{1 - \mathrm{sn}^2(z)}, \quad \mathrm{dn}(z) = \sqrt{1 - k^2\mathrm{sn}^2(z)}, \quad \mathrm{tn}(z) = \mathrm{sn}(z)/\mathrm{cn}(z)$$

$\mathrm{tn}(z)$ is often written as $\mathrm{sc}(z)$. The inverse functions of sn, cn, dn, tn are denoted by sn^{-1}, cn^{-1}, dn^{-1}, tn^{-1}.

1.1 Complete elliptic integral

1.1.1 Property of complete elliptic integral

The complete elliptic integral of the first kind:

$$K(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2\theta}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right),$$

The complete elliptic integral of the second kind:

$$E(k) = E\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\theta}\, \mathrm{d}\theta = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1; k^2\right).$$

Power series expansion of K, E $\quad [|k| < 1]$

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots + \left(\frac{(2r-1)!!}{(2r)!!}\right)^2 k^{2r} + \cdots\right]$$

$$= \frac{(1+\kappa)\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 \kappa^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \kappa^4 + \cdots + \left(\frac{(2r-1)!!}{(2r)!!}\right)^2 \kappa^{2r} + \cdots\right],$$

$$E(k) = \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \cdots - \left(\frac{(2r-1)!!}{(2r)!!}\right)^2 \frac{k^{2r}}{2r-1} - \cdots\right]$$

$$= \frac{\pi}{2(1+\kappa)}\left[1 + \left(\frac{1}{2}\right)^2 \kappa^2 + \left(\frac{1}{2\cdot 4}\right)^2 \kappa^4 + \cdots + \left(\frac{(2r-3)!!}{(2r)!!}\right)^2 \kappa^{2r} + \cdots\right],$$

where $\kappa = \dfrac{1 - \sqrt{1-k^2}}{1 + \sqrt{1-k^2}}$. When k is real, the inequality $\kappa < k$ holds, for which the power series of κ can converge faster.

Relation between K and E:

$$K'(k) = K(k') = K\left(\sqrt{1-k^2}\right), \quad E'(k) = E(k') = E\left(\sqrt{1-k^2}\right),$$

Putting $k' = \sqrt{1-k^2}$ and $\imath = \sqrt{-1}$, we have

$$EK' + E'K - KK' = \frac{\pi}{2} \quad \text{(Legendre's relation)},$$

$$K\left(\frac{1-k'}{1+k'}\right) = \frac{1+k'}{2}K(k), \quad E\left(\frac{1-k'}{1+k'}\right) = \frac{1}{1+k'}\left[E(k) + k'K(k)\right],$$

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)\,K(k), \quad E\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1}{1+k}\left[2E(k) - k'^2K(k)\right],$$

$$K\left(\imath\frac{k}{k'}\right) = k'K(k), \quad K'\left(\imath\frac{k}{k'}\right) = k'\left[K\left(k'\right) - \imath K(k)\right],$$

$$K\left(\frac{1}{k}\right) = k\left[K(k) + \imath K'(k)\right].$$

When $k \to 1-0$, we have

$$\frac{K(k)}{\log\frac{1}{1-k^2}} \to \frac{1}{2}, \quad \frac{1}{2}\int_0^1 K(k)\mathrm{d}k = 0.91596\ 5594 \quad \text{(Catalan's constant)}.$$

2 Potential flow around an ellipsoid with $a = 1/2$, $b = 1$, $c = 3/2$

The velocity potential ϕ (where $\phi \equiv \phi(x,\lambda)$ with $\lambda \equiv \lambda(x,y,z)$ in $0 \le \lambda < \infty$) for a flow around an ellipsoid of the semi-axes $a = 1/2$, $b = 1$, $c = 3/2$ (or $a = 1$, $b = 2$, $c = 3$) and moving with a uniform velocity U in the x-direction in the cartesian frame (x,y,z) is given by the following expression (Eq(4) in §17.52 of Milne-Thomson's book):

$$\phi = \frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)^{\frac{3}{2}}\left(b^2+\lambda\right)^{\frac{1}{2}}\left(c^2+\lambda\right)^{\frac{1}{2}}}, \tag{1}$$

where α_0 is given by

$$\alpha_0 = abc\int_0^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)^{\frac{3}{2}}\left(b^2+\lambda\right)^{\frac{1}{2}}\left(c^2+\lambda\right)^{\frac{1}{2}}}, \tag{2}$$

and λ is on the ellipsoid:

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1, \tag{3}$$

or

$$f(\lambda) = x^2\left(b^2+\lambda\right)\left(c^2+\lambda\right) + y^2\left(c^2+\lambda\right)\left(a^2+\lambda\right) + z^2\left(a^2+\lambda\right)\left(b^2+\lambda\right) - \left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right) = 0. \tag{4}$$

By taking $\lambda = 1/s$ for which $\mathrm{d}\lambda = -\mathrm{d}s/s^2$ as in the conventional manner shown before, the integral in Eq(1) may be transformed as

$$\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}} = \int_s^0 \frac{-\mathrm{d}s/s^2}{\left(a^2+1/s\right)\sqrt{\left(a^2+1/s\right)\left(b^2+1/s\right)\left(c^2+1/s\right)}}$$

$$= \int_0^s \frac{\mathrm{d}s}{\left(a^2+1/s\right)\sqrt{s^4\left(a^2+1/s\right)\left(b^2+1/s\right)\left(c^2+1/s\right)}} = \int_0^s \frac{1}{a^2}\left(\frac{-1}{a^2s+1}+1\right)\frac{\mathrm{d}s}{abc\sqrt{\varphi(s)}}$$

$$= \frac{1}{a^2abc}\int_0^s\left[1 - \frac{1/a^2}{s+1/a^2}\right]\frac{\mathrm{d}s}{\sqrt{\varphi(s)}} \tag{5}$$

where $\varphi(s)$ is defined as

$$\varphi(s) = s\left(s+\frac{1}{a^2}\right)\left(s+\frac{1}{b^2}\right)\left(s+\frac{1}{c^2}\right) = \left(s^2+\frac{1}{c^2}s\right)\left(s^2+\left(\frac{1}{a^2}+\frac{1}{b^2}\right)s+\frac{1}{a^2b^2}\right). \tag{6}$$

The characteristic equation for $\varphi(s)$ is given by the quadratic equation in ξ

$$\begin{vmatrix} \xi^2 & -2\xi & 1 \\ 0 & 1/c^2 & 1 \\ 1/\left(a^2b^2\right) & 1/a^2+1/b^2 & 1 \end{vmatrix} = \left(\frac{1}{c^2}-\frac{1}{a^2}-\frac{1}{b^2}\right)\xi^2+\frac{-2}{a^2b^2}\xi-\frac{1}{a^2b^2c^2}=0, \tag{7}$$

which is then denoted as

$$c_0\xi^2+2c_1\xi+c_2=0, \tag{8}$$

with

$$c_0=\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2},\quad c_1=\frac{1}{a^2b^2},\quad c_2=\frac{1}{a^2b^2c^2}. \tag{9}$$

The solution to Eq(8) is given by

$$\xi=-\frac{c_1}{c_0}\pm\sqrt{\left(\frac{c_1}{c_0}\right)^2-\frac{c_2}{c_0}}=\xi_1,\ \xi_2,\ \text{say}. \tag{10}$$

Using the transformation given by $s=\left(\xi_1t+\xi_2\right)/\left(t+1\right)$, we have

$$\frac{\mathrm{d}s}{\sqrt{\varphi(s)}}=\frac{\left(\xi_1-\xi_2\right)\mathrm{d}t}{\sqrt{\varphi(\xi_1)\left(t^2\pm\mu^2\right)\left(t^2\pm\nu^2\right)}}=\frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\frac{\mathrm{d}t}{\sqrt{\pm\left(t^2\pm\mu^2\right)\left(t^2\pm\nu^2\right)}}, \tag{11}$$

where μ and ν $(\nu^2>\mu^2)$ are defined as

$$\pm\nu^2=\frac{\xi_2^2+\xi_2/c^2}{\xi_1^2+\xi_1/c^2}=-\left(\frac{\xi_2}{\xi_1}\right)^2,\quad \pm\mu^2=\frac{\left(\xi_2+1/a^2\right)\left(\xi_2+1/b^2\right)}{\left(\xi_1+1/a^2\right)\left(\xi_1+1/b^2\right)}=-\left(\frac{\xi_2+1/a^2}{\xi_1+1/a^2}\right)^2. \tag{12}$$

For $a=1$, $b=2$, $c=3$, we have some typical values in the transformation given by $s=\left(\xi_1t+\xi_2\right)/\left(t+1\right)$

$$\xi_1=-6.5254748E-02,\quad \xi_2=-3.7376964E-01,\quad \xi_1-\xi_2=3.0851489E-01,\quad \xi_2/\xi_1=5.7278536E+00,$$
$$\varphi(\xi_1)=-5.1674737E-04,\quad -\nu^2=-3.2808306E+01,\quad -\mu^2=-4.4882982E-01,$$
$$\nu=5.7278536E+00,\quad \mu=6.6994762E-01,\quad k^2=\left(\nu^2-\mu^2\right)/\nu^2\quad\rightarrow\quad k=9.9313626E-01.$$

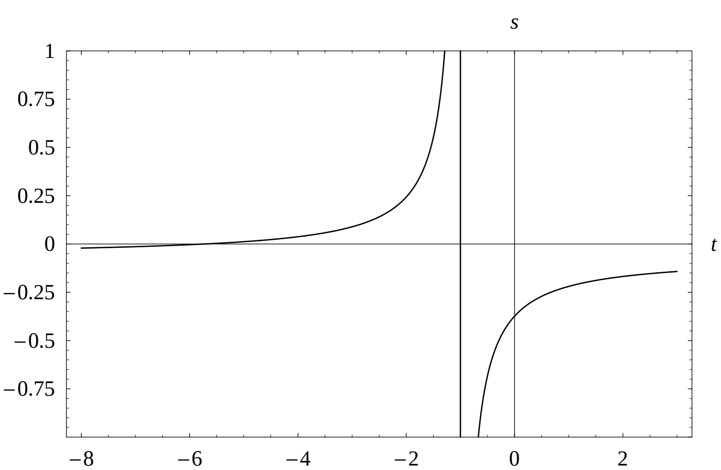

Fig.1 Transformation given by $s=\left(\xi_1t+\xi_2\right)/\left(t+1\right)$; $-\nu<t<-\mu$ (ν =5.7278536E+00 and μ =6.6994762E-01). The region $0<s<\infty$ corresponds to $-\nu<t<-1$.

Therefore, the transformation given by $t^2=\nu^2\left(1-k^2u^2\right)$ in $\mu^2<t^2<\nu^2$ leads to $t=-\sqrt{\nu^2\left(1-k^2u^2\right)}$ in $-\nu<t<-1$ corresponding to the region $0<s<\infty$, for which $0<u<u_0$ (where $u_0\equiv\sqrt{\left(\nu^2-1\right)/\left(\nu^2-\mu^2\right)}$), $2t\mathrm{d}t=-\nu^2k^22u\mathrm{d}u$ and the integral in u is

$$\int_0^s\frac{\mathrm{d}s}{\sqrt{\varphi(s)}}=\int_{-\xi_2/\xi_1}^t\frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\frac{\mathrm{d}t}{\sqrt{\left(t^2-\mu^2\right)\left(\nu^2-t^2\right)}}=\int_0^u\frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\frac{\mathrm{d}u/\nu}{\sqrt{\left(1-u^2\right)\left(1-k^2u^2\right)}}. \tag{13}$$

Then we have

$$1-\frac{1/a^2}{s+1/a^2}=1-\frac{1/a^2}{(\xi_1 t+\xi_2)/(t+1)+1/a^2}=1-\frac{(t+1)/a^2}{(\xi_1+1/a^2)t+\xi_2+1/a^2}$$

$$=1-\frac{1/a^2}{\xi_1+1/a^2}\left[1+\frac{\xi_1-\xi_2}{(\xi_1+1/a^2)t+\xi_2+1/a^2}\right]=1-\frac{1/a^2}{\xi_1+1/a^2}-\frac{(\xi_1-\xi_2)/a^2}{\xi_1+1/a^2}\frac{\left(\xi_1+1/a^2\right)t-\left(\xi_2+1/a^2\right)}{(\xi_1+1/a^2)^2t^2-(\xi_2+1/a^2)^2}$$

$$=1-\frac{1/a^2}{\xi_1+1/a^2}+\frac{(\xi_1-\xi_2)/a^2}{\xi_1+1/a^2}\frac{\left(\xi_1+1/a^2\right)\sqrt{\nu^2(1-k^2u^2)}+\left(\xi_2+1/a^2\right)}{(\xi_1+1/a^2)^2\nu^2(1-k^2u^2)-(\xi_2+1/a^2)^2}$$

$$=1-\frac{1/a^2}{\xi_1+1/a^2}+A\frac{b_1\sqrt{(1-k^2u^2)}+b_0}{a_0-a_2u^2} \tag{14}$$

with

$$A=\frac{(\xi_1-\xi_2)/a^2}{\xi_1+1/a^2},\quad a_0=\left(\xi_1+1/a^2\right)^2\nu^2-\left(\xi_2+1/a^2\right)^2,\quad a_2=\left(\xi_1+1/a^2\right)^2\nu^2k^2=a_0,$$

$$b_1=\left(\xi_1+1/a^2\right)\nu,\quad b_0=\left(\xi_2+1/a^2\right)=\left(\xi_1+1/a^2\right)\mu=b_1\frac{\mu}{\nu}=b_1k'.$$

Therefore the integral is now expressed as

$$\frac{1}{a^2abc}\frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\int_0^u\left[1-\frac{1/a^2}{\xi_1+1/a^2}+\frac{A}{a_0}\frac{b_0+b_1\sqrt{1-k^2u^2}}{1-u^2}\right]\frac{\mathrm{d}u/\nu}{\sqrt{(1-u^2)(1-k^2u^2)}}$$

$$=\frac{1}{a^2abc}\frac{\xi_1-\xi_2}{\nu\ \sqrt{|\varphi(\xi_1)|}}\left\{\left[1-\frac{1/a^2}{\xi_1+1/a^2}\right]F(\theta,k)\ +\frac{Ab_0}{a_0}\left[F(\theta,k)-\frac{1}{k'^2}E(\theta,k)+\frac{1}{k'^2}\tan\theta\sqrt{1-k^2\sin^2\theta}\right]\right.$$
$$\left.-2\frac{Ab_1}{a_0}\left[\sqrt{1-u^2}-1\right]\right\}, \tag{15}$$

by taking $u=\sin\theta$, $0<k^2<1$ and $k'^2=1-k^2$. In the derivation, we have used the elliptic integral of the first kind

$$\int_0^u\frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-k^2u^2)}}\equiv\int_0^\theta\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}=F(\theta,k), \tag{16a}$$

the second kind,

$$\int_0^u\sqrt{\frac{1-k^2u^2}{1-u^2}}\ \mathrm{d}u\equiv\int_0^\theta\sqrt{1-k^2\sin^2\theta}\ \mathrm{d}\theta=E(\theta,k), \tag{16b}$$

and

$$\int_0^u\frac{\mathrm{d}u}{\sqrt{(1-u^2)^3(1-k^2u^2)}}\equiv\int_0^\theta\frac{\mathrm{d}\theta}{\cos^2\theta\ \sqrt{1-k^2\sin^2\theta}}=F(\theta,k)-\frac{1}{k'^2}E(\theta,k)+\frac{1}{k'^2}\tan\theta\sqrt{1-k^2\sin^2\theta}. \tag{16c}$$

The transformation $\lambda\to s\to t\to u\to\theta$ and the inverse $\theta\to u\to t\to s\to\lambda$ are arranged in Table 1.

Table 1 Transformation $\lambda\to s\to t\to u\to\theta$ and the inverse $\theta\to u\to t\to s\to\lambda$.

λ	$s=1/\lambda$	$t=-\frac{s-\xi_2}{s-\xi_1}$	$u=\sqrt{\frac{\nu^2-t^2}{\nu^2-\mu^2}}$	$\theta=\sin^{-1}u$
0	∞	-1	$u_0=\sqrt{\frac{\nu^2-1}{\nu^2-\mu^2}}$	$\theta_0=\sin^{-1}u_0$
$\lambda=1/s$	$s=(\xi_1t+\xi_2)/(t+1)$	$t=-\sqrt{\nu^2(1-k^2u^2)}$	$u=\sin\theta$	θ
∞	0	$-\xi_2/\xi_1=-\nu$	0	0
$0<\lambda<\infty$	$0<s<\infty$	$-\nu<t<-1<-\mu$	$0<u<u_0<1$	$0<\theta<\theta_0<\pi/2$

[Complementary calculation] The solutions:

$$\xi = \frac{-c_1 \pm \sqrt{c_1^2 - c_2 c_0}}{c_0} = \xi_1,\ \xi_2$$

are then expressed as

$$\xi = \frac{-c_1 \pm \sqrt{\dfrac{1}{a^2 b^2}\left(\dfrac{1}{a^2} - \dfrac{1}{c^2}\right)\left(\dfrac{1}{b^2} - \dfrac{1}{c^2}\right)}}{c_0} = \frac{-c_1 \pm \sqrt{\alpha\beta}}{c_0} = \frac{-\gamma \pm \sqrt{\alpha\beta}}{c_0}, \quad \frac{\xi_2}{\xi_1} = \frac{-\gamma - \sqrt{\alpha\beta}}{-\gamma + \sqrt{\alpha\beta}}$$

with α, β, γ

$$\alpha = \frac{1}{a^2}\left(\frac{1}{a^2} - \frac{1}{c^2}\right), \quad \beta = \frac{1}{b^2}\left(\frac{1}{b^2} - \frac{1}{c^2}\right), \quad \gamma = \frac{1}{a^2 b^2}.$$

Using these, we have the following expressions:

$$\xi_1 + 1/c^2 = \frac{-c_1 + \sqrt{\alpha\beta}}{c_0} + 1/c^2 = \frac{-\alpha\beta/\gamma + \sqrt{\alpha\beta}}{c_0} = \frac{\sqrt{\alpha\beta}}{c_0}\left(1 - \frac{\sqrt{\alpha\beta}}{\gamma}\right)$$

$$\xi_2 + 1/c^2 = \frac{-c_1 - \sqrt{\alpha\beta}}{c_0} + 1/c^2 = \frac{-\alpha\beta/\gamma - \sqrt{\alpha\beta}}{c_0} = -\frac{\sqrt{\alpha\beta}}{c_0}\left(1 + \frac{\sqrt{\alpha\beta}}{\gamma}\right),$$

whence

$$-\nu^2 = \frac{\xi_2}{\xi_1}\frac{\xi_2 + 1/c^2}{\xi_1 + 1/c^2} = \frac{\xi_2}{\xi_1}\frac{-1 - \sqrt{\alpha\beta}/\gamma}{1 - \sqrt{\alpha\beta}/\gamma} = -\left(\frac{\xi_2}{\xi_1}\right)^2.$$

Moreover, we have the following expressions:

$$\xi_1 + 1/a^2 = \frac{\alpha + \sqrt{\alpha\beta}}{c_0}, \quad \xi_2 + 1/a^2 = \frac{\alpha - \sqrt{\alpha\beta}}{c_0},$$

$$\xi_1 + 1/b^2 = \frac{\beta + \sqrt{\alpha\beta}}{c_0}, \quad \xi_2 + 1/b^2 = \frac{\beta - \sqrt{\alpha\beta}}{c_0},$$

whence

$$-\mu^2 = \frac{\left(\xi_2 + 1/a^2\right)\left(\xi_2 + 1/b^2\right)}{\left(\xi_1 + 1/a^2\right)\left(\xi_1 + 1/b^2\right)} = \frac{\left(\alpha - \sqrt{\alpha\beta}\right)\left(\beta - \sqrt{\alpha\beta}\right)}{\left(\alpha + \sqrt{\alpha\beta}\right)\left(\beta + \sqrt{\alpha\beta}\right)} = \frac{\left(\sqrt{\alpha} - \sqrt{\beta}\right)\left(\sqrt{\beta} - \sqrt{\alpha}\right)}{\left(\sqrt{\alpha} + \sqrt{\beta}\right)\left(\sqrt{\beta} + \sqrt{\alpha}\right)} = -\left(\frac{\sqrt{\alpha} - \sqrt{\beta}}{\sqrt{\alpha} + \sqrt{\beta}}\right)^2,$$

$$a_0 - a_2 = -\left(\xi_2 + 1/a^2\right)^2 + \mu^2\left(\xi_1 + 1/a^2\right)^2 = -\left(\frac{\sqrt{\alpha}}{c_0}\right)^2\left(\sqrt{\alpha} - \sqrt{\beta}\right)^2 + \left(\frac{\sqrt{\alpha} - \sqrt{\beta}}{\sqrt{\alpha} + \sqrt{\beta}}\right)^2\left(\frac{\sqrt{\alpha}}{c_0}\right)^2\left(\sqrt{\alpha} + \sqrt{\beta}\right)^2 = 0.$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\tan\theta\sqrt{1 - k^2\sin^2\theta}\right) &= \frac{1}{\cos^2\theta}\sqrt{1 - k^2\sin^2\theta} - \tan\theta k^2 \sin\theta\cos\theta\frac{1}{\sqrt{1 - k^2\sin^2\theta}} \\
&= \frac{1}{\cos^2\theta}\sqrt{1 - k^2\sin^2\theta} + \sqrt{1 - k^2\sin^2\theta} - \frac{1}{\sqrt{1 - k^2\sin^2\theta}} \\
&= \frac{1 - k^2\sin^2\theta}{\cos^2\theta}\frac{1}{\sqrt{1 - k^2\sin^2\theta}} + \sqrt{1 - k^2\sin^2\theta} - \frac{1}{\sqrt{1 - k^2\sin^2\theta}} \\
&= \frac{1 - k^2}{\cos^2\theta}\frac{1}{\sqrt{1 - k^2\sin^2\theta}} + \sqrt{1 - k^2\sin^2\theta} - \frac{1 - k^2}{\sqrt{1 - k^2\sin^2\theta}} \\
&= \frac{k'^2}{\cos^2\theta}\frac{1}{\sqrt{1 - k^2\sin^2\theta}} + \sqrt{1 - k^2\sin^2\theta} - \frac{k'^2}{\sqrt{1 - k^2\sin^2\theta}}
\end{aligned}$$

$$\int_0^\theta \frac{\mathrm{d}\theta}{\cos^2\theta\sqrt{1-k^2\sin^2\theta}} = \frac{1}{k'^2}\tan\theta\sqrt{1-k^2\sin^2\theta} - \int_0^\theta \frac{1}{k'^2}\sqrt{1-k^2\sin^2\theta}\mathrm{d}\theta + \int_0^\theta \frac{1}{\sqrt{1-k^2\sin^2\theta}}\mathrm{d}\theta$$
$$= \frac{1}{k'^2}\tan\theta\sqrt{1-k^2\sin^2\theta} - \frac{1}{k'^2}E(\theta,k) + F(\theta,k)$$

with $k'^2 = 1-k^2$.
[**calculations**]

$$\xi = -\frac{c_1}{c_0} \pm \sqrt{\left(\frac{c_1}{c_0}\right)^2 - \frac{c_2}{c_0}} = \frac{1}{c_0a^2b^2c^2}\left[-c^2 \pm \sqrt{(c^2-a^2)(c^2-b^2)}\right] = \xi_1,\ \xi_2,\ \text{say.}$$

$$\xi_1 + 1/a^2 = \frac{c^2-a^2}{c_0a^2b^2c^2}\left[\frac{b^2}{a^2} + \sqrt{\frac{c^2-b^2}{c^2-a^2}}\right],\quad \xi_2 + 1/a^2 = \frac{c^2-a^2}{c_0a^2b^2c^2}\left[\frac{b^2}{a^2} - \sqrt{\frac{c^2-b^2}{c^2-a^2}}\right]$$

$$2\xi_1\xi_2 + \frac{1}{c^2}(\xi_1+\xi_2) = 0,\quad 2\xi_1\xi_2 + \left(\frac{1}{a^2}+\frac{1}{b^2}\right)(\xi_1+\xi_2) + \frac{2}{a^2b^2} = 0,$$

$$\xi_1\left(\xi_2 + 1/c^2\right) + \xi_2\left(\xi_1 + 1/c^2\right) = 0 \quad\to\quad \nu = \frac{\xi_2}{\xi_1} = -\frac{\xi_2 + 1/c^2}{\xi_1 + 1/c^2},$$

$$\left(\xi_1 + 1/a^2\right)\left(\xi_2 + 1/b^2\right) + \left(\xi_2 + 1/a^2\right)\left(\xi_1 + 1/b^2\right) = 0 \quad\to\quad \mu = \frac{\xi_2 + 1/a^2}{\xi_1 + 1/a^2} = -\frac{\xi_2 + 1/b^2}{\xi_1 + 1/b^2},$$

$$\phi = \frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)^{\frac{3}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}} = \frac{abcUx}{2-\alpha_0}\int_s^0 \frac{-\mathrm{d}s/s^2}{(a^2+1/s)\sqrt{(a^2+1/s)(b^2+1/s)(c^2+1/s)}}$$
$$= \frac{abcUx}{2-\alpha_0}\int_0^s \frac{\mathrm{d}s}{(a^2+1/s)\sqrt{s^4(a^2+1/s)(b^2+1/s)(c^2+1/s)}} = \frac{abcUx}{2-\alpha_0}\int_0^s \frac{1}{a^2}\left(\frac{-1}{a^2s+1}+1\right)\frac{\mathrm{d}s}{abc\sqrt{\varphi(s)}}$$
$$= \frac{Ux}{a^2(2-\alpha_0)}\int_0^s \left[1 - \frac{1/a^2}{s+1/a^2}\right]\frac{\mathrm{d}s}{\sqrt{\varphi(s)}}$$

A note on flows of a second order fluid around an ellipsoid

T.Funada, May 4, 2004 / ellipsoid-may04.tex / printed July 16, 2004

The terms "Spheroid (oblate spheroid and prolate spheroid)" are due to Mathematica. The terms "Ellipsoid (ovary ellipsoid and planetary ellipsoid)" are due to Milne-Thomson.

1 "17.52 Translational motion of an ellipsoid" Milne-Thomson (1974)

Consider the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \tag{1}$$

moving in the direction of x-axis with velocity U, whence the velocity potential ϕ for which $\boldsymbol{v}=\nabla\phi$ is given by

$$\phi=-\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}, \tag{2}$$

where

$$\alpha_0=abc\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}, \tag{3}$$

with the ellipsoidal coordinates $(\lambda,\,\mu,\,\nu)=(\lambda_1,\,\lambda_2,\,\lambda_3)=\lambda_1\boldsymbol{e}_1+\lambda_2\boldsymbol{e}_2+\lambda_3\boldsymbol{e}_3$ which are three solutions of the following equation through cartesian coordinates $(x,\,y,\,z)=x\boldsymbol{e}_x+y\boldsymbol{e}_y+z\boldsymbol{e}_z$

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1. \tag{4}$$

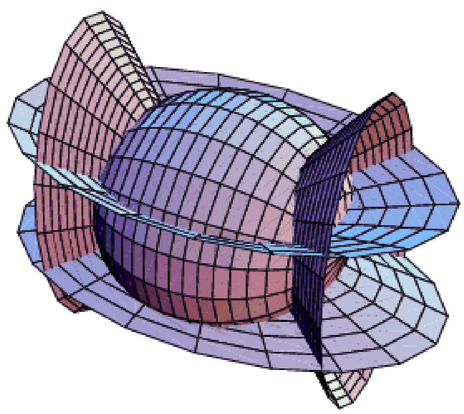

Fig.1 Confocal ellipsoidal coordinates.

1.1 Orthogonal curvilinear coordinates

Cartesian coordinates (x,y,z) and an orthogonal curvilinear coordinates (ξ_1,ξ_2,ξ_3) are related as

$$\delta\boldsymbol{x}=\delta x\boldsymbol{e}_x+\delta y\boldsymbol{e}_y+\delta z\boldsymbol{e}_z=h_1\delta\xi_1\boldsymbol{e}_1+h_2\delta\xi_2\boldsymbol{e}_2+h_3\delta\xi_3\boldsymbol{e}_3,$$

where

$$\frac{\partial\boldsymbol{x}}{\partial\xi_1}=h_1\boldsymbol{e}_1,\quad \frac{\partial\boldsymbol{x}}{\partial\xi_2}=h_2\boldsymbol{e}_2,\quad \frac{\partial\boldsymbol{x}}{\partial\xi_3}=h_3\boldsymbol{e}_3,$$

with the scale factors defined as

$$h_1=\sqrt{\frac{\partial\boldsymbol{x}}{\partial\xi_1}\cdot\frac{\partial\boldsymbol{x}}{\partial\xi_1}},\quad h_2=\sqrt{\frac{\partial\boldsymbol{x}}{\partial\xi_2}\cdot\frac{\partial\boldsymbol{x}}{\partial\xi_2}},\quad h_3=\sqrt{\frac{\partial\boldsymbol{x}}{\partial\xi_3}\cdot\frac{\partial\boldsymbol{x}}{\partial\xi_3}},$$

and the orthogonal condition

$$\frac{\partial \boldsymbol{x}}{\partial \xi_1} \cdot \frac{\partial \boldsymbol{x}}{\partial \xi_2} = h_1 h_2 \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0, \quad \frac{\partial \boldsymbol{x}}{\partial \xi_2} \cdot \frac{\partial \boldsymbol{x}}{\partial \xi_3} = h_2 h_3 \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = 0, \quad \frac{\partial \boldsymbol{x}}{\partial \xi_3} \cdot \frac{\partial \boldsymbol{x}}{\partial \xi_1} = h_3 h_1 \boldsymbol{e}_3 \cdot \boldsymbol{e}_1 = 0.$$

The derivatives of unit vectors are given by

$$\frac{\partial^2 \boldsymbol{x}}{\partial \xi_1 \partial \xi_2} = \frac{\partial}{\partial \xi_1}(h_2 \boldsymbol{e}_2) = \frac{\partial}{\partial \xi_2}(h_1 \boldsymbol{e}_1) = \frac{\partial h_1}{\partial \xi_2}\boldsymbol{e}_1 + h_1 \frac{\partial \boldsymbol{e}_1}{\partial \xi_2} = \frac{\partial h_2}{\partial \xi_1}\boldsymbol{e}_2 + h_2 \frac{\partial \boldsymbol{e}_2}{\partial \xi_1},$$

$$\frac{\partial^2 \boldsymbol{x}}{\partial \xi_2 \partial \xi_3} = \frac{\partial}{\partial \xi_2}(h_3 \boldsymbol{e}_3) = \frac{\partial}{\partial \xi_3}(h_2 \boldsymbol{e}_2) = \frac{\partial h_2}{\partial \xi_3}\boldsymbol{e}_2 + h_2 \frac{\partial \boldsymbol{e}_2}{\partial \xi_3} = \frac{\partial h_3}{\partial \xi_2}\boldsymbol{e}_3 + h_3 \frac{\partial \boldsymbol{e}_3}{\partial \xi_2},$$

$$\frac{\partial^2 \boldsymbol{x}}{\partial \xi_3 \partial \xi_1} = \frac{\partial}{\partial \xi_3}(h_1 \boldsymbol{e}_1) = \frac{\partial}{\partial \xi_1}(h_3 \boldsymbol{e}_3) = \frac{\partial h_3}{\partial \xi_1}\boldsymbol{e}_3 + h_3 \frac{\partial \boldsymbol{e}_3}{\partial \xi_1} = \frac{\partial h_1}{\partial \xi_3}\boldsymbol{e}_1 + h_1 \frac{\partial \boldsymbol{e}_1}{\partial \xi_3},$$

$$\frac{\partial \boldsymbol{e}_1}{\partial \xi_2} = \frac{1}{h_1}\frac{\partial h_2}{\partial \xi_1}\boldsymbol{e}_2, \quad \frac{\partial \boldsymbol{e}_2}{\partial \xi_1} = \frac{1}{h_2}\frac{\partial h_1}{\partial \xi_2}\boldsymbol{e}_1,$$

$$\frac{\partial \boldsymbol{e}_2}{\partial \xi_3} = \frac{1}{h_2}\frac{\partial h_3}{\partial \xi_2}\boldsymbol{e}_3, \quad \frac{\partial \boldsymbol{e}_3}{\partial \xi_2} = \frac{1}{h_3}\frac{\partial h_2}{\partial \xi_3}\boldsymbol{e}_2,$$

$$\frac{\partial \boldsymbol{e}_3}{\partial \xi_1} = \frac{1}{h_3}\frac{\partial h_1}{\partial \xi_3}\boldsymbol{e}_1, \quad \frac{\partial \boldsymbol{e}_1}{\partial \xi_3} = \frac{1}{h_1}\frac{\partial h_3}{\partial \xi_1}\boldsymbol{e}_3.$$

$$\frac{\partial \boldsymbol{e}_i}{\partial \xi_j} = \frac{1}{h_i}\frac{\partial h_j}{\partial \xi_i}\boldsymbol{e}_j \quad i \neq j$$

(6)

$$\frac{\partial \boldsymbol{e}_1}{\partial \xi_1} = \frac{\partial}{\partial \xi_1}(\boldsymbol{e}_2 \times \boldsymbol{e}_3) = -\frac{1}{h_2}\frac{\partial h_1}{\partial \xi_2}\boldsymbol{e}_2 - \frac{1}{h_3}\frac{\partial h_1}{\partial \xi_3}\boldsymbol{e}_3,$$

$$\frac{\partial \boldsymbol{e}_2}{\partial \xi_2} = \frac{\partial}{\partial \xi_2}(\boldsymbol{e}_3 \times \boldsymbol{e}_1) = -\frac{1}{h_3}\frac{\partial h_2}{\partial \xi_3}\boldsymbol{e}_3 - \frac{1}{h_1}\frac{\partial h_2}{\partial \xi_1}\boldsymbol{e}_1,$$

$$\frac{\partial \boldsymbol{e}_3}{\partial \xi_3} = \frac{\partial}{\partial \xi_3}(\boldsymbol{e}_1 \times \boldsymbol{e}_2) = -\frac{1}{h_1}\frac{\partial h_3}{\partial \xi_1}\boldsymbol{e}_1 - \frac{1}{h_2}\frac{\partial h_3}{\partial \xi_2}\boldsymbol{e}_2.$$

$$\frac{\partial \boldsymbol{e}_i}{\partial \xi_i} = \sum_{j=1\ (j\neq i)}^{3} \frac{-1}{h_j}\frac{\partial h_i}{\partial \xi_j}\boldsymbol{e}_j$$

$$\nabla \equiv \frac{\boldsymbol{e}_1}{h_1}\frac{\partial}{\partial \xi_1} + \frac{\boldsymbol{e}_2}{h_2}\frac{\partial}{\partial \xi_2} + \frac{\boldsymbol{e}_3}{h_3}\frac{\partial}{\partial \xi_3}, \quad (\boldsymbol{n}\cdot\nabla) F = DF \equiv \left(\frac{n_1}{h_1}\frac{\partial F}{\partial \xi_1} + \frac{n_2}{h_2}\frac{\partial F}{\partial \xi_2} + \frac{n_3}{h_3}\frac{\partial F}{\partial \xi_3}\right)$$

$$\boldsymbol{e}_1 \cdot (\boldsymbol{n}\cdot\nabla)\boldsymbol{F} = (\boldsymbol{n}\cdot\nabla) F_1 + \frac{F_2}{h_1 h_2}\left(n_1 \frac{\partial h_1}{\partial \xi_2} - n_2 \frac{\partial h_2}{\partial \xi_1}\right) + \frac{F_3}{h_3 h_1}\left(n_1 \frac{\partial h_1}{\partial \xi_3} - n_3 \frac{\partial h_3}{\partial \xi_1}\right),$$

$$\boldsymbol{e}_2 \cdot (\boldsymbol{n}\cdot\nabla)\boldsymbol{F} = (\boldsymbol{n}\cdot\nabla) F_2 + \frac{F_1}{h_1 h_2}\left(n_2 \frac{\partial h_2}{\partial \xi_1} - n_1 \frac{\partial h_1}{\partial \xi_2}\right) + \frac{F_3}{h_2 h_3}\left(n_2 \frac{\partial h_2}{\partial \xi_3} - n_3 \frac{\partial h_3}{\partial \xi_2}\right),$$

$$\boldsymbol{e}_3 \cdot (\boldsymbol{n}\cdot\nabla)\boldsymbol{F} = (\boldsymbol{n}\cdot\nabla) F_3 + \frac{F_1}{h_3 h_1}\left(n_3 \frac{\partial h_3}{\partial \xi_1} - n_1 \frac{\partial h_1}{\partial \xi_3}\right) + \frac{F_2}{h_2 h_3}\left(n_3 \frac{\partial h_3}{\partial \xi_2} - n_2 \frac{\partial h_2}{\partial \xi_3}\right).$$

$$D\boldsymbol{e}_i \equiv \sum_{j=1\ (j\neq i)}^{3} b_{ij}\boldsymbol{e}_j, \quad b_{ij} = -b_{ji} \quad i \neq j$$

$$D\boldsymbol{e}_1 = \frac{n_1}{h_1}\frac{\partial \boldsymbol{e}_1}{\partial \xi_1} + \frac{n_2}{h_2}\frac{\partial \boldsymbol{e}_1}{\partial \xi_2} + \frac{n_3}{h_3}\frac{\partial \boldsymbol{e}_1}{\partial \xi_3} = \frac{n_1}{h_1}\left(\frac{-1}{h_2}\frac{\partial h_1}{\partial \xi_2}\boldsymbol{e}_2 + \frac{-1}{h_3}\frac{\partial h_1}{\partial \xi_3}\boldsymbol{e}_3\right) + \frac{n_2}{h_1 h_2}\frac{\partial h_2}{\partial \xi_1}\boldsymbol{e}_2 + \frac{n_3}{h_3 h_1}\frac{\partial h_3}{\partial \xi_1}\boldsymbol{e}_3$$

$$= \frac{1}{h_1 h_2}\left(-n_1 \frac{\partial h_1}{\partial \xi_2} + n_2 \frac{\partial h_2}{\partial \xi_1}\right)\boldsymbol{e}_2 + \frac{1}{h_3 h_1}\left(-n_1 \frac{\partial h_1}{\partial \xi_3} + n_3 \frac{\partial h_3}{\partial \xi_1}\right)\boldsymbol{e}_3 = b_{12}\boldsymbol{e}_2 + b_{13}\boldsymbol{e}_3$$

$$\begin{aligned}
D\boldsymbol{e}_2 &= \frac{n_1}{h_1}\frac{\partial \boldsymbol{e}_2}{\partial \xi_1} + \frac{n_2}{h_2}\frac{\partial \boldsymbol{e}_2}{\partial \xi_2} + \frac{n_3}{h_3}\frac{\partial \boldsymbol{e}_2}{\partial \xi_3} = \frac{n_1}{h_1 h_2}\frac{\partial h_1}{\partial \xi_2}\boldsymbol{e}_1 + \frac{n_2}{h_2}\left(\frac{-1}{h_1}\frac{\partial h_2}{\partial \xi_1}\boldsymbol{e}_1 + \frac{-1}{h_3}\frac{\partial h_2}{\partial \xi_3}\boldsymbol{e}_3\right) + \frac{n_3}{h_2 h_3}\frac{\partial h_3}{\partial \xi_2}\boldsymbol{e}_3 \\
&= \frac{1}{h_2 h_3}\left(-n_2\frac{\partial h_2}{\partial \xi_3} + n_3\frac{\partial h_3}{\partial \xi_2}\right)\boldsymbol{e}_3 + \frac{1}{h_1 h_2}\left(-n_2\frac{\partial h_2}{\partial \xi_1} + n_1\frac{\partial h_1}{\partial \xi_2}\right)\boldsymbol{e}_1 = b_{23}\boldsymbol{e}_3 + b_{21}\boldsymbol{e}_1 \\
D\boldsymbol{e}_3 &= \frac{n_1}{h_1}\frac{\partial \boldsymbol{e}_3}{\partial \xi_1} + \frac{n_2}{h_2}\frac{\partial \boldsymbol{e}_3}{\partial \xi_2} + \frac{n_3}{h_3}\frac{\partial \boldsymbol{e}_3}{\partial \xi_3} = \frac{n_1}{h_3 h_1}\frac{\partial h_1}{\partial \xi_3}\boldsymbol{e}_1 + \frac{n_2}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3}\boldsymbol{e}_2 + \frac{n_3}{h_3}\left(\frac{-1}{h_1}\frac{\partial h_3}{\partial \xi_1}\boldsymbol{e}_1 + \frac{-1}{h_2}\frac{\partial h_3}{\partial \xi_2}\boldsymbol{e}_2\right) \\
&= \frac{1}{h_3 h_1}\left(-n_3\frac{\partial h_3}{\partial \xi_1} + n_1\frac{\partial h_1}{\partial \xi_3}\right)\boldsymbol{e}_1 + \frac{1}{h_2 h_3}\left(-n_3\frac{\partial h_3}{\partial \xi_2} + n_2\frac{\partial h_2}{\partial \xi_3}\right)\boldsymbol{e}_2 = b_{31}\boldsymbol{e}_1 + b_{32}\boldsymbol{e}_2
\end{aligned}$$

$$(L_{ij}) = \nabla \boldsymbol{v}, \quad A_{ij} = L_{ij} + L_{ji},$$

$$\begin{aligned}
\left[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\right] \;=\;& \boldsymbol{e}_1\boldsymbol{e}_1 2\left(\frac{1}{h_1}\frac{\partial v_1}{\partial \xi_1} + \frac{v_2}{h_1 h_2}\frac{\partial h_1}{\partial \xi_2} + \frac{v_3}{h_3 h_1}\frac{\partial h_1}{\partial \xi_3}\right) \\
+\;& \boldsymbol{e}_1\boldsymbol{e}_2\left[\left(\frac{-v_1}{h_1 h_2}\frac{\partial h_1}{\partial \xi_2} + \frac{1}{h_1}\frac{\partial v_2}{\partial \xi_1}\right) + \left(\frac{1}{h_2}\frac{\partial v_1}{\partial \xi_2} - \frac{v_2}{h_1 h_2}\frac{\partial h_2}{\partial \xi_1}\right)\right] \\
+\;& \boldsymbol{e}_1\boldsymbol{e}_3\left[\left(-\frac{v_1}{h_3 h_1}\frac{\partial h_1}{\partial \xi_3} + \frac{1}{h_1}\frac{\partial v_3}{\partial \xi_1}\right) + \left(\frac{1}{h_3}\frac{\partial v_1}{\partial \xi_3} - \frac{v_3}{h_3 h_1}\frac{\partial h_3}{\partial \xi_1}\right)\right] \\
+\;& \boldsymbol{e}_2\boldsymbol{e}_1\left[\left(\frac{1}{h_2}\frac{\partial v_1}{\partial \xi_2} - \frac{v_2}{h_1 h_2}\frac{\partial h_2}{\partial \xi_1}\right) + \left(\frac{-v_1}{h_1 h_2}\frac{\partial h_1}{\partial \xi_2} + \frac{1}{h_1}\frac{\partial v_2}{\partial \xi_1}\right)\right] \\
+\;& \boldsymbol{e}_2\boldsymbol{e}_2 2\left(\frac{v_1}{h_1 h_2}\frac{\partial h_2}{\partial \xi_1} + \frac{1}{h_2}\frac{\partial v_2}{\partial \xi_2} + \frac{v_3}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3}\right) \\
+\;& \boldsymbol{e}_2\boldsymbol{e}_3\left[\left(-\frac{v_2}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3} + \frac{1}{h_2}\frac{\partial v_3}{\partial \xi_2}\right) + \left(\frac{1}{h_3}\frac{\partial v_2}{\partial \xi_3} - \frac{v_3}{h_2 h_3}\frac{\partial h_3}{\partial \xi_2}\right)\right] \\
+\;& \boldsymbol{e}_3\boldsymbol{e}_1\left[\left(\frac{1}{h_3}\frac{\partial v_1}{\partial \xi_3} - \frac{v_3}{h_3 h_1}\frac{\partial h_3}{\partial \xi_1}\right) + \left(-\frac{v_1}{h_3 h_1}\frac{\partial h_1}{\partial \xi_3} + \frac{1}{h_1}\frac{\partial v_3}{\partial \xi_1}\right)\right] \\
+\;& \boldsymbol{e}_3\boldsymbol{e}_2\left[\left(\frac{1}{h_3}\frac{\partial v_2}{\partial \xi_3} - \frac{v_3}{h_2 h_3}\frac{\partial h_3}{\partial \xi_2}\right) + \left(-\frac{v_2}{h_2 h_3}\frac{\partial h_2}{\partial \xi_3} + \frac{1}{h_2}\frac{\partial v_3}{\partial \xi_2}\right)\right] \\
+\;& \boldsymbol{e}_3\boldsymbol{e}_3 2\left(\frac{v_1}{h_3 h_1}\frac{\partial h_3}{\partial \xi_1} + \frac{v_2}{h_2 h_3}\frac{\partial h_3}{\partial \xi_2} + \frac{1}{h_3}\frac{\partial v_3}{\partial \xi_3}\right)
\end{aligned}$$

(20) $(A_{ij}) = (A_{ji}) = (\boldsymbol{e}_i a_{ij} \boldsymbol{e}_j)$

$$\begin{aligned}
\sum_{i=1}^{3}\sum_{j=1}^{3}(\boldsymbol{n}\cdot\nabla)A_{ij} &= \sum_{i=1}^{3}\sum_{j=1}^{3} D(\boldsymbol{e}_i a_{ij}\boldsymbol{e}_j) = \sum_{i=1}^{3}\sum_{j=1}^{3}\left[\boldsymbol{e}_i\boldsymbol{e}_j(Da_{ij}) + a_{ij}\left((D\boldsymbol{e}_i)\boldsymbol{e}_j + \boldsymbol{e}_i(D\boldsymbol{e}_j)\right)\right] \\
&= \boldsymbol{e}_1\boldsymbol{e}_1(Da_{11}) + a_{11}2(D\boldsymbol{e}_1)\boldsymbol{e}_1 \\
&\quad + \boldsymbol{e}_2\boldsymbol{e}_2(Da_{22}) + a_{22}2(D\boldsymbol{e}_2)\boldsymbol{e}_2 \\
&\quad + \boldsymbol{e}_3\boldsymbol{e}_3(Da_{33}) + a_{33}2(D\boldsymbol{e}_3)\boldsymbol{e}_3 \\
&\quad + 2\boldsymbol{e}_1\boldsymbol{e}_2(Da_{12}) + 2a_{12}\left((D\boldsymbol{e}_1)\boldsymbol{e}_2 + \boldsymbol{e}_1(D\boldsymbol{e}_2)\right) \\
&\quad + 2\boldsymbol{e}_1\boldsymbol{e}_3(Da_{13}) + 2a_{13}\left((D\boldsymbol{e}_1)\boldsymbol{e}_3 + \boldsymbol{e}_1(D\boldsymbol{e}_3)\right) \\
&\quad + 2\boldsymbol{e}_2\boldsymbol{e}_3(Da_{23}) + 2a_{23}\left((D\boldsymbol{e}_2)\boldsymbol{e}_3 + \boldsymbol{e}_2(D\boldsymbol{e}_3)\right) \\
&= \boldsymbol{e}_1\boldsymbol{e}_1(Da_{11}) + a_{11}2(b_{12}\boldsymbol{e}_2 + b_{13}\boldsymbol{e}_3)\boldsymbol{e}_1 \\
&\quad + \boldsymbol{e}_2\boldsymbol{e}_2(Da_{22}) + a_{22}2(b_{23}\boldsymbol{e}_3 + b_{21}\boldsymbol{e}_1)\boldsymbol{e}_2 \\
&\quad + \boldsymbol{e}_3\boldsymbol{e}_3(Da_{33}) + a_{33}2(b_{31}\boldsymbol{e}_1 + b_{32}\boldsymbol{e}_2)\boldsymbol{e}_3
\end{aligned}$$

$$
\begin{aligned}
&+2\boldsymbol{e}_1\boldsymbol{e}_2\left(Da_{12}\right)+2a_{12}\left[\left(b_{12}\boldsymbol{e}_2+b_{13}\boldsymbol{e}_3\right)\boldsymbol{e}_2+\boldsymbol{e}_1\left(b_{23}\boldsymbol{e}_3+b_{21}\boldsymbol{e}_1\right)\right]\\
&+2\boldsymbol{e}_1\boldsymbol{e}_3\left(Da_{13}\right)+2a_{13}\left[\left(b_{12}\boldsymbol{e}_2+b_{13}\boldsymbol{e}_3\right)\boldsymbol{e}_3+\boldsymbol{e}_1\left(b_{31}\boldsymbol{e}_1+b_{32}\boldsymbol{e}_2\right)\right]\\
&+2\boldsymbol{e}_2\boldsymbol{e}_3\left(Da_{23}\right)+2a_{23}\left[\left(b_{23}\boldsymbol{e}_3+b_{21}\boldsymbol{e}_1\right)\boldsymbol{e}_3+\boldsymbol{e}_2\left(b_{31}\boldsymbol{e}_1+b_{32}\boldsymbol{e}_2\right)\right]\\
&=\boldsymbol{e}_1\boldsymbol{e}_1\left[\left(Da_{11}\right)+2a_{12}b_{21}+2a_{13}b_{31}\right]\\
&+\boldsymbol{e}_2\boldsymbol{e}_2\left[\left(Da_{22}\right)+2a_{12}b_{12}+2a_{23}b_{32}\right]\\
&+\boldsymbol{e}_3\boldsymbol{e}_3\left[\left(Da_{33}\right)+2a_{13}b_{13}+2a_{23}b_{23}\right]\\
&+\boldsymbol{e}_1\boldsymbol{e}_2\left[2a_{11}b_{12}+2a_{22}b_{21}+2\left(Da_{12}\right)+2a_{13}b_{32}+2a_{23}b_{31}\right]\\
&+\boldsymbol{e}_1\boldsymbol{e}_3\left[2a_{11}b_{13}+2a_{33}b_{31}+2a_{12}b_{23}+2\left(Da_{13}\right)+2a_{23}b_{21}\right]\\
&+\boldsymbol{e}_2\boldsymbol{e}_3\left[2a_{22}b_{23}+2a_{33}b_{32}+2a_{12}b_{13}+2a_{13}b_{12}+2\left(Da_{23}\right)\right]
\end{aligned}
$$

2 Second order fluid

For the second order fluid, the gradient of velocity, the lower convective derivative and the stress tensor are given by

$$
L_{ij}=\frac{\partial v_i}{\partial x_j},\quad A_{ij}=L_{ij}+L_{ji},\quad B_{ij}=\frac{\partial A_{ij}}{\partial t}+v_k\frac{\partial A_{ij}}{\partial x_k}+A_{ik}L_{kj}+L_{ik}A_{kj},\tag{2.1}
$$

$$
T_{ij}=-p\delta_{ij}+\mu A_{ij}+\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj}.\tag{2.2}
$$

Using the velocity potential for which

$$
\boldsymbol{v}=\nabla\phi,\quad v_i=\frac{\partial\phi}{\partial x_i},\quad \nabla\cdot\boldsymbol{v}=\nabla^2\phi=0,\quad L_{ij}=\frac{\partial^2\phi}{\partial x_i\partial x_j},\quad A_{ij}=2\frac{\partial^2\phi}{\partial x_i\partial x_j},\tag{2.3}
$$

we have the following expressions

$$
\begin{aligned}
B_{ij}&=2\frac{\partial^3\phi}{\partial t\partial x_i\partial x_j}+2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}+2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}+2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\\
&=2\frac{\partial^3\phi}{\partial t\partial x_i\partial x_j}+2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}+4\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j},
\end{aligned}\tag{2.4}
$$

$$
\frac{\partial B_{ij}}{\partial x_j}=\frac{\partial}{\partial x_j}\left[2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}+4\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right],\quad \frac{\partial T_{ij}}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial A_{ij}}{\partial x_j}+\alpha_1\frac{\partial B_{ij}}{\partial x_j}+\alpha_2\frac{\partial\left(A_{ik}A_{kj}\right)}{\partial x_j}.\tag{2.5}
$$

$$
\chi\equiv\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k},\quad \frac{\partial}{\partial x_j}\left(v_k\frac{\partial A_{ij}}{\partial x_k}\right)=\frac{\partial\chi}{\partial x_i},\quad \frac{\partial}{\partial x_j}\left(A_{ik}L_{kj}\right)=\frac{\partial\chi}{\partial x_i},\quad \frac{\partial}{\partial x_j}A_{ik}A_{kj}=2\frac{\partial\chi}{\partial x_i},\tag{2.6}
$$

$$
\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_k}2\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial^2\phi}{\partial x_k\partial x_j}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right),\quad \frac{\partial}{\partial x_j}\left(2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}\right),
$$

$$
\frac{\partial}{\partial x_j}\left(2\frac{\partial^2\phi}{\partial x_i\partial x_k}2\frac{\partial^2\phi}{\partial x_k\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(2\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}\right),\quad \frac{\partial}{\partial x_j}\left(\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj}\right)=\frac{\partial}{\partial x_i}\left[\left(3\alpha_1+2\alpha_2\right)\chi\right].
$$

The equation of motion is given by

$$
\rho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\nabla^2 v_i+\frac{\partial}{\partial x_j}\left(\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj}\right),\tag{2.7}
$$

whence the Bernoulli function is expressed as

$$\frac{\partial}{\partial x_i}\left[\rho\frac{\partial\phi}{\partial t}+\frac{\rho}{2}\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_j}+p-\hat{\beta}\chi\right]=0 \quad \text{with } \hat{\beta}=3\alpha_1+2\alpha_2\geq 0, \tag{2.8}$$

$$\rho\frac{\partial\phi}{\partial t}+\frac{\rho}{2}\left|\nabla\phi\right|^2+p-\hat{\beta}\chi=C(t). \tag{2.9}$$

The stress tensor is given by

$$T_{ij}=-\left[C+\hat{\beta}\chi-\frac{\rho}{2}\left|\nabla\phi\right|^2-\rho\frac{\partial\phi}{\partial t}\right]\delta_{ij}+\left[\mu+\alpha_1\left(\frac{\partial}{\partial t}+\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\right)\right]A_{ij}+\left(\alpha_1+\alpha_2\right)A_{ik}A_{kj}. \tag{2.10}$$

3 Sphere

For a sphere for which $a^2=b^2=c^2$, we use the spherical coordinates $(r,\,\theta,\,\varphi)$ together with the cartesian coordinates $(x,\,y,\,z)=(r\cos\theta,\,r\sin\theta\cos\varphi,\,r\sin\theta\sin\varphi)$ to give the equation of λ

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1 \quad\rightarrow\quad x^2+y^2+z^2=a^2+\lambda\equiv r^2. \tag{3.1}$$

The velocity potential ϕ for which $\boldsymbol{v}=\nabla\phi$ is given by

$$\phi=-\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty\frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}}$$

$$\rightarrow\quad \phi=-\frac{a^3Ux}{2-\alpha_0}\int_\lambda^\infty\frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)^3}}=\frac{-a^3Ux}{2-\alpha_0}\frac{2}{3}\left(a^2+\lambda\right)^{-3/2}=\frac{-Uxa^3}{2\sqrt{\left(a^2+\lambda\right)^3}}=-\frac{Uxa^3}{2r^3}, \tag{3.2}$$

with α_0 defined as

$$\alpha_0=abc\int_0^\infty\frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}}\quad\rightarrow\quad\alpha_0=a^3\int_0^\infty\frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)^3}}=\frac{2}{3}. \tag{3.3}$$

The velocity $\boldsymbol{v}=(u,\,v,\,w)$ in the $(x,\,y,\,z)$ coordinates or $\boldsymbol{v}=(v_r,\,v_\theta,\,v_\varphi)$ in the $(r,\,\theta,\,\varphi)$ coordinates are given by

$$\begin{aligned}\boldsymbol{v}=(u,\,v,\,w)=\nabla\phi&=-\boldsymbol{e}_x\frac{\partial}{\partial x}\left(Ux\frac{a^3}{2r^3}\right)-\boldsymbol{e}_y\frac{\partial}{\partial y}\left(Ux\frac{a^3}{2r^3}\right)-\boldsymbol{e}_z\frac{\partial}{\partial z}\left(Ux\frac{a^3}{2r^3}\right)\\&=-\boldsymbol{e}_xU\left(\frac{a^3}{2r^3}-\frac{3a^3}{2r^4}\frac{x^2}{r}\right)+\boldsymbol{e}_yU\left(\frac{3a^3}{2r^4}\frac{xy}{r}\right)+\boldsymbol{e}_zU\left(\frac{3a^3}{2r^4}\frac{xz}{r}\right),\end{aligned} \tag{3.4}$$

$$\begin{aligned}\boldsymbol{v}=(v_r,\,v_\theta,\,v_\varphi)=\nabla\phi&=-\boldsymbol{e}_r\frac{\partial}{\partial r}\left(Ux\frac{a^3}{2r^3}\right)-\boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left(Ux\frac{a^3}{2r^3}\right)-\boldsymbol{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(Ux\frac{a^3}{2r^3}\right)\\&=\boldsymbol{e}_rU\cos\theta\frac{a^3}{r^3}+\boldsymbol{e}_\theta U\sin\theta\frac{a^3}{2r^3}.\end{aligned} \tag{3.5}$$

If the frame is taken on the ellipsoid, the velocity potential is modified as

$$\phi=-Ux\left(1+\frac{a^3}{2r^3}\right). \tag{3.6}$$

The unit vectors in the spherical coordinates (r,θ,φ) are given by

$$\boldsymbol{e}=\cos\varphi\boldsymbol{e}_1+\sin\varphi\boldsymbol{e}_2,\quad \boldsymbol{e}_r=\sin\theta\cos\varphi\boldsymbol{e}_1+\sin\theta\sin\varphi\boldsymbol{e}_2+\cos\theta\boldsymbol{e}_3=\sin\theta\boldsymbol{e}+\cos\theta\boldsymbol{e}_3, \tag{3.7}$$

$$\frac{\partial\boldsymbol{e}_r}{\partial\theta}=\cos\theta\cos\varphi\boldsymbol{e}_1+\cos\theta\sin\varphi\boldsymbol{e}_2-\sin\theta\boldsymbol{e}_3=\boldsymbol{e}_\theta, \tag{3.8}$$

$$\frac{\partial\boldsymbol{e}_r}{\partial\varphi}=-\sin\theta\sin\varphi\boldsymbol{e}_1+\sin\theta\cos\varphi\boldsymbol{e}_2=\sin\theta\frac{\partial\boldsymbol{e}}{\partial\varphi},\quad\frac{\partial\boldsymbol{e}_\theta}{\partial\theta}=-\boldsymbol{e}_r. \tag{3.9}$$

3.1 Normal stress on the sphere

The potential is given by

$$\phi = -U\cos\theta\left(r + \frac{a^3}{2r^2}\right), \tag{3.10}$$

whence the velocity $\boldsymbol{v} = \nabla\phi$

$$(v_r,\, v_\theta,\, v_\varphi) = \left(\frac{\partial\phi}{\partial r},\, \frac{1}{r}\frac{\partial\phi}{\partial\theta},\, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right) = (v_r,\, v_\theta,\, 0)\,, \tag{3.11}$$

$$v_r = \frac{\partial\phi}{\partial r} = -U\cos\theta\left(1 - \frac{a^3}{r^3}\right), \quad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\sin\theta\left(1 + \frac{a^3}{2r^3}\right), \tag{3.12}$$

$$\frac{\partial v_r}{\partial r} = -U\cos\theta\frac{3a^3}{r^4}, \quad \frac{1}{r}\frac{\partial v_r}{\partial\theta} = U\sin\theta\left(\frac{1}{r} - \frac{a^3}{r^4}\right), \tag{3.13}$$

$$\frac{\partial v_\theta}{\partial r} = -U\sin\theta\frac{3a^3}{2r^4}, \quad \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} = U\cos\theta\left(\frac{1}{r} + \frac{a^3}{2r^4}\right), \tag{3.14}$$

$$-\frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial\theta} = -U\sin\theta\left(\frac{1}{r} + \frac{a^3}{2r^4}\right) + U\sin\theta\left(\frac{1}{r} - \frac{a^3}{r^4}\right) = -U\sin\theta\frac{3a^3}{2r^4}, \tag{3.15}$$

$$\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r} = U\cos\theta\left(\frac{1}{r} + \frac{a^3}{2r^4}\right) - U\cos\theta\left(\frac{1}{r} - \frac{a^3}{r^4}\right) = U\cos\theta\frac{3a^3}{2r^4}, \tag{3.16}$$

$$\frac{v_r}{r} + \frac{v_\theta}{r}\cot\theta = -U\cos\theta\left(\frac{1}{r} - \frac{a^3}{r^4}\right) + U\sin\theta\left(\frac{1}{r} + \frac{a^3}{2r^4}\right)\cot\theta = U\cos\theta\frac{3a^3}{2r^4}. \tag{3.17}$$

The gradient of the velocity is given by

$$(L_{ij}) = \begin{pmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{\partial v_\theta}{\partial r} & \dfrac{\partial v_\varphi}{\partial r} \\ -\dfrac{v_\theta}{r} + \dfrac{1}{r}\dfrac{\partial v_r}{\partial\theta} & \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial\theta} + \dfrac{v_r}{r} & \dfrac{1}{r}\dfrac{\partial v_\varphi}{\partial\theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial v_r}{\partial\varphi} - \dfrac{v_\varphi}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial v_\theta}{\partial\varphi} - \dfrac{v_\varphi}{r}\cot\theta & \dfrac{1}{r\sin\theta}\dfrac{\partial v_\varphi}{\partial\varphi} + \dfrac{v_r}{r} + \dfrac{v_\theta}{r}\cot\theta \end{pmatrix} \tag{3.18}$$

$$(L_{ij}) = \frac{3a^3U}{2r^4}\begin{pmatrix} -2\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}. \tag{3.19}$$

The strain tensor is given by

$$A_{ij} = L_{ij} + L_{ji} = 2L_{ij},$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

$$c_{11} = \frac{9a^6U^2}{r^8}\left(4\cos^2\theta + \sin^2\theta\right), \quad c_{12} = \frac{9a^6U^2}{r^8}\sin\theta\cos\theta, \quad c_{13} = 0,$$

$$c_{21} = \frac{9a^6U^2}{r^8}\sin\theta\cos\theta, \quad c_{22} = \frac{9a^6U^2}{r^8}, \quad c_{23} = 0,$$

$$c_{31} = c_{32} = 0, \quad c_{33} = \frac{9a^6U^2}{r^8}\cos^2\theta,$$

$$\boldsymbol{A}^2 = (A_{ik}A_{kj}) = \frac{9a^6U^2}{r^8}\begin{pmatrix} 4\cos^2\theta + \sin^2\theta & \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta & 1 & 0 \\ 0 & 0 & \cos^2\theta \end{pmatrix} \tag{3.20}$$

$$\frac{1}{4}\mathrm{tr}\boldsymbol{A}^2 = \frac{9U^2}{4a^2}\left(5\cos^2\theta + \sin^2\theta + 1\right) = \frac{9U^2}{4a^2}\left(4\cos^2\theta + 2\right) = \frac{9U^2}{2a^2}\left(1 + 2\cos^2\theta\right) \tag{3.21}$$

$$(3\alpha_1 + 2\alpha_2)\,\frac{9}{2} = \left(\frac{27}{2}\alpha_1 + 9\alpha_2\right) \tag{3.22}$$

$$p = p_\infty + \frac{\rho}{2}U^2 - \rho\frac{\partial\phi}{\partial t} - \frac{1}{2}\rho\left|\nabla\phi\right|^2 + \hat{\beta}\chi = p_\infty + \frac{\rho}{2}U^2\left(1 - \frac{9}{4}\sin^2\theta\right) + \frac{U^2}{a^2}\left(\frac{27}{2}\alpha_1 + 9\alpha_2\right)\left(1 + 2\cos^2\theta\right) \tag{3.23}$$

The normal component of $(\boldsymbol{v}\cdot\nabla)\,A_{ij}$ is

$$\boldsymbol{e}_1\boldsymbol{e}_1\left(v_r\frac{\partial c_{11}}{\partial r} + \frac{v_\theta}{r}\frac{\partial c_{11}}{\partial\theta}\right) + c_{12}\frac{v_\theta}{r}\boldsymbol{e}_1\frac{\partial\boldsymbol{e}_2}{\partial\theta} + c_{21}\frac{v_\theta}{r}\boldsymbol{e}_1\frac{\partial\boldsymbol{e}_2}{\partial\theta} = \boldsymbol{e}_1\boldsymbol{e}_1\left[v_r\frac{\partial c_{11}}{\partial r} + \frac{v_\theta}{r}\frac{\partial c_{11}}{\partial\theta} - \frac{v_\theta}{r}\left(c_{12} + c_{21}\right)\right]$$
$$= \boldsymbol{e}_1\boldsymbol{e}_1\left[v_r\frac{\partial^2 v_r}{\partial r^2} + \frac{v_\theta}{r}\frac{\partial^2 v_r}{\partial\theta\partial r} - \frac{v_\theta}{r}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\right] = \boldsymbol{e}_1\boldsymbol{e}_1\frac{9U^2}{a^2}\sin^2\theta \tag{3.24}$$

$$n_i n_j\left(\boldsymbol{v}\cdot\nabla\right)A_{ij} = \frac{18U^2}{a^2}\sin^2\theta \tag{3.25}$$

$$\begin{aligned} n_i T_{ij} n_j &= -p + n_i n_j\left[\left[\mu + \alpha_1\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\right]A_{ij} + \left(\alpha_1 + \alpha_2\right)A_{ik}A_{kj}\right] \\ &= -p + \mu n_i n_j A_{ij} + n_i n_j\alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij} + \left(\alpha_1 + \alpha_2\right)n_i n_j A_{ik}A_{kj} \\ &= -p + \mu\left(-\frac{6a^3U}{r^4}\cos\theta\right) + n_i n_j\alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij} + \left(\alpha_1 + \alpha_2\right)\frac{9a^6U^2}{r^8}\left(4\cos^2\theta + \sin^2\theta\right) \end{aligned}$$

$$\begin{aligned} n_i n_j\left(T_{ij}\right)_{r=a} &= -p_\infty - \left(\frac{27}{2}\alpha_1 + 9\alpha_2\right)\frac{U^2}{a^2}\left(1 + 2\cos^2\theta\right) - \frac{\rho}{2}U^2\left(1 - \frac{9}{4}\sin^2\theta\right) \\ &\quad -\frac{6U}{a}\mu\cos\theta + n_i n_j\alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij} + \left(\alpha_1 + \alpha_2\right)\frac{9U^2}{a^2}\left(4\cos^2\theta + \sin^2\theta\right) \\ &= -p_\infty - \frac{\rho}{2}U^2\left(1 - \frac{9}{4}\sin^2\theta\right) - \frac{6U}{a}\mu\cos\theta + n_i n_j\alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij} \\ &\quad +\frac{U^2}{a^2}\alpha_1\left(36\cos^2\theta + 9\sin^2\theta - \frac{27}{2} - 27\cos^2\theta\right) + \frac{U^2}{a^2}\alpha_2\left(36\cos^2\theta + 9\sin^2\theta - 9 - 18\cos^2\theta\right) \\ &= -p_\infty - \frac{\rho}{2}U^2\left(1 - \frac{9}{4}\sin^2\theta\right) - \frac{6U}{a}\mu\cos\theta + \frac{18U^2}{a^2}\alpha_1\sin^2\theta - \frac{U^2}{a^2}\frac{9}{2}\alpha_1 + \frac{U^2}{a^2}\alpha_2 9\cos^2\theta \end{aligned}$$

Therefore the normalised normal stress on the sphere is now expressed as

$$T^*_{rr} = \frac{T_{rr} + p_\infty}{\rho U^2/2} = -\left(1 - \frac{9}{4}\sin^2\theta\right) - \frac{12}{R}\cos\theta + \frac{\alpha_1}{\rho a^2}\left(36\sin^2\theta - 9\right) + \frac{\alpha_2}{\rho a^2}18\cos^2\theta. \tag{3.26}$$

4 Spheroid (oblate spheroid and prolate spheroid)

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1 \quad\rightarrow\quad \frac{x^2+y^2}{a^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1, \tag{4.1}$$

$$\phi = -\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}} \quad\rightarrow\quad \phi = -\frac{a^2cUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)^2\sqrt{c^2+\lambda}}, \tag{4.2}$$

$$\alpha_0 = abc\int_0^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}} \quad\rightarrow\quad \alpha_0 = a^2c\int_0^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)^2\sqrt{c^2+\lambda}}. \tag{4.3}$$

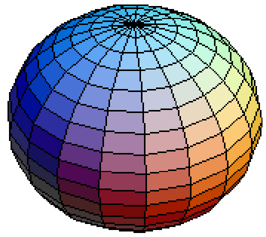

Fig.1 Oblate spheroid, $\dfrac{x^2+y^2}{a^2}+\dfrac{z^2}{c^2}=1$ with $a>c$.

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1 \quad\rightarrow\quad \frac{x^2+y^2}{a^2+\lambda}+\frac{z^2}{c^2+\lambda}=1, \tag{4.4}$$

$$\phi=-\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad\rightarrow\quad \phi=-\frac{a^2cUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)^2\sqrt{c^2+\lambda}}, \tag{4.5}$$

$$\alpha_0=abc\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad\rightarrow\quad \alpha_0=a^2c\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)^2\sqrt{c^2+\lambda}}. \tag{4.6}$$

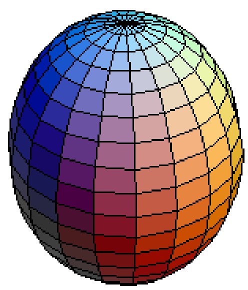

Fig.1 Prolate spheroid, $\dfrac{x^2+y^2}{a^2}+\dfrac{z^2}{c^2}=1$ with $a<c$.

$$\int\frac{\mathrm{d}\lambda}{(a^2+\lambda)^2\sqrt{c^2+\lambda}}=\frac{-1}{a^2+\lambda}\left(c^2+\lambda\right)^{-\frac{1}{2}}-\int\frac{-1}{a^2+\lambda}\frac{-1}{2}\left(c^2+\lambda\right)^{-\frac{3}{2}}\mathrm{d}\lambda$$

$$=\frac{-1}{a^2+\lambda}\left(c^2+\lambda\right)^{-\frac{1}{2}}+\frac{1}{2}\frac{1}{a^2-c^2}\int\left(\frac{1}{a^2+\lambda}-\frac{1}{c^2+\lambda}\right)\left(c^2+\lambda\right)^{-\frac{1}{2}}\mathrm{d}\lambda$$

$$=\frac{-1}{a^2+\lambda}\left(c^2+\lambda\right)^{-\frac{1}{2}}+\frac{1}{2}\frac{1}{a^2-c^2}\left[2\left(c^2+\lambda\right)^{-\frac{1}{2}}+\int\frac{\left(c^2+\lambda\right)^{-\frac{1}{2}}}{a^2+\lambda}\mathrm{d}\lambda\right]$$

$$\xi=\sqrt{c^2+\lambda}\quad \mathrm{d}\xi=\frac{1}{2}\frac{\mathrm{d}\lambda}{\sqrt{c^2+\lambda}}$$

$$\int\frac{\left(c^2+\lambda\right)^{-\frac{1}{2}}}{a^2+\lambda}\mathrm{d}\lambda=\int\frac{2\mathrm{d}\xi}{a^2-c^2+\xi^2}$$

$$a^2-c^2>0\quad \xi=\sqrt{a^2-c^2}\tan\theta\quad \mathrm{d}\xi=\sqrt{a^2-c^2}\frac{\mathrm{d}\theta}{\cos^2\theta}$$

$$\int\frac{2\mathrm{d}\xi}{a^2-c^2+\xi^2}=\int\frac{2\sqrt{a^2-c^2}\frac{\mathrm{d}\theta}{\cos^2\theta}}{(a^2-c^2)\left(1+\tan^2\theta\right)}=\frac{2}{\sqrt{a^2-c^2}}\theta=\frac{2}{\sqrt{a^2-c^2}}\arctan\left(\frac{\sqrt{c^2+\lambda}}{\sqrt{a^2-c^2}}\right)$$

$$\int \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)^2\sqrt{c^2+\lambda}} = \frac{-1}{a^2+\lambda}\left(c^2+\lambda\right)^{-\frac{1}{2}} + \frac{\left(c^2+\lambda\right)^{-\frac{1}{2}}}{a^2-c^2} + \left(a^2-c^2\right)^{-3/2}\arctan\left(\frac{\sqrt{c^2+\lambda}}{\sqrt{a^2-c^2}}\right)$$

$$a^2-c^2<0 \quad \xi=\sqrt{c^2-a^2}\coth\theta \quad \mathrm{d}\xi=\sqrt{c^2-a^2}\frac{-\mathrm{d}\theta}{\sinh^2\theta}$$

$$\int \frac{2\mathrm{d}\xi}{a^2-c^2+\xi^2} = \int \frac{2\sqrt{c^2-a^2}\frac{-\mathrm{d}\theta}{\sinh^2\theta}}{\left(c^2-a^2\right)\left(-1+\coth^2\theta\right)} = \frac{-2}{\sqrt{c^2-a^2}}\theta = \frac{-2}{\sqrt{c^2-a^2}}\mathrm{arccoth}\left(\frac{\sqrt{c^2+\lambda}}{\sqrt{c^2-a^2}}\right)$$

$$\frac{X+1/X}{X-1/X} = \frac{\sqrt{c^2+\lambda}}{\sqrt{c^2-a^2}} \quad \rightarrow \quad X^2=e^{2\theta}=\frac{\sqrt{c^2+\lambda}+\sqrt{c^2-a^2}}{\sqrt{c^2+\lambda}-\sqrt{c^2-a^2}}$$

$$\int \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)^2\sqrt{c^2+\lambda}} = \frac{-1}{a^2+\lambda}\left(c^2+\lambda\right)^{-\frac{1}{2}} + \frac{\left(c^2+\lambda\right)^{-\frac{1}{2}}}{a^2-c^2} + \left(c^2-a^2\right)^{-3/2}\frac{1}{2}\ln\left(\frac{\sqrt{c^2+\lambda}+\sqrt{c^2-a^2}}{\sqrt{c^2+\lambda}-\sqrt{c^2-a^2}}\right)$$

5 Ellipsoid of revolution (ovary ellipsoid and planetary ellipsoid)

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1 \quad \rightarrow \quad \frac{x^2}{a^2+\lambda}+\frac{y^2+z^2}{c^2+\lambda}=1, \tag{5.1}$$

$$\phi=-\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}} \quad \rightarrow \quad \phi=-\frac{ac^2Ux}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\sqrt{\left(a^2+\lambda\right)^3\left(c^2+\lambda\right)^2}}, \tag{5.2}$$

$$\alpha_0=abc\int_0^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}} \quad \rightarrow \quad \alpha_0=ac^2\int_0^\infty \frac{\mathrm{d}\lambda}{\sqrt{\left(a^2+\lambda\right)^3\left(c^2+\lambda\right)^2}}. \tag{5.3}$$

Fig.1 Ovary ellipsoid, $\frac{x^2}{a^2}+\frac{y^2+z^2}{c^2}=1$ with $a>c$.

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1 \quad \rightarrow \quad \frac{x^2}{a^2+\lambda}+\frac{y^2+z^2}{c^2+\lambda}=1, \tag{5.4}$$

$$\phi=-\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}} \quad \rightarrow \quad \phi=-\frac{ac^2Ux}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\sqrt{\left(a^2+\lambda\right)^3\left(c^2+\lambda\right)^2}}, \tag{5.5}$$

$$\alpha_0=abc\int_0^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}} \quad \rightarrow \quad \alpha_0=ac^2\int_0^\infty \frac{\mathrm{d}\lambda}{\sqrt{\left(a^2+\lambda\right)^3\left(c^2+\lambda\right)^2}}. \tag{5.6}$$

Fig.1 Planetary ellipsoid, $\frac{x^2}{a^2}+\frac{y^2+z^2}{c^2}=1$ with $a<c$.

$$\int \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)^{\frac{3}{2}}\left(c^2+\lambda\right)} = \frac{1}{c^2-a^2}\int\left(\frac{1}{a^2+\lambda}-\frac{1}{c^2+\lambda}\right)\left(a^2+\lambda\right)^{-\frac{1}{2}}\mathrm{d}\lambda$$

$$=\frac{1}{a^2-c^2}\left[2\left(a^2+\lambda\right)^{-\frac{1}{2}}+\int\frac{\left(a^2+\lambda\right)^{-\frac{1}{2}}}{c^2+\lambda}\mathrm{d}\lambda\right]$$

$$\xi = \sqrt{a^2+\lambda} \quad \mathrm{d}\xi = \frac{1}{2}\frac{\mathrm{d}\lambda}{\sqrt{a^2+\lambda}}$$

$$\int \frac{\left(a^2+\lambda\right)^{-\frac{1}{2}}}{c^2+\lambda}\mathrm{d}\lambda = \int \frac{2\mathrm{d}\xi}{c^2-a^2+\xi^2}$$

$$c^2-a^2>0 \quad \xi = \sqrt{c^2-a^2}\tan\theta \quad \mathrm{d}\xi = \sqrt{c^2-a^2}\frac{\mathrm{d}\theta}{\cos^2\theta}$$

$$\int \frac{2\mathrm{d}\xi}{c^2-a^2+\xi^2} = \int \frac{2\sqrt{c^2-a^2}\frac{\mathrm{d}\theta}{\cos^2\theta}}{\left(c^2-a^2\right)\left(1+\tan^2\theta\right)} = \frac{2}{\sqrt{c^2-a^2}}\theta = \frac{2}{\sqrt{c^2-a^2}}\arctan\left(\frac{\sqrt{a^2+\lambda}}{\sqrt{c^2-a^2}}\right)$$

$$c^2-a^2<0 \quad \xi = \sqrt{a^2-c^2}\coth\theta \quad \mathrm{d}\xi = \sqrt{a^2-c^2}\frac{-\mathrm{d}\theta}{\sinh^2\theta}$$

$$\int \frac{2\mathrm{d}\xi}{c^2-a^2+\xi^2} = \int \frac{2\sqrt{c^2-a^2}\frac{\mathrm{d}\theta}{\cos^2\theta}}{\left(c^2-a^2\right)\left(1+\tan^2\theta\right)} = \int \frac{2\sqrt{a^2-c^2}\frac{-\mathrm{d}\theta}{\sinh^2\theta}}{\left(a^2-c^2\right)\left(-1+\coth^2\theta\right)} = \frac{-2}{\sqrt{a^2-c^2}}\theta$$

$$= \frac{-1}{\sqrt{a^2-c^2}}\ln\left(\frac{\sqrt{a^2+\lambda}+\sqrt{a^2-c^2}}{\sqrt{a^2+\lambda}-\sqrt{a^2-c^2}}\right)$$

$$\int \frac{\left(a^2+\lambda\right)^{-\frac{1}{2}}}{c^2+\lambda}\mathrm{d}\lambda = \frac{1}{a^2-c^2}\left[2\left(a^2+\lambda\right)^{-\frac{1}{2}} + \frac{2}{\sqrt{c^2-a^2}}\arctan\left(\frac{\sqrt{a^2+\lambda}}{\sqrt{c^2-a^2}}\right)\right] \quad \text{in } c^2-a^2>0$$

$$\int \frac{\left(a^2+\lambda\right)^{-\frac{1}{2}}}{c^2+\lambda}\mathrm{d}\lambda = \frac{1}{a^2-c^2}\left[2\left(a^2+\lambda\right)^{-\frac{1}{2}} - \frac{1}{\sqrt{a^2-c^2}}\ln\left(\frac{\sqrt{a^2+\lambda}+\sqrt{a^2-c^2}}{\sqrt{a^2+\lambda}-\sqrt{a^2-c^2}}\right)\right] \quad \text{in } a^2-c^2>0$$

6 Circular cylinder

For a circular cylinder for which $a^2 = b^2$ and $c^2 \to \infty$, we have

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda} = 1 \quad \to \quad \frac{x^2}{a^2+\lambda}+\frac{y^2}{a^2+\lambda} = 1 \quad \to \quad x^2+y^2 = r^2 = a^2+\lambda, \tag{6.1}$$

$$\phi = -\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}} \quad \to \quad \phi = -\frac{a^2Ux}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)^2} = -\frac{a^2Ux}{a^2+\lambda} = -\frac{a^2Ux}{r^2}, \tag{6.2}$$

$$\alpha_0 = abc\int_0^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}} \quad \to \quad \alpha_0 = a^2\int_0^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)^2} = 1. \tag{6.3}$$

If the frame is taken on the ellipsoid, the velocity potential is modified as

$$\phi = -Ux\left(1+\frac{a^2}{r^2}\right). \tag{6.4}$$

The unit vectors in the cylindrical coordinates (r, θ, z) are given by

$$\boldsymbol{e}_r = \cos\theta\boldsymbol{e}_1+\sin\theta\boldsymbol{e}_2, \quad \frac{\partial\boldsymbol{e}_r}{\partial\theta} = -\sin\theta\boldsymbol{e}_1+\cos\theta\boldsymbol{e}_2 = \boldsymbol{e}_\theta, \quad \frac{\partial\boldsymbol{e}_\theta}{\partial\theta} = -\boldsymbol{e}_r. \tag{6.5}$$

The relations between this coordinates and the cartesian coordinates (x, y) are given by

$$r^2 = x^2+y^2, \quad 2r\mathrm{d}r = 2x\mathrm{d}x+2y\mathrm{d}y, \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}. \tag{6.6}$$

The velocity (u, v) and the relevant relations are

$$u = -\frac{\partial \phi}{\partial x} = U\left(1 + \frac{a^2}{r^2}\right) - Ux\frac{2a^2}{r^3}\frac{\partial r}{\partial x} = U\left(1 + \frac{a^2}{r^2}\right) - 2U\frac{a^2x^2}{r^4}, \tag{6.7}$$

$$v = -\frac{\partial \phi}{\partial y} = -Ux\frac{2a^2}{r^3}\frac{y}{r} = -2U\frac{a^2xy}{r^4}, \tag{6.8}$$

$$\frac{\partial u}{\partial x} = U\frac{-2a^2}{r^3}\frac{\partial r}{\partial x} - \frac{4Ua^2x}{r^4} + \frac{8Ua^2x^2}{r^5}\frac{x}{r} = -\frac{6Ua^2x}{r^4} + \frac{8Ua^2x^3}{r^6}, \tag{6.9}$$

$$\frac{\partial u}{\partial y} = \frac{-2Ua^2}{r^3}\frac{y}{r} + \frac{8Ua^2x^2}{r^5}\frac{y}{r} = \frac{-2Ua^2y}{r^4} + \frac{8Ua^2x^2y}{r^6}, \tag{6.10}$$

$$\frac{\partial v}{\partial x} = -2U\frac{a^2y}{r^4} + \frac{8Ua^2xy}{r^5}\frac{x}{r} = -2U\frac{a^2y}{r^4} + \frac{8Ua^2x^2y}{r^6} = \frac{\partial u}{\partial y}, \tag{6.11}$$

$$\frac{\partial v}{\partial y} = -2U\frac{a^2x}{r^4} + \frac{8Ua^2xy}{r^5}\frac{y}{r} = -2U\frac{a^2x}{r^4} + \frac{8Ua^2xy^2}{r^6} = -\frac{\partial u}{\partial x}, \tag{6.12}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{6Ua^2}{r^4} + \frac{48Ua^2x^2}{r^6} - \frac{48Ua^2x^4}{r^8}, \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{24Ua^2xy}{r^6} - \frac{48Ua^2x^3y}{r^8}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{6Ua^2}{r^4} - \frac{48Ua^2x^2y^2}{r^8}, \tag{6.13}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \rightarrow \boldsymbol{\omega} = \mathbf{0} \quad \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial/\partial x & \partial/\partial y & 0 \\ u & v & 0 \end{vmatrix} = \mathbf{0}, \tag{6.14}$$

$$(L_{ij}) = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial y} & -\dfrac{\partial u}{\partial x} \end{pmatrix}, \tag{6.15}$$

$$(A_{ij}) = (L_{ij} + L_{ji}) = \begin{pmatrix} 2\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} & 2\dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2\dfrac{\partial u}{\partial x} & 2\dfrac{\partial u}{\partial y} \\ 2\dfrac{\partial u}{\partial y} & -2\dfrac{\partial u}{\partial x} \end{pmatrix}, \tag{6.16}$$

$$\begin{aligned}(A_{ik}A_{kj}) &= \begin{pmatrix} 4\left(\dfrac{\partial^2 \phi}{\partial x^2}\right)^2 + 4\left(\dfrac{\partial^2 \phi}{\partial y \partial x}\right)^2 & 4\dfrac{\partial^2 \phi}{\partial y \partial x}\left(\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right) \\ 4\dfrac{\partial^2 \phi}{\partial y \partial x}\left(\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right) & 4\left(\dfrac{\partial^2 \phi}{\partial y \partial x}\right)^2 + 4\left(\dfrac{\partial^2 \phi}{\partial y^2}\right)^2 \end{pmatrix} \\ &= 4\left[\left(\frac{\partial^2 \phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2 \phi}{\partial y \partial x}\right)^2\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\chi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\end{aligned} \tag{6.17}$$

with

$$\chi = \frac{\partial^2 \phi}{\partial x_i \partial x_j}\frac{\partial^2 \phi}{\partial x_j \partial x_i} = \left(\frac{\partial^2 \phi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 \phi}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 \phi}{\partial y^2}\right)^2 = 2\left(\frac{\partial^2 \phi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 \phi}{\partial x \partial y}\right)^2. \tag{6.18}$$

The stress tensor for Newtonian is given by

$$T_{ij} = -p\delta_{ij} + \mu A_{ij}, \tag{6.19}$$

and the Bernoulli function is given by

$$-\frac{\partial\phi}{\partial t}+\frac{1}{2}\left|\nabla\phi\right|^2+\frac{p}{\rho}=\frac{1}{2}U^2+\frac{p_\infty}{\rho}\quad\rightarrow\quad p=p_\infty+\frac{\rho}{2}U^2+\rho\frac{\partial\phi}{\partial t}-\frac{\rho}{2}\left(u^2+v^2\right). \tag{6.20}$$

Using this, the normal stress on the cylinder surface ($r=a$) is given by

$$n_in_jT_{ij}=-p+\mu n_in_jA_{ij}=-\left[p_\infty+\frac{\rho}{2}U^2-\frac{\rho}{2}\left(u^2+v^2\right)\right]+\mu n_in_jA_{ij}, \tag{6.21}$$

with the normal viscous stress specified as

$$n_in_jA_{ij}=n_x^2A_{11}+2n_xn_yA_{12}+n_y^2A_{22}=2\left(n_x^2-n_y^2\right)\frac{\partial u}{\partial x}+4n_xn_y\frac{\partial u}{\partial y}. \tag{6.22}$$

The normalized form T^*_{nn} is given by

$$T^*_{nn}=\frac{n_in_jT_{ij}+p_\infty}{\frac{\rho}{2}U^2}=-1+\left(u^2+v^2\right)+\frac{4}{R_e}\left[\left(n_x^2-n_y^2\right)\frac{\partial u}{\partial x}+2n_xn_y\frac{\partial u}{\partial y}\right], \tag{6.23}$$

where u, v, $\partial u/\partial x$, $\partial u/\partial y$ should be read as the normalized ones.

From the paper of Wang & Joseph (2003), the stress tensor for the second order fluid is given by

$$B_{ij}=\frac{\partial A_{ij}}{\partial t}+v_k\frac{\partial A_{ij}}{\partial x_k}+A_{ik}L_{kj}+L_{ik}A_{kj}, \tag{6.24}$$

$$T_{ij}=-p\delta_{ij}+\mu A_{ij}+\alpha_1B_{ij}+\alpha_2A_{ik}A_{kj}, \tag{6.25}$$

and the Bernoulli function is given by

$$\begin{aligned}&-\frac{\partial\phi}{\partial t}+\frac{1}{2}\left|\nabla\phi\right|^2+\frac{p}{\rho}-\frac{1}{\rho}\left(3\alpha_1+2\alpha_2\right)\chi=\frac{1}{2}U^2+\frac{p_\infty}{\rho}\\ &\rightarrow\quad p=p_\infty+\frac{\rho}{2}U^2+\rho\frac{\partial\phi}{\partial t}-\frac{\rho}{2}\left(u^2+v^2\right)+\left(3\alpha_1+2\alpha_2\right)\chi.\end{aligned} \tag{6.26}$$

Using these, the normal stress on the cylinder surface ($r=a$) is given by

$$\begin{aligned}n_in_jT_{ij}&=-p+\mu n_in_jA_{ij}+\alpha_1n_in_jB_{ij}+\alpha_2n_in_jA_{ik}A_{kj}\\ &=-\left[p_\infty+\frac{\rho}{2}U^2-\frac{\rho}{2}\left(u^2+v^2\right)+\left(3\alpha_1+2\alpha_2\right)\chi\right]+\mu n_in_jA_{ij}+\alpha_1n_in_jv_k\frac{\partial A_{ij}}{\partial x_k}+2\left(\alpha_1+\alpha_2\right)\chi\\ &=-p_\infty-\frac{\rho}{2}U^2+\frac{\rho}{2}\left(u^2+v^2\right)+\mu\left[2\left(n_x^2-n_y^2\right)\frac{\partial u}{\partial x}+4n_xn_y\frac{\partial u}{\partial y}\right]+\alpha_1n_in_jv_k\frac{\partial A_{ij}}{\partial x_k}-\alpha_1\chi,\end{aligned} \tag{6.27}$$

for which

$$\begin{aligned}n_in_jv_k\frac{\partial A_{ij}}{\partial x_k}&=n_x^2\left(u\frac{\partial A_{11}}{\partial x}+v\frac{\partial A_{11}}{\partial y}\right)+2n_xn_y\left(u\frac{\partial A_{12}}{\partial x}+v\frac{\partial A_{12}}{\partial y}\right)+n_y^2\left(u\frac{\partial A_{22}}{\partial x}+v\frac{\partial A_{22}}{\partial y}\right)\\ &=2\left(n_x^2-n_y^2\right)\left(u\frac{\partial^2u}{\partial x^2}+v\frac{\partial^2u}{\partial x\partial y}\right)+4n_xn_y\left(u\frac{\partial^2u}{\partial x\partial y}+v\frac{\partial^2u}{\partial y^2}\right).\end{aligned} \tag{6.28}$$

The normalized form T^*_{nn} is given by

$$\begin{aligned}T^*_{nn}=\frac{n_in_jT_{ij}+p_\infty}{\frac{\rho}{2}U^2}&=-1+\left(u^2+v^2\right)+\frac{4}{R_e}\left[\left(n_x^2-n_y^2\right)\frac{\partial u}{\partial x}+2n_xn_y\frac{\partial u}{\partial y}\right]\\ &+\frac{4\alpha_1}{\rho a^2}\left[\left(n_x^2-n_y^2\right)\left(u\frac{\partial^2u}{\partial x^2}+v\frac{\partial^2u}{\partial x\partial y}\right)+2n_xn_y\left(u\frac{\partial^2u}{\partial x\partial y}+v\frac{\partial^2u}{\partial y^2}\right)-\chi/2\right],\end{aligned} \tag{6.29}$$

where u, v, $\partial u/\partial x$, $\partial u/\partial y$, $\chi=2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial y}\right)^2$ should be read as the normalized ones.

7 Elliptic cylinder

For an elliptic cylinder for which $a^2 \neq b^2$ and $c^2 \to \infty$, we have

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1 \quad \to \quad \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \tag{7.1}$$

$$\phi=-\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad \to \quad \phi=-\frac{abUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)}}, \tag{7.2}$$

$$\alpha_0=abc\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad \to \quad \alpha_0=ab\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)}}. \tag{7.3}$$

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[5] oblate spheroid http://mathworld.wolfram.com/OblateSpheroid.html

[6] prolate spheroid http://mathworld.wolfram.com/ProlateSpheroid.html

Contents

A note on "Potential flow of a second order fluid over a sphere or an ellipse" by Wang & Joseph (2003)

T.Funada, April 27, 2004 / ellipse-wang-joseph-apr27.tex / printed August 28, 2004

1 Introduction

$$L_{ij}=\frac{\partial v_i}{\partial x_j},\quad A_{ij}=L_{ij}+L_{ji}$$

$$B_{ij}=\frac{\partial A_{ij}}{\partial t}+v_k\frac{\partial A_{ij}}{\partial x_k}+A_{ik}L_{kj}+L_{ik}A_{kj}$$

$$T_{ij}=-p\delta_{ij}+\mu A_{ij}+\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj}.$$

$$\boldsymbol{v}=\nabla\phi,\quad v_i=\frac{\partial\phi}{\partial x_i},\quad \nabla\cdot\boldsymbol{v}=\nabla^2\phi=0$$

$$L_{ij}=\frac{\partial^2\phi}{\partial x_i\partial x_j}=L_{ji},\quad A_{ij}=2\frac{\partial^2\phi}{\partial x_i\partial x_j}.$$

$$\begin{aligned}B_{ij}&=2\frac{\partial^3\phi}{\partial t\partial x_i\partial x_j}+2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}+2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}+2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\\&=2\frac{\partial^3\phi}{\partial t\partial x_i\partial x_j}+2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}+4\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}=\left(\frac{\partial}{\partial t}+\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\right)A_{ij}+A_{ik}A_{kj}.\end{aligned}$$

$$T_{ij}=-p\delta_{ij}+\left[\mu+\alpha_1\left(\frac{\partial}{\partial t}+\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\right)\right]A_{ij}+(\alpha_1+\alpha_2)A_{ik}A_{kj}.$$

$$\frac{\partial B_{ij}}{\partial x_j}=\frac{\partial}{\partial x_j}\left[2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}+4\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right].$$

$$\frac{\partial T_{ij}}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial A_{ij}}{\partial x_j}+\alpha_1\frac{\partial B_{ij}}{\partial x_j}+\alpha_2\frac{\partial(A_{ik}A_{kj})}{\partial x_j}.$$

$$\frac{\partial}{\partial x_j}\left(v_k\frac{\partial A_{ij}}{\partial x_k}\right)=\frac{\partial\chi}{\partial x_i},\quad \frac{\partial}{\partial x_j}(A_{ik}L_{kj})=\frac{\partial\chi}{\partial x_i},\quad \frac{\partial}{\partial x_j}A_{ik}A_{kj}=2\frac{\partial\chi}{\partial x_i}$$

$$\chi\equiv\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}.$$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_k}2\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial^2\phi}{\partial x_k\partial x_j}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right)$$

$$\frac{\partial}{\partial x_j}\left(2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}\right)$$

$$\frac{\partial}{\partial x_j}\left(2\frac{\partial^2\phi}{\partial x_i\partial x_k}2\frac{\partial^2\phi}{\partial x_k\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(2\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}\right)$$

$$\frac{\partial}{\partial x_j}(\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj})=\frac{\partial}{\partial x_i}\left[(3\alpha_1+2\alpha_2)\chi\right].$$

$$\rho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\nabla^2 v_i+\frac{\partial}{\partial x_j}(\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj})$$

$$\frac{\partial}{\partial x_i}\left[\rho\frac{\partial\phi}{\partial t}+\frac{\rho}{2}\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_j}+p-\hat\beta\chi\right]=0$$

$$\hat\beta=3\alpha_1+2\alpha_2\geq 0$$

$$\rho\frac{\partial\phi}{\partial t}+\frac{\rho}{2}|\nabla\phi|^2+p-\hat\beta\chi=C(t).$$

$$T_{ij}=-\left[C+\hat\beta\chi-\frac{\rho}{2}|\nabla\phi|^2-\rho\frac{\partial\phi}{\partial t}\right]\delta_{ij}+\left[\mu+\alpha_1\left(\frac{\partial}{\partial t}+\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\right)\right]A_{ij}+(\alpha_1+\alpha_2)A_{ik}A_{kj}.$$

2 Potential flow over a sphere

The unit vectors in the spherical coordinates (r, θ, φ) are given by

$$\boldsymbol{e} = \cos\varphi \boldsymbol{e}_1 + \sin\varphi \boldsymbol{e}_2, \quad \boldsymbol{e}_r = \sin\theta\cos\varphi \boldsymbol{e}_1 + \sin\theta\sin\varphi \boldsymbol{e}_2 + \cos\theta \boldsymbol{e}_3 = \sin\theta \boldsymbol{e} + \cos\theta \boldsymbol{e}_3,$$

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \cos\theta\cos\varphi \boldsymbol{e}_1 + \cos\theta\sin\varphi \boldsymbol{e}_2 - \sin\theta \boldsymbol{e}_3 = \boldsymbol{e}_\theta$$

$$\frac{\partial \boldsymbol{e}_r}{\partial \varphi} = -\sin\theta\sin\varphi \boldsymbol{e}_1 + \sin\theta\cos\varphi \boldsymbol{e}_2 = \sin\theta \frac{\partial \boldsymbol{e}}{\partial \varphi}, \quad \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r$$

The potential is given by

$$\phi = -U\cos\theta\left(r + \frac{a^3}{2r^2}\right)$$

whence the velocity $\boldsymbol{v} = \nabla\phi$

$$(v_r,\, v_\theta,\, v_\varphi) = \left(\frac{\partial\phi}{\partial r},\, \frac{1}{r}\frac{\partial\phi}{\partial\theta},\, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right) = (v_r,\, v_\theta,\, 0)$$

$$v_r = \frac{\partial\phi}{\partial r} = -U\cos\theta\left(1 - \frac{a^3}{r^3}\right), \quad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\sin\theta\left(1 + \frac{a^3}{2r^3}\right),$$

$$\frac{\partial v_r}{\partial r} = -U\cos\theta\frac{3a^3}{r^4}, \quad \frac{1}{r}\frac{\partial v_r}{\partial\theta} = U\sin\theta\left(\frac{1}{r} - \frac{a^3}{r^4}\right),$$

$$\frac{\partial v_\theta}{\partial r} = -U\sin\theta\frac{3a^3}{2r^4}, \quad \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} = U\cos\theta\left(\frac{1}{r} + \frac{a^3}{2r^4}\right),$$

$$-\frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial\theta} = -U\sin\theta\left(\frac{1}{r} + \frac{a^3}{2r^4}\right) + U\sin\theta\left(\frac{1}{r} - \frac{a^3}{r^4}\right) = -U\sin\theta\frac{3a^3}{2r^4},$$

$$\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r} = U\cos\theta\left(\frac{1}{r} + \frac{a^3}{2r^4}\right) - U\cos\theta\left(\frac{1}{r} - \frac{a^3}{r^4}\right) = U\cos\theta\frac{3a^3}{2r^4},$$

$$\frac{v_r}{r} + \frac{v_\theta}{r}\cot\theta = -U\cos\theta\left(\frac{1}{r} - \frac{a^3}{r^4}\right) + U\sin\theta\left(\frac{1}{r} + \frac{a^3}{2r^4}\right)\cot\theta = U\cos\theta\frac{3a^3}{2r^4},$$

The gradient of the velocity is given by

$$(L_{ij}) = \begin{pmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{\partial v_\theta}{\partial r} & \dfrac{\partial v_\varphi}{\partial r} \\ -\dfrac{v_\theta}{r} + \dfrac{1}{r}\dfrac{\partial v_r}{\partial\theta} & \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial\theta} + \dfrac{v_r}{r} & \dfrac{1}{r}\dfrac{\partial v_\varphi}{\partial\theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial v_r}{\partial\varphi} - \dfrac{v_\varphi}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial v_\theta}{\partial\varphi} - \dfrac{v_\varphi}{r}\cot\theta & \dfrac{1}{r\sin\theta}\dfrac{\partial v_\varphi}{\partial\varphi} + \dfrac{v_r}{r} + \dfrac{v_\theta}{r}\cot\theta \end{pmatrix}$$

$$(L_{ij}) = \frac{3a^3U}{2r^4}\begin{pmatrix} -2\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}$$

The strain tensor is given by

$$A_{ij} = L_{ij} + L_{ji} = 2L_{ij},$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

$$c_{11} = \frac{9a^6U^2}{r^8}\left(4\cos^2\theta + \sin^2\theta\right), \quad c_{12} = \frac{9a^6U^2}{r^8}\sin\theta\cos\theta, \quad c_{13} = 0,$$

$$c_{21}=\frac{9a^6U^2}{r^8}\sin\theta\cos\theta,\quad c_{22}=\frac{9a^6U^2}{r^8},\quad c_{23}=0,$$

$$c_{31}=c_{32}=0,\quad c_{33}=\frac{9a^6U^2}{r^8}\cos^2\theta,$$

$$\boldsymbol{A}^2=(A_{ik}A_{kj})=\frac{9a^6U^2}{r^8}\begin{pmatrix}4\cos^2\theta+\sin^2\theta & \sin\theta\cos\theta & 0\\ \sin\theta\cos\theta & 1 & 0\\ 0 & 0 & \cos^2\theta\end{pmatrix}$$

$$\frac{1}{4}\mathrm{tr}\boldsymbol{A}^2=\frac{9U^2}{4a^2}\left(5\cos^2\theta+\sin^2\theta+1\right)=\frac{9U^2}{4a^2}\left(4\cos^2\theta+2\right)=\frac{9U^2}{2a^2}\left(1+2\cos^2\theta\right)$$

$$(3\alpha_1+2\alpha_2)\,\frac{9}{2}=\left(\frac{27}{2}\alpha_1+9\alpha_2\right)$$

$$\begin{aligned}p&=p_\infty+\frac{\rho}{2}U^2-\rho\frac{\partial\phi}{\partial t}-\frac{1}{2}\rho\left|\nabla\phi\right|^2+\hat{\beta}\chi\\&=p_\infty+\frac{\rho}{2}U^2\left(1-\frac{9}{4}\sin^2\theta\right)+\frac{U^2}{a^2}\left(\frac{27}{2}\alpha_1+9\alpha_2\right)\left(1+2\cos^2\theta\right)\end{aligned}$$

The normal component of $(\boldsymbol{v}\cdot\nabla)\,L_{ij}$ is

$$\boldsymbol{e}_1\boldsymbol{e}_1\left(v_r\frac{\partial c_{11}}{\partial r}+\frac{v_\theta}{r}\frac{\partial c_{11}}{\partial\theta}\right)+c_{12}\frac{v_\theta}{r}\boldsymbol{e}_1\frac{\partial\boldsymbol{e}_2}{\partial\theta}+c_{21}\frac{v_\theta}{r}\boldsymbol{e}_1\frac{\partial\boldsymbol{e}_2}{\partial\theta}=\boldsymbol{e}_1\boldsymbol{e}_1\left[v_r\frac{\partial c_{11}}{\partial r}+\frac{v_\theta}{r}\frac{\partial c_{11}}{\partial\theta}-\frac{v_\theta}{r}\left(c_{12}+c_{21}\right)\right]$$

$$=\boldsymbol{e}_1\boldsymbol{e}_1\left[v_r\frac{\partial^2 v_r}{\partial r^2}+\frac{v_\theta}{r}\frac{\partial^2 v_r}{\partial\theta\partial r}-\frac{v_\theta}{r}\left(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}+\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\right]=\boldsymbol{e}_1\boldsymbol{e}_1\frac{6a^3U^2}{r^5}\left[-2\cos^2\theta\left(1-\frac{a^3}{r^3}\right)+\sin^2\theta\left(1+\frac{a^3}{2r^3}\right)\right]$$

$$\left[\frac{6a^3U^2}{r^5}\left[-2\cos^2\theta\left(1-\frac{a^3}{r^3}\right)+\sin^2\theta\left(1+\frac{a^3}{2r^3}\right)\right]\right]_{r=a}=\frac{9U^2}{a^2}\sin^2\theta$$

$$\left[n_in_j\left(\boldsymbol{v}\cdot\nabla\right)A_{ij}\right]_{r=a}=\left[n_in_j\left(\boldsymbol{v}\cdot\nabla\right)2L_{ij}\right]_{r=a}=\frac{18U^2}{a^2}\sin^2\theta$$

$$\begin{aligned}n_iT_{ij}n_j&=-p+n_in_j\left[\left[\mu+\alpha_1\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\right]A_{ij}+\left(\alpha_1+\alpha_2\right)A_{ik}A_{kj}\right]\\&=-p+\mu n_in_jA_{ij}+n_in_j\alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij}+\left(\alpha_1+\alpha_2\right)n_in_jA_{ik}A_{kj}\\&=-p+\mu\left(-\frac{6a^3U}{r^4}\cos\theta\right)+n_in_j\alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij}+\left(\alpha_1+\alpha_2\right)\frac{9a^6U^2}{r^8}\left(4\cos^2\theta+\sin^2\theta\right)\end{aligned}$$

$$\begin{aligned}n_in_j\left(T_{ij}\right)_{r=a}&=-p_\infty-\left(\frac{27}{2}\alpha_1+9\alpha_2\right)\frac{U^2}{a^2}\left(1+2\cos^2\theta\right)-\frac{\rho}{2}U^2\left(1-\frac{9}{4}\sin^2\theta\right)\\&-\frac{6U}{a}\mu\cos\theta+n_in_j\alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij}+\left(\alpha_1+\alpha_2\right)\frac{9U^2}{a^2}\left(4\cos^2\theta+\sin^2\theta\right)\\&=-p_\infty-\frac{\rho}{2}U^2\left(1-\frac{9}{4}\sin^2\theta\right)-\frac{6U}{a}\mu\cos\theta+n_in_j\alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij}\\&+\frac{U^2}{a^2}\alpha_1\left(36\cos^2\theta+9\sin^2\theta-\frac{27}{2}-27\cos^2\theta\right)+\frac{U^2}{a^2}\alpha_2\left(36\cos^2\theta+9\sin^2\theta-9-18\cos^2\theta\right)\\&=-p_\infty-\frac{\rho}{2}U^2\left(1-\frac{9}{4}\sin^2\theta\right)-\frac{6U}{a}\mu\cos\theta+\frac{18U^2}{a^2}\alpha_1\sin^2\theta-\frac{U^2}{a^2}\frac{9}{2}\alpha_1+\frac{U^2}{a^2}\alpha_2 9\cos^2\theta\end{aligned}$$

$$T_{rr}^*=\frac{T_{rr}+p_\infty}{\rho U^2/2}=-\left(1-\frac{9}{4}\sin^2\theta\right)-\frac{12}{R_e}\cos\theta+\frac{\alpha_1}{\rho a^2}\left(36\sin^2\theta-9\right)+\frac{\alpha_2}{\rho a^2}18\cos^2\theta$$

$R_e=\rho Ua/\mu$.

$$T_{rr}^*=-1\mp\frac{12}{R_e}+\frac{9}{\rho a^2}\left(2\alpha_2-\alpha_1\right)\quad\text{at }\theta=0,\pi$$

3 Potential flow over an ellipse

For flows over an ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with $c^2=a^2-b^2$, the complex velocity potential w is given by

$$w=-\frac{1}{2}U\left(a+b\right)\left[\frac{e^{-\imath\alpha}\left(z+\sqrt{z^2-c^2}\right)}{a+b}+\frac{e^{\imath\alpha}\left(z-\sqrt{z^2-c^2}\right)}{a-b}\right]$$

$$w=\phi+\imath\psi,\quad z=x+\imath y,\quad \bar{z}=x-\imath y$$

$$\boldsymbol{v}=\nabla\phi=u\boldsymbol{e}_1+v\boldsymbol{e}_2,\quad \boldsymbol{v}=\nabla\times(\boldsymbol{e}_3\psi)=\boldsymbol{e}_1\frac{\partial\psi}{\partial y}-\boldsymbol{e}_2\frac{\partial\psi}{\partial x},$$

$$\nabla\cdot\boldsymbol{v}=0,\quad \nabla\times\boldsymbol{v}=\boldsymbol{0},\quad \nabla^2\phi=0,\quad \nabla^2\psi=0.$$

$$\frac{\mathrm{d}w}{\mathrm{d}z}=\frac{\partial}{\partial x}\left(\phi+\imath\psi\right)=u-\imath v=\frac{1}{\imath}\frac{\partial}{\partial y}\left(\phi+\imath\psi\right)=-\imath v+u$$

$$u-\imath v=\frac{\mathrm{d}w}{\mathrm{d}z},\quad u+\imath v=\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}$$

$$u=\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}z}+\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right),\quad v=\frac{\imath}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}z}-\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)$$

$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=\mathrm{d}x+\imath\mathrm{d}y,\quad \mathrm{d}\bar{z}=\frac{\partial\bar{z}}{\partial x}\mathrm{d}x+\frac{\partial\bar{z}}{\partial y}\mathrm{d}y=\mathrm{d}x-\imath\mathrm{d}y,$$

$$\mathrm{d}x=\frac{\partial x}{\partial z}\mathrm{d}z+\frac{\partial x}{\partial\bar{z}}\mathrm{d}\bar{z},\quad \mathrm{d}y=\frac{\partial y}{\partial z}\mathrm{d}z+\frac{\partial y}{\partial\bar{z}}\mathrm{d}\bar{z},$$

$$\frac{\partial}{\partial x}=\frac{\partial z}{\partial x}\frac{\partial}{\partial z}+\frac{\partial\bar{z}}{\partial x}\frac{\partial}{\partial\bar{z}}=\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar{z}},\quad \frac{\partial}{\partial y}=\frac{\partial z}{\partial y}\frac{\partial}{\partial z}+\frac{\partial\bar{z}}{\partial y}\frac{\partial}{\partial\bar{z}}=\imath\frac{\partial}{\partial z}-\imath\frac{\partial}{\partial\bar{z}},$$

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\imath\frac{\partial}{\partial y}\right),\quad \frac{\partial}{\partial\bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\imath\frac{\partial}{\partial y}\right),$$

$$\frac{\partial u}{\partial x}=\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar{z}}\right)\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}z}+\frac{\partial\bar{w}}{\partial\bar{z}}\right)=\frac{1}{2}\left(\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2}\right),$$

$$\frac{\partial u}{\partial y}=\imath\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial\bar{z}}\right)\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}z}+\frac{\partial\bar{w}}{\partial\bar{z}}\right)=\frac{\imath}{2}\left(\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2}\right),$$

$$\frac{\partial v}{\partial x}=\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial\bar{z}}\right)\frac{\imath}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}z}-\frac{\partial\bar{w}}{\partial\bar{z}}\right)=\frac{\imath}{2}\left(\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2}\right)=\frac{\partial u}{\partial y},$$

$$\frac{\partial v}{\partial y}=\imath\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial\bar{z}}\right)\frac{\imath}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}z}-\frac{\partial\bar{w}}{\partial\bar{z}}\right)=-\frac{1}{2}\left(\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2}\right)=-\frac{\partial u}{\partial x}.$$

$$\boldsymbol{L}=\nabla\boldsymbol{v}=\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{pmatrix}$$

$$\boldsymbol{A}=\boldsymbol{L}+\boldsymbol{L}^T=\begin{pmatrix}2\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y}\end{pmatrix}=\begin{pmatrix}n & s\\ s & -n\end{pmatrix}$$

$$n=2\frac{\partial u}{\partial x}=-2\frac{\partial v}{\partial y}=\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2},\quad s=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=\imath\left(\frac{\mathrm{d}^2w}{\mathrm{d}z^2}-\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2}\right)$$

$$\boldsymbol{A}^2 = \begin{pmatrix} n^2+s^2 & 0 \\ 0 & n^2+s^2 \end{pmatrix}, \quad \mathrm{tr}\boldsymbol{A}^2 = 2\left(n^2+s^2\right)$$

$$\chi = \frac{\partial^2\phi}{\partial x_j \partial x_k}\frac{\partial^2\phi}{\partial x_j \partial x_k} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y}\right)^2 = \frac{1}{2}\left(n^2+s^2\right).$$

$$p = p_\infty + \frac{\rho}{2}U^2 - \frac{\rho}{2}\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}} + \frac{1}{2}\left(3\alpha_1 + 2\alpha_2\right)\left(n^2+s^2\right)$$

$$T_{ij} = \left[-p_\infty - \frac{\rho}{2}U^2 + \frac{\rho}{2}\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}} - \frac{1}{2}\left(3\alpha_1 + 2\alpha_2\right)\left(n^2+s^2\right)\right]\delta_{ij}$$

$$+\mu A_{ij} + \alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij} + \left(\alpha_1+\alpha_2\right)\left(n^2+s^2\right)\delta_{ij}$$

$$\left(\boldsymbol{v}\cdot\nabla\right)A_{ij} = u\frac{\partial}{\partial x}A_{ij} + v\frac{\partial}{\partial y}A_{ij} = u\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}\right)\begin{pmatrix} n & s \\ s & -n \end{pmatrix} + v\imath\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)\begin{pmatrix} n & s \\ s & -n \end{pmatrix}$$

$$k \equiv \left(\frac{\mathrm{d}n}{\mathrm{d}z}+\frac{\mathrm{d}n}{\mathrm{d}\bar{z}}\right) = \frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \frac{\mathrm{d}^3\bar{w}}{\mathrm{d}\bar{z}^3}, \quad q = \left(\frac{\mathrm{d}s}{\mathrm{d}z}+\frac{\mathrm{d}s}{\mathrm{d}\bar{z}}\right) = \imath\left(\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} - \frac{\mathrm{d}^3\bar{w}}{\mathrm{d}\bar{z}^3}\right)$$

$$\imath\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)n = \imath\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}+\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2}\right) = \imath\left(\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} - \frac{\mathrm{d}^3\bar{w}}{\mathrm{d}\bar{z}^3}\right) = q$$

$$\imath\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)\imath\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}-\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2}\right) = -\left(\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \frac{\mathrm{d}^3\bar{w}}{\mathrm{d}\bar{z}^3}\right) = -k$$

$$\left(\boldsymbol{v}\cdot\nabla\right)A_{ij} = u\begin{pmatrix} k & q \\ q & -k \end{pmatrix} + v\begin{pmatrix} q & -k \\ -k & -q \end{pmatrix}$$

$$n_i T_{ij} n_j = -p_\infty - \frac{\rho}{2}U^2 + \frac{\rho}{2}\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}} - \frac{1}{2}\alpha_1\left(n^2+s^2\right)$$

$$+\left(\mu n + \alpha_1 uk + \alpha_1 vq\right)\left(n_x^2 - n_y^2\right) + \left(\mu s + \alpha_1 uq - \alpha_1 vk\right)2n_x n_y$$

$$\nabla\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right) = \frac{2x}{a^2}\boldsymbol{e}_x + \frac{2y}{b^2}\boldsymbol{e}_y \quad\rightarrow\quad \boldsymbol{n} = \left(n_x,\, n_y\right) = \left(\frac{x}{a^2}\boldsymbol{e}_x + \frac{y}{b^2}\boldsymbol{e}_y\right)\Big/\sqrt{\left(\frac{x}{a^2}\right)^2+\left(\frac{y}{b^2}\right)^2}$$

$$w = -\frac{1}{2}U\left(a+b\right)\left[\frac{e^{-\imath\alpha}}{a+b}\left(z+\sqrt{z^2-c^2}\right) + \frac{e^{\imath\alpha}}{a-b}\left(z-\sqrt{z^2-c^2}\right)\right]$$

$$\bar{w} = -\frac{1}{2}U\left(a+b\right)\left[\frac{e^{\imath\alpha}}{a+b}\left(\bar{z}+\sqrt{\bar{z}^2-c^2}\right) + \frac{e^{-\imath\alpha}}{a-b}\left(\bar{z}-\sqrt{\bar{z}^2-c^2}\right)\right]$$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = -\frac{1}{2}U\left(a+b\right)\left[\frac{e^{-\imath\alpha}}{a+b}\left(1+z\left(z^2-c^2\right)^{-\frac{1}{2}}\right) + \frac{e^{\imath\alpha}}{a-b}\left(1-z\left(z^2-c^2\right)^{-\frac{1}{2}}\right)\right]$$

$$\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}} = -\frac{1}{2}U\left(a+b\right)\left[\frac{e^{\imath\alpha}}{a+b}\left(1+\bar{z}\left(\bar{z}^2-c^2\right)^{-\frac{1}{2}}\right) + \frac{e^{-\imath\alpha}}{a-b}\left(1-\bar{z}\left(\bar{z}^2-c^2\right)^{-\frac{1}{2}}\right)\right]$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = -\frac{1}{2}U\left(a+b\right)\left[\frac{e^{-\imath\alpha}}{a+b}\left(-c^2\left(z^2-c^2\right)^{-\frac{3}{2}}\right) + \frac{e^{\imath\alpha}}{a-b}c^2\left(z^2-c^2\right)^{-\frac{3}{2}}\right]$$

$$\frac{\mathrm{d}^2 \bar{w}}{\mathrm{d}\bar{z}^2} = -\frac{1}{2}U\left(a+b\right)\left[\frac{e^{\imath\alpha}}{a+b}\left(-c^2\left(\bar{z}^2-c^2\right)^{-\frac{3}{2}}\right) + \frac{e^{-\imath\alpha}}{a-b}c^2\left(\bar{z}^2-c^2\right)^{-\frac{3}{2}}\right]$$

$$\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} = -\frac{1}{2}U\left(a+b\right)\left[\frac{e^{-\imath\alpha}}{a+b}3c^2 z\left(z^2-c^2\right)^{-\frac{5}{2}} + \frac{e^{\imath\alpha}}{a-b}\left(-3c^2\right)z\left(z^2-c^2\right)^{-\frac{5}{2}}\right]$$

$$\frac{\mathrm{d}^3 \bar{w}}{\mathrm{d}\bar{z}^3} = -\frac{1}{2}U\left(a+b\right)\left[\frac{e^{\imath\alpha}}{a+b}3c^2 \bar{z}\left(\bar{z}^2-c^2\right)^{-\frac{5}{2}} + \frac{e^{-\imath\alpha}}{a-b}\left(-3c^2\right)\bar{z}\left(\bar{z}^2-c^2\right)^{-\frac{5}{2}}\right]$$

3.1 Normalization

In terms of a and U, the normaslization is made, and the normal stress is expressed as

$$\frac{n_i T_{ij} n_j + p_\infty}{\frac{\rho}{2}U^2} = -1 + \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}} - \frac{\alpha_1}{\rho a^2}\left(n^2+s^2\right)$$

$$+2\left(\frac{1}{R_e}n + \frac{\alpha_1}{\rho a^2}\left(uk+vq\right)\right)\left(n_x^2-n_y^2\right) + \left(\frac{1}{R_e}s + \frac{\alpha_1}{\rho a^2}\left(uq-vk\right)\right)4n_x n_y,$$

in which the normal vector is defined as

$$\nabla\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right) = \frac{2x}{a^2}\boldsymbol{e}_x + \frac{2y}{b^2}\boldsymbol{e}_y \quad \rightarrow \quad \boldsymbol{n} = (n_x,\, n_y) = \left(\frac{x}{a^2}\boldsymbol{e}_x + \frac{y}{b^2}\boldsymbol{e}_y\right)/\sqrt{\left(\frac{x}{a^2}\right)^2 + \left(\frac{y}{b^2}\right)^2}.$$

$$u = \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right), \quad v = \frac{\imath}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right),$$

$$n = 2\frac{\partial u}{\partial x} = \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2}, \quad s = 2\frac{\partial u}{\partial y} = \imath\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2}\right),$$

$$k = \frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \frac{\mathrm{d}^3\bar{w}}{\mathrm{d}\bar{z}^3}, \quad q = \imath\left(\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} - \frac{\mathrm{d}^3\bar{w}}{\mathrm{d}\bar{z}^3}\right),$$

$$w = -\frac{1}{2}\frac{(a+b)}{a}\left[\frac{e^{-\imath\alpha}}{a+b}\left(z+\sqrt{z^2-c^2}\right) + \frac{e^{\imath\alpha}}{a-b}\left(z-\sqrt{z^2-c^2}\right)\right]$$

$$\bar{w} = -\frac{1}{2}\frac{(a+b)}{a}\left[\frac{e^{\imath\alpha}}{a+b}\left(\bar{z}+\sqrt{\bar{z}^2-c^2}\right) + \frac{e^{-\imath\alpha}}{a-b}\left(\bar{z}-\sqrt{\bar{z}^2-c^2}\right)\right]$$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = -\frac{1}{2}\frac{(a+b)}{a}\left[\frac{e^{-\imath\alpha}}{a+b}\left(1+z\left(z^2-c^2\right)^{-\frac{1}{2}}\right) + \frac{e^{\imath\alpha}}{a-b}\left(1-z\left(z^2-c^2\right)^{-\frac{1}{2}}\right)\right]$$

$$\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}} = -\frac{1}{2}\frac{(a+b)}{a}\left[\frac{e^{\imath\alpha}}{a+b}\left(1+\bar{z}\left(\bar{z}^2-c^2\right)^{-\frac{1}{2}}\right) + \frac{e^{-\imath\alpha}}{a-b}\left(1-\bar{z}\left(\bar{z}^2-c^2\right)^{-\frac{1}{2}}\right)\right]$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = -\frac{1}{2}\frac{(a+b)}{a}\left[\frac{e^{-\imath\alpha}}{a+b}\left(-c^2\left(z^2-c^2\right)^{-\frac{3}{2}}\right) + \frac{e^{\imath\alpha}}{a-b}c^2\left(z^2-c^2\right)^{-\frac{3}{2}}\right]$$

$$\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2} = -\frac{1}{2}\frac{(a+b)}{a}\left[\frac{e^{\imath\alpha}}{a+b}\left(-c^2\left(\bar{z}^2-c^2\right)^{-\frac{3}{2}}\right) + \frac{e^{-\imath\alpha}}{a-b}c^2\left(\bar{z}^2-c^2\right)^{-\frac{3}{2}}\right]$$

$$\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} = -\frac{1}{2}\frac{(a+b)}{a}\left[\frac{e^{-\imath\alpha}}{a+b}3c^2 z\left(z^2-c^2\right)^{-\frac{5}{2}} + \frac{e^{\imath\alpha}}{a-b}\left(-3c^2\right)z\left(z^2-c^2\right)^{-\frac{5}{2}}\right]$$

$$\frac{\mathrm{d}^3\bar{w}}{\mathrm{d}\bar{z}^3} = -\frac{1}{2}\frac{(a+b)}{a}\left[\frac{e^{\imath\alpha}}{a+b}3c^2\bar{z}\left(\bar{z}^2-c^2\right)^{-\frac{5}{2}} + \frac{e^{-\imath\alpha}}{a-b}\left(-3c^2\right)\bar{z}\left(\bar{z}^2-c^2\right)^{-\frac{5}{2}}\right]$$

4 Flow past a circular cylinder with circulation

$$w = -U\left(z + \frac{a^2}{z}\right) - \imath\kappa \ln\left(\frac{z}{a}\right)$$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - \imath v = -U\left(1 - \frac{a^2}{z^2}\right) - \imath\kappa\frac{1}{z} = -U\left[1 - \frac{a^2}{r^2}(\cos 2\theta - \imath\sin 2\theta)\right] - \frac{\imath\kappa}{r}(\cos\theta - \imath\sin\theta)$$

$$= -U\left(1 - \frac{a^2}{r^2}\cos 2\theta\right) - \frac{\kappa}{r}\sin\theta + \imath\left[-U\frac{a^2}{r^2}\sin 2\theta - \frac{\kappa}{r}\cos\theta\right]$$

$$= -U\left(1 - \frac{a^2}{r^2}\cos 2\theta\right) - \frac{\kappa}{r}\sin\theta - \imath\left[\frac{Ua^2}{r^2}\sin 2\theta + \frac{\kappa}{r}\cos\theta\right]$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = -\frac{2Ua^2}{z^3} + \imath\kappa\frac{1}{z^2} = -\frac{2Ua^2}{r^3}(\cos 3\theta - \imath\sin 3\theta) + \frac{\imath\kappa}{r^2}(\cos 2\theta - \imath\sin 2\theta)$$

$$= -\frac{2Ua^2}{r^3}\cos 3\theta + \frac{\kappa}{r^2}\sin 2\theta + \imath\left(\frac{2Ua^2}{r^3}\sin 3\theta + \frac{\kappa}{r^2}\cos 2\theta\right)$$

$$n = \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2} = -\frac{4Ua^2}{r^3}\cos 3\theta + \frac{2\kappa}{r^2}\sin 2\theta$$

$$s = \imath\left(\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - \frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\bar{z}^2}\right) = -\frac{4Ua^2}{r^3}\sin 3\theta - \frac{2\kappa}{r^2}\cos 2\theta$$

$$\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} = \frac{6Ua^2}{z^4} - \frac{2\imath\kappa}{z^3} = \frac{6Ua^2}{r^4}(\cos 4\theta - \imath\sin 4\theta) - \frac{2\imath\kappa}{r^3}(\cos 3\theta - \imath\sin 3\theta)$$

$$= \frac{6Ua^2}{r^4}\cos 4\theta - \frac{2\kappa}{r^3}\sin 3\theta - \imath\left(\frac{6Ua^2}{r^4}\sin 4\theta + \frac{2\kappa}{r^3}\cos 3\theta\right)$$

$$k = \frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \frac{\mathrm{d}^3\bar{w}}{\mathrm{d}\bar{z}^3} = \frac{12Ua^2}{r^4}\cos 4\theta - \frac{4\kappa}{r^3}\sin 3\theta$$

$$q = \imath\left(\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} - \frac{\mathrm{d}^3\bar{w}}{\mathrm{d}\bar{z}^3}\right) = \frac{12Ua^2}{r^4}\sin 4\theta + \frac{4\kappa}{r^3}\cos 3\theta$$

$$n_i T_{ij} n_j = -p_\infty - \frac{\rho}{2}U^2 + \frac{\rho}{2}\frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}} - \frac{1}{2}\alpha_1\left(n^2 + s^2\right)$$
$$+ (\mu n + \alpha_1 uk + \alpha_1 vq)\left(n_x^2 - n_y^2\right) + (\mu s + \alpha_1 uq - \alpha_1 vk)\, 2n_x n_y$$

$$\frac{T_{nn} + p_\infty}{\frac{1}{2}\rho U^2} = -1 + \left[-U\left(1 - \frac{a^2}{r^2}\cos 2\theta\right) - \frac{\kappa}{r}\sin\theta\right]^2 + \left[\frac{Ua^2}{r^2}\sin 2\theta + \frac{\kappa}{r}\cos\theta\right]^2$$
$$-\frac{1}{2}\alpha_1\left[\left(\frac{-4Ua^2}{r^3}\cos 3\theta + \frac{2\kappa}{r^2}\sin 2\theta\right)^2 + \left(\frac{4Ua^2}{r^3}\sin 3\theta + \frac{2\kappa}{r^2}\cos 2\theta\right)^2\right]$$

$$= -1 + U^2\left(1 - \frac{a^2}{r^2}\cos 2\theta\right)^2 - 2U\left(1 - \frac{a^2}{r^2}\cos 2\theta\right)\frac{\kappa}{r}\sin\theta + \frac{\kappa^2}{r^2} + \frac{U^2a^4}{r^4}\sin^2 2\theta + 2\frac{Ua^2}{r^2}\sin 2\theta\frac{\kappa}{r}\cos\theta$$
$$-\frac{1}{2}\alpha_1\left(\frac{16U^2a^4}{r^6} - 2\frac{4Ua^2}{r^3}\frac{2\kappa}{r^2}\sin\theta + \frac{4\kappa^2}{r^4}\right)$$

A note on flows of a second order fluid around an ellipsoid

T.Funada, May 31, 2004 / ellipse-may31a.tex / printed July 16, 2004

1 "17.52 Translational motion of an ellipsoid" Milne-Thomson (1974)

Consider the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \tag{1}$$

moving in the negative direction of the x-axis with velocity U in a fluid at rest, whence the velocity potential ϕ for which $\boldsymbol{v}=\nabla\phi$ is given by

$$\phi=\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}}=xf_0, \tag{2}$$

where

$$\alpha_0=abc\int_0^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}}, \tag{3}$$

$$f_0\equiv f_0(\lambda)=\frac{abcU}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\left(a^2+\lambda\right)\sqrt{\left(a^2+\lambda\right)\left(b^2+\lambda\right)\left(c^2+\lambda\right)}},$$

with the ellipsoidal coordinates $(\lambda,\,\mu,\,\nu)=(\lambda_1,\,\lambda_2,\,\lambda_3)=\lambda_1\boldsymbol{e}_1+\lambda_2\boldsymbol{e}_2+\lambda_3\boldsymbol{e}_3$ which are three solutions of the following equation through the cartesian coordinates $(x,\,y,\,z)=x\boldsymbol{e}_x+y\boldsymbol{e}_y+z\boldsymbol{e}_z$

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1. \tag{4}$$

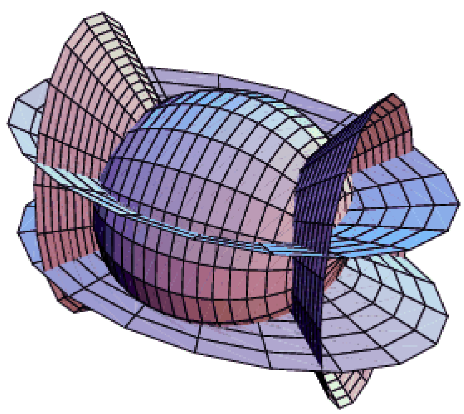

Fig.1 Confocal ellipsoidal coordinates.

1.1 Cartesian coordinates (x,y,z) fixed on the ellipsoid

In the cartesian coordinates (x,y,z) fixed on the ellipsoid, the potential is given by

$$\phi=Ux+xf_0. \tag{1.1}$$

The ellipsoid with a papameter θ is expressed as

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1, \tag{1.2}$$

which then gives

$$\begin{aligned}&x^2\left(b^2+\theta\right)\left(c^2+\theta\right)+y^2\left(a^2+\theta\right)\left(c^2+\theta\right)+z^2\left(a^2+\theta\right)\left(b^2+\theta\right)\\&-\left(a^2+\theta\right)\left(b^2+\theta\right)\left(c^2+\theta\right)=-\left(\theta-\lambda\right)\left(\theta-\mu\right)\left(\theta-\nu\right)\equiv f(\theta),\end{aligned} \tag{1.3}$$

thus

$$f(\lambda) = f(\mu) = f(\nu) = 0, \tag{1.4}$$

$$\begin{aligned} f(\lambda) &= -\lambda^3 - \left(a^2+b^2+c^2-x^2-y^2-z^2\right)\lambda^2 \\ &\quad - \left(a^2b^2+b^2c^2+c^2a^2-x^2\left(b^2+c^2\right)-y^2\left(c^2+a^2\right)-z^2\left(a^2+b^2\right)\right)\lambda \\ &\quad -a^2b^2c^2+x^2b^2c^2+y^2c^2a^2+z^2a^2b^2=0 \\ &= -\lambda^3 - A_2\lambda^2 - A_1\lambda - A_0 = 0, \end{aligned} \tag{1.5}$$

$$x^2\left(b^2-a^2\right)\left(c^2-a^2\right) = \left(a^2+\lambda\right)\left(a^2+\mu\right)\left(a^2+\nu\right) = f(-a^2) \quad \rightarrow \quad x^2 = \frac{\left(a^2+\lambda\right)\left(a^2+\mu\right)\left(a^2+\nu\right)}{(b^2-a^2)(c^2-a^2)}. \tag{1.6}$$

The velocity $\boldsymbol{v} = (u, v, w) = \nabla\phi$ in the frame (x, y, z) is given by

$$u = U + \frac{\partial(xf_0)}{\partial x} = U + f_0 + x\frac{\partial\lambda}{\partial x}\frac{\partial f_0}{\partial\lambda} = U + f_0 + x\frac{\partial\lambda}{\partial x}f_1, \tag{1.7}$$

$$v = \frac{\partial(xf_0)}{\partial y} = x\frac{\partial\lambda}{\partial y}\frac{\partial f_0}{\partial\lambda} = x\frac{\partial\lambda}{\partial y}f_1, \quad w = \frac{\partial(xf_0)}{\partial z} = x\frac{\partial\lambda}{\partial z}\frac{\partial f_0}{\partial\lambda} = x\frac{\partial\lambda}{\partial z}f_1, \tag{1.8}$$

where $f_1 \equiv \partial f_0/\partial\lambda$ and the relevant expressions will be shown in Appendix.

1.2 Newtonian fluid

For irrotational flows of Newtonian fluid, we have

$$\nabla\times\boldsymbol{v} = \nabla\times\nabla\phi = \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}, \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) = 0, \tag{1.9}$$

$$(L_{ij}) = \left(\frac{\partial^2\phi}{\partial x_i\partial x_j}\right) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix}, \tag{1.10}$$

$$(A_{ij}) = (L_{ij}+L_{ji}) = 2\left(\frac{\partial^2\phi}{\partial x_i\partial x_j}\right) = \begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{pmatrix} = 2\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix}. \tag{1.11}$$

On the boundary of the ellipsoid, the normal vector $\boldsymbol{n}$ is defined as

$$\nabla\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right) = 2\left(\frac{x}{a^2}\boldsymbol{e}_x+\frac{y}{b^2}\boldsymbol{e}_y+\frac{z}{c^2}\boldsymbol{e}_z\right), \tag{1.12}$$

$$\boldsymbol{n} = n_x\boldsymbol{e}_x+n_y\boldsymbol{e}_y+n_z\boldsymbol{e}_z = \left(\frac{x}{a^2}\boldsymbol{e}_x+\frac{y}{b^2}\boldsymbol{e}_y+\frac{z}{c^2}\boldsymbol{e}_z\right)/\sqrt{\left(\frac{x}{a^2}\right)^2+\left(\frac{y}{b^2}\right)^2+\left(\frac{z}{c^2}\right)^2}. \tag{1.13}$$

The boundary conditions at the boundary are given by the kinematic condition

$$n_iv_i = 0, \tag{1.14}$$

and by the normal stress balance for the stress tensor $T_{ij} = -p\delta_{ij} + \mu A_{ij}$

$$n_iT_{ij}n_j = \gamma\nabla\cdot\boldsymbol{n}, \tag{1.15}$$

where γ is the surface tension coefficient.

2 Second order fluid

For the second order fluid, the gradient of velocity, the lower convective derivative and the stress tensor are given by

$$L_{ij} = \frac{\partial v_i}{\partial x_j}, \quad A_{ij} = L_{ij} + L_{ji}, \quad B_{ij} = \frac{\partial A_{ij}}{\partial t} + v_k \frac{\partial A_{ij}}{\partial x_k} + A_{ik}L_{kj} + L_{ik}A_{kj}, \tag{2.1}$$

$$T_{ij} = -p\delta_{ij} + \mu A_{ij} + \alpha_1 B_{ij} + \alpha_2 A_{ik}A_{kj}. \tag{2.2}$$

Using the velocity potential for which

$$\boldsymbol{v} = \nabla\phi, \quad v_i = \frac{\partial \phi}{\partial x_i}, \quad \nabla\cdot\boldsymbol{v} = \nabla^2\phi = 0, \quad L_{ij} = \frac{\partial^2\phi}{\partial x_i \partial x_j}, \quad A_{ij} = 2\frac{\partial^2\phi}{\partial x_i \partial x_j}, \quad C_{ij} = A_{ik}A_{kj}, \tag{2.3}$$

we have the following expressions

$$\begin{aligned} B_{ij} &= 2\frac{\partial^3\phi}{\partial t\partial x_i\partial x_j} + 2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j} + 2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j} + 2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j} \\ &= 2\frac{\partial^3\phi}{\partial t\partial x_i\partial x_j} + 2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j} + 4\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}, \end{aligned} \tag{2.4}$$

$$\frac{\partial B_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j} + 4\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right], \quad \frac{\partial T_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial A_{ij}}{\partial x_j} + \alpha_1\frac{\partial B_{ij}}{\partial x_j} + \alpha_2\frac{\partial\left(A_{ik}A_{kj}\right)}{\partial x_j}. \tag{2.5}$$

Thus we have the following expressions:

$$\chi \equiv \frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}, \quad \frac{\partial}{\partial x_j}\left(v_k\frac{\partial A_{ij}}{\partial x_k}\right) = \frac{\partial\chi}{\partial x_i}, \quad \frac{\partial}{\partial x_j}\left(A_{ik}L_{kj}\right) = \frac{\partial\chi}{\partial x_i}, \quad \frac{\partial}{\partial x_j}\left(A_{ik}A_{kj}\right) = 2\frac{\partial\chi}{\partial x_i}, \tag{2.6}$$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_k}2\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial^2\phi}{\partial x_k\partial x_j}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right), \quad \frac{\partial}{\partial x_j}\left(2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}\right),$$

$$\frac{\partial}{\partial x_j}\left(2\frac{\partial^2\phi}{\partial x_i\partial x_k}2\frac{\partial^2\phi}{\partial x_k\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(2\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}\right), \quad \frac{\partial}{\partial x_j}\left(\alpha_1 B_{ij} + \alpha_2 A_{ik}A_{kj}\right) = \frac{\partial}{\partial x_i}\left[\left(3\alpha_1 + 2\alpha_2\right)\chi\right].$$

The equation of motion is given by

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 v_i + \frac{\partial}{\partial x_j}\left(\alpha_1 B_{ij} + \alpha_2 A_{ik}A_{kj}\right), \tag{2.7}$$

whence the Bernoulli function is expressed as

$$\frac{\partial}{\partial x_i}\left[\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_j} + p - \hat{\beta}\chi\right] = 0 \ \text{ with } \hat{\beta} = 3\alpha_1 + 2\alpha_2 \geq 0, \tag{2.8}$$

$$\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}\left|\nabla\phi\right|^2 + p - \hat{\beta}\chi = C(t) = \frac{\rho}{2}U^2 + p_\infty. \tag{2.9}$$

The stress tensor is given by

$$T_{ij} = -\left[C + \hat{\beta}\chi - \frac{\rho}{2}\left|\nabla\phi\right|^2 - \rho\frac{\partial\phi}{\partial t}\right]\delta_{ij} + \left[\mu + \alpha_1\left(\frac{\partial}{\partial t} + \frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\right)\right]A_{ij} + \left(\alpha_1 + \alpha_2\right)A_{ik}A_{kj}, \tag{2.10}$$

which has the normal component at the boundary

$$n_i n_j T_{ij} = -\left[C + \hat{\beta}\chi - \frac{\rho}{2}\left|\nabla\phi\right|^2\right] + \mu n_i n_j A_{ij} + \alpha_1 n_i n_j\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}A_{ij} + \left(\alpha_1 + \alpha_2\right)n_i n_j A_{ik}A_{kj}. \tag{2.11}$$

The term is expressed as

$$\frac{\partial \phi}{\partial x_k}\frac{\partial}{\partial x_k}A_{ij} = u\frac{\partial A_{ij}}{\partial x} + v\frac{\partial A_{ij}}{\partial y} + w\frac{\partial A_{ij}}{\partial z}, \tag{2.12}$$

for which

$$u\frac{\partial A_{11}}{\partial x} + v\frac{\partial A_{11}}{\partial y} + w\frac{\partial A_{11}}{\partial z} = 2u\frac{\partial^2 u}{\partial x^2} + 2v\frac{\partial^2 u}{\partial x\partial y} + 2w\frac{\partial^2 u}{\partial x\partial z}, \tag{2.13}$$

$$u\frac{\partial A_{12}}{\partial x} + v\frac{\partial A_{12}}{\partial y} + w\frac{\partial A_{12}}{\partial z} = 2u\frac{\partial^2 u}{\partial x\partial y} + 2v\frac{\partial^2 u}{\partial y^2} + 2w\frac{\partial^2 u}{\partial y\partial z}, \tag{2.14}$$

$$u\frac{\partial A_{13}}{\partial x} + v\frac{\partial A_{13}}{\partial y} + w\frac{\partial A_{13}}{\partial z} = 2u\frac{\partial^2 u}{\partial x\partial z} + 2v\frac{\partial^2 u}{\partial y\partial z} + 2w\frac{\partial^2 u}{\partial z^2}, \tag{2.15}$$

$$u\frac{\partial A_{22}}{\partial x} + v\frac{\partial A_{22}}{\partial y} + w\frac{\partial A_{22}}{\partial z} = 2u\frac{\partial v}{\partial x\partial y} + 2v\frac{\partial^2 v}{\partial y^2} + 2w\frac{\partial^2 v}{\partial y\partial z}, \tag{2.16}$$

$$u\frac{\partial A_{23}}{\partial x} + v\frac{\partial A_{23}}{\partial y} + w\frac{\partial A_{23}}{\partial z} = 2u\frac{\partial^2 v}{\partial x\partial z} + 2v\frac{\partial^2 v}{\partial y\partial z} + 2w\frac{\partial^2 v}{\partial z^2}, \tag{2.17}$$

$$u\frac{\partial A_{33}}{\partial x} + v\frac{\partial A_{33}}{\partial y} + w\frac{\partial A_{33}}{\partial z} = 2u\frac{\partial^2 w}{\partial x\partial z} + 2v\frac{\partial^2 w}{\partial y\partial z} + 2w\frac{\partial^2 w}{\partial z^2}. \tag{2.18}$$

The stress tensor and the second order term are

$$(A_{ij}) = \begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{pmatrix} = 2\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} = 2\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix}, \tag{2.19}$$

$$(A_{ik}A_{kj}) = (A_{ik}A_{kj})^T = (C_{ij}) = 4\left(\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right) = 4\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$

$$= 4\begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}\right) & \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}\right) & \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}\right) + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}\right) & \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}\right) & \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial x}\right) + \left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial x}\right) & \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial y}\right) & \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial z}\right) \end{pmatrix}$$

$$= 4\begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 & \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) & \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) & \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 & \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right) & \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\right) & \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \end{pmatrix}$$

$$= 4\begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 & \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 & \frac{\partial v}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} & \frac{\partial v}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} & \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \end{pmatrix}. \tag{2.20}$$

$$\chi \equiv \frac{\partial^2\phi}{\partial x_j \partial x_k}\frac{\partial^2\phi}{\partial x_j \partial x_k} = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 + 2\left(\frac{\partial^2\phi}{\partial x\partial z}\right)^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2\phi}{\partial y\partial z}\right)^2 + \left(\frac{\partial^2\phi}{\partial z^2}\right)^2$$
$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2$$
$$= \frac{1}{4}\text{trace}\left(A_{ik}A_{kj}\right) = \frac{1}{4}\text{trace}\left(C_{ij}\right) \quad (2.21)$$

$$n_i n_j A_{ij} = n_x^2 A_{11} + 2n_x n_y A_{12} + 2n_x n_z A_{13} + n_y^2 A_{22} + 2n_y n_z A_{23} + n_z^2 A_{33}$$
$$= 2\left[n_x^2\frac{\partial u}{\partial x} + 2n_x n_y\frac{\partial u}{\partial y} + 2n_x n_z\frac{\partial u}{\partial z} + n_y^2\frac{\partial v}{\partial y} + 2n_y n_z\frac{\partial v}{\partial z} + n_z^2\frac{\partial w}{\partial z}\right] \quad (2.22)$$

$$n_i n_j \frac{\partial\phi}{\partial x_k}\frac{\partial A_{ij}}{\partial x_k} = n_x^2\left(u\frac{\partial A_{11}}{\partial x} + v\frac{\partial A_{11}}{\partial y} + w\frac{\partial A_{11}}{\partial z}\right) + 2n_x n_y\left(u\frac{\partial A_{12}}{\partial x} + v\frac{\partial A_{12}}{\partial y} + w\frac{\partial A_{12}}{\partial z}\right)$$
$$+2n_x n_z\left(u\frac{\partial A_{13}}{\partial x} + v\frac{\partial A_{13}}{\partial y} + w\frac{\partial A_{13}}{\partial z}\right) + n_y^2\left(u\frac{\partial A_{22}}{\partial x} + v\frac{\partial A_{22}}{\partial y} + w\frac{\partial A_{22}}{\partial z}\right)$$
$$+2n_y n_z\left(u\frac{\partial A_{23}}{\partial x} + v\frac{\partial A_{23}}{\partial y} + w\frac{\partial A_{23}}{\partial z}\right) + n_z^2\left(u\frac{\partial A_{33}}{\partial x} + v\frac{\partial A_{33}}{\partial y} + w\frac{\partial A_{33}}{\partial z}\right)$$
$$= 2n_x^2\left(u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial x\partial y} + w\frac{\partial^2 u}{\partial x\partial z}\right) + 4n_x n_y\left(u\frac{\partial^2 u}{\partial x\partial y} + v\frac{\partial^2 u}{\partial y^2} + w\frac{\partial^2 u}{\partial y\partial z}\right)$$
$$+4n_x n_z\left(u\frac{\partial^2 u}{\partial x\partial z} + v\frac{\partial^2 u}{\partial y\partial z} + w\frac{\partial^2 u}{\partial z^2}\right) + 2n_y^2\left(u\frac{\partial v}{\partial x\partial y} + v\frac{\partial^2 v}{\partial y^2} + w\frac{\partial^2 v}{\partial y\partial z}\right)$$
$$+4n_y n_z\left(u\frac{\partial^2 v}{\partial x\partial z} + v\frac{\partial^2 v}{\partial y\partial z} + w\frac{\partial^2 v}{\partial z^2}\right) + 2n_z^2\left(u\frac{\partial^2 w}{\partial x\partial z} + v\frac{\partial^2 w}{\partial y\partial z} + w\frac{\partial^2 w}{\partial z^2}\right) \quad (2.23)$$

$$n_i n_j T_{ij} = -\left[C + \hat{\beta}\chi - \frac{\rho}{2}\left|\nabla\phi\right|^2\right] + \mu n_i n_j A_{ij} + \alpha_1 n_i n_j\frac{\partial\phi}{\partial x_k}\frac{\partial A_{ij}}{\partial x_k} + \left(\alpha_1 + \alpha_2\right)n_i n_j A_{ik}A_{kj}$$
$$= -\left[p_\infty + \frac{\rho}{2}U^2 + \hat{\beta}\chi - \frac{\rho}{2}\left|\nabla\phi\right|^2\right] + \mu n_i n_j A_{ij} + \alpha_1 n_i n_j\frac{\partial\phi}{\partial x_k}\frac{\partial A_{ij}}{\partial x_k} + \left(\alpha_1 + \alpha_2\right)n_i n_j C_{ij}$$
$$= -\left[p_\infty + \frac{\rho}{2}U^2 + \left(3\alpha_1 + 2\alpha_2\right)\chi - \frac{\rho}{2}\left(u^2 + v^2 + w^2\right)\right]$$
$$+2\mu\left(n_x^2\frac{\partial u}{\partial x} + 2n_x n_y\frac{\partial u}{\partial y} + 2n_x n_z\frac{\partial u}{\partial z} + n_y^2\frac{\partial v}{\partial y} + 2n_y n_z\frac{\partial v}{\partial z} + n_z^2\frac{\partial w}{\partial z}\right)$$
$$+2\alpha_1\left[n_x^2\left(u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial x\partial y} + w\frac{\partial^2 u}{\partial x\partial z}\right) + 2n_x n_y\left(u\frac{\partial^2 u}{\partial x\partial y} + v\frac{\partial^2 u}{\partial y^2} + w\frac{\partial^2 u}{\partial y\partial z}\right)\right.$$
$$+2n_x n_z\left(u\frac{\partial^2 u}{\partial x\partial z} + v\frac{\partial^2 u}{\partial y\partial z} + w\frac{\partial^2 u}{\partial z^2}\right) + n_y^2\left(u\frac{\partial v}{\partial x\partial y} + v\frac{\partial^2 v}{\partial y^2} + w\frac{\partial^2 v}{\partial y\partial z}\right)$$
$$\left.+2n_y n_z\left(u\frac{\partial^2 v}{\partial x\partial z} + v\frac{\partial^2 v}{\partial y\partial z} + w\frac{\partial^2 v}{\partial z^2}\right) + n_z^2\left(u\frac{\partial^2 w}{\partial x\partial z} + v\frac{\partial^2 w}{\partial y\partial z} + w\frac{\partial^2 w}{\partial z^2}\right)\right]$$
$$+\left(\alpha_1 + \alpha_2\right)\left[n_x^2 C_{11} + 2n_x n_y C_{12} + 2n_x n_z C_{13} + n_y^2 C_{22} + 2n_y n_z C_{23} + n_z^2 C_{33}\right]. \quad (2.24)$$

A Expressions

Consider the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad (1)$$

moving in the negative direction of the x-axis with velocity U in a fluid at rest, whence the velocity potential ϕ for which $\boldsymbol{v} = \nabla\phi$ is given by

$$\phi = \frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} = xf_0, \tag{2}$$

where

$$\alpha_0 = abc\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}, \tag{3}$$

$$f_0 \equiv f_0(\lambda) = \frac{abcU}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

where f_0 is evaluated using the elliptic integral (see the report "ellipsoid-cal-apr22-apr29.tex"). Then

$$f_1 = \frac{\partial f_0}{\partial\lambda} = -\frac{abcU}{2-\alpha_0}\left(a^2+\lambda\right)^{-3/2}\left(b^2+\lambda\right)^{-1/2}\left(c^2+\lambda\right)^{-1/2},$$

$$f_2 = \frac{\partial^2 f_0}{\partial\lambda^2} = \frac{\partial f_1}{\partial\lambda} = -f_1\left[\frac{3}{2}\left(a^2+\lambda\right)^{-1} + \frac{1}{2}\left(b^2+\lambda\right)^{-1} + \frac{1}{2}\left(c^2+\lambda\right)^{-1}\right],$$

$$f_3 = \frac{\partial^3 f_0}{\partial\lambda^3} = \frac{\partial f_2}{\partial\lambda} = f_1\left[\frac{3}{2}\left(a^2+\lambda\right)^{-1} + \frac{1}{2}\left(b^2+\lambda\right)^{-1} + \frac{1}{2}\left(c^2+\lambda\right)^{-1}\right]^2$$
$$+f_1\left[\frac{3}{2}\left(a^2+\lambda\right)^{-2} + \frac{1}{2}\left(b^2+\lambda\right)^{-2} + \frac{1}{2}\left(c^2+\lambda\right)^{-2}\right].$$

In the cartesian coordinates (x, y, z) fixed on the ellipsoid, the potential is given by

$$\phi = Ux + xf_0. \tag{A.1}$$

The velocity (u, v, w) in the frame (x, y, z) is given by

$$u = U + \frac{\partial(xf_0)}{\partial x} = U + f_0 + x\frac{\partial\lambda}{\partial x}\frac{\partial f_0}{\partial\lambda} = U + f_0 + x\frac{\partial\lambda}{\partial x}f_1, \tag{A.2}$$

$$v = \frac{\partial(xf_0)}{\partial y} = x\frac{\partial\lambda}{\partial y}\frac{\partial f_0}{\partial\lambda} = x\frac{\partial\lambda}{\partial y}f_1, \quad w = \frac{\partial(xf_0)}{\partial z} = x\frac{\partial\lambda}{\partial z}\frac{\partial f_0}{\partial\lambda} = x\frac{\partial\lambda}{\partial z}f_1. \tag{A.3}$$

The components of the strain tensor are expressed as

$$\frac{\partial u}{\partial x} = 2\frac{\partial\lambda}{\partial x}\frac{\partial f_0}{\partial\lambda} + x\frac{\partial^2\lambda}{\partial x^2}\frac{\partial f_0}{\partial\lambda} + x\left(\frac{\partial\lambda}{\partial x}\right)^2\frac{\partial^2 f_0}{\partial\lambda^2} = 2\frac{\partial\lambda}{\partial x}f_1 + x\frac{\partial^2\lambda}{\partial x^2}f_1 + x\left(\frac{\partial\lambda}{\partial x}\right)^2 f_2, \tag{A.4}$$

$$\frac{\partial v}{\partial x} = \frac{\partial\lambda}{\partial y}\frac{\partial f_0}{\partial\lambda} + x\frac{\partial^2\lambda}{\partial x\partial y}\frac{\partial f_0}{\partial\lambda} + x\left(\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}\right)\frac{\partial^2 f_0}{\partial\lambda^2} = \frac{\partial\lambda}{\partial y}f_1 + x\frac{\partial^2\lambda}{\partial x\partial y}f_1 + x\left(\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}\right)f_2 = \frac{\partial u}{\partial y}, \tag{A.5}$$

$$\frac{\partial w}{\partial x} = \frac{\partial\lambda}{\partial z}\frac{\partial f_0}{\partial\lambda} + x\frac{\partial^2\lambda}{\partial x\partial z}\frac{\partial f_0}{\partial\lambda} + x\left(\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial z}\right)\frac{\partial^2 f_0}{\partial\lambda^2} = \frac{\partial\lambda}{\partial z}f_1 + x\frac{\partial^2\lambda}{\partial x\partial z}f_1 + x\left(\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial z}\right)f_2 = \frac{\partial u}{\partial z}, \tag{A.6}$$

$$\frac{\partial v}{\partial y} = x\frac{\partial^2\lambda}{\partial y^2}\frac{\partial f_0}{\partial\lambda} + x\left(\frac{\partial\lambda}{\partial y}\right)^2\frac{\partial^2 f_0}{\partial\lambda^2} = x\frac{\partial^2\lambda}{\partial y^2}f_1 + x\left(\frac{\partial\lambda}{\partial y}\right)^2 f_2, \tag{A.7}$$

$$\frac{\partial w}{\partial y} = x\frac{\partial^2\lambda}{\partial y\partial z}\frac{\partial f_0}{\partial\lambda} + x\left(\frac{\partial\lambda}{\partial y}\frac{\partial\lambda}{\partial z}\right)\frac{\partial^2 f_0}{\partial\lambda^2} = x\frac{\partial^2\lambda}{\partial y\partial z}f_1 + x\left(\frac{\partial\lambda}{\partial y}\frac{\partial\lambda}{\partial z}\right)f_2 = \frac{\partial v}{\partial z}, \tag{A.8}$$

$$\frac{\partial w}{\partial z} = x\frac{\partial^2\lambda}{\partial z^2}\frac{\partial f_0}{\partial\lambda} + x\left(\frac{\partial\lambda}{\partial z}\right)^2\frac{\partial^2 f_0}{\partial\lambda^2} = x\frac{\partial^2\lambda}{\partial z^2}f_1 + x\left(\frac{\partial\lambda}{\partial z}\right)^2 f_2. \tag{A.9}$$

For the second order fluid, we then need the following expressions

$$\frac{\partial^2 u}{\partial x^2} = 3\frac{\partial^2\lambda}{\partial x^2}f_1 + 3\left(\frac{\partial\lambda}{\partial x}\right)^2 f_2 + x\frac{\partial^3\lambda}{\partial x^3}f_1 + 3x\frac{\partial^2\lambda}{\partial x^2}\frac{\partial\lambda}{\partial x}f_2 + x\left(\frac{\partial\lambda}{\partial x}\right)^3 f_3, \quad (A.10)$$

$$\frac{\partial^2 u}{\partial x\partial y} = 2\frac{\partial^2\lambda}{\partial x\partial y}f_1 + 2\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}f_2 + x\frac{\partial^3\lambda}{\partial x^2\partial y}f_1 + x\frac{\partial^2\lambda}{\partial x^2}\frac{\partial\lambda}{\partial y}f_2 + 2x\frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial x\partial y}f_2 + x\left(\frac{\partial\lambda}{\partial x}\right)^2\frac{\partial\lambda}{\partial y}f_3, \quad (A.11)$$

$$\frac{\partial^2 u}{\partial x\partial z} = 2\frac{\partial^2\lambda}{\partial x\partial z}f_1 + 2\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial z}f_2 + x\frac{\partial^3\lambda}{\partial x^2\partial z}f_1 + x\frac{\partial^2\lambda}{\partial x^2}\frac{\partial\lambda}{\partial z}f_2 + 2x\frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial x\partial z}f_2 + x\left(\frac{\partial\lambda}{\partial x}\right)^2\frac{\partial\lambda}{\partial z}f_3, \quad (A.12)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2\lambda}{\partial y^2}f_1 + \left(\frac{\partial\lambda}{\partial y}\right)^2 f_2 + x\frac{\partial^3\lambda}{\partial x\partial y^2}f_1 + x\left(2\frac{\partial^2\lambda}{\partial x\partial y}\frac{\partial\lambda}{\partial y} + \frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial y^2}\right)f_2 + x\frac{\partial\lambda}{\partial x}\left(\frac{\partial\lambda}{\partial y}\right)^2 f_3 = \frac{\partial^2 v}{\partial x\partial y}, \quad (A.13)$$

$$\begin{aligned}\frac{\partial^2 u}{\partial y\partial z} = \frac{\partial^2\lambda}{\partial y\partial z}f_1 + \frac{\partial\lambda}{\partial y}\frac{\partial\lambda}{\partial z}f_2 + x\frac{\partial^3\lambda}{\partial x\partial y\partial z}f_1 + x\left(\frac{\partial^2\lambda}{\partial x\partial y}\frac{\partial\lambda}{\partial z} + \frac{\partial^2\lambda}{\partial x\partial z}\frac{\partial\lambda}{\partial y} + \frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial y\partial z}\right)f_2 \\ +x\left(\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}\frac{\partial\lambda}{\partial z}\right)f_3 = \frac{\partial^2 v}{\partial x\partial z}, \quad (A.14)\end{aligned}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2\lambda}{\partial z^2}f_1 + \left(\frac{\partial\lambda}{\partial z}\right)^2 f_2 + x\frac{\partial^3\lambda}{\partial x\partial z^2}f_1 + x\left(2\frac{\partial^2\lambda}{\partial x\partial z}\frac{\partial\lambda}{\partial z} + \frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial z^2}\right)f_2 + x\frac{\partial\lambda}{\partial x}\left(\frac{\partial\lambda}{\partial z}\right)^2 f_3, \quad (A.15)$$

$$\frac{\partial^2 v}{\partial y^2} = x\frac{\partial^3\lambda}{\partial y^3}f_1 + 3x\frac{\partial\lambda}{\partial y}\frac{\partial^2\lambda}{\partial y^2}f_2 + x\left(\frac{\partial\lambda}{\partial y}\right)^3 f_3, \quad (A.16)$$

$$\frac{\partial^2 v}{\partial y\partial z} = x\frac{\partial^3\lambda}{\partial y^2\partial z}f_1 + x\frac{\partial^2\lambda}{\partial y^2}\frac{\partial\lambda}{\partial z}f_2 + 2x\frac{\partial\lambda}{\partial y}\frac{\partial^2\lambda}{\partial y\partial z}f_2 + x\left(\frac{\partial\lambda}{\partial y}\right)^2\frac{\partial\lambda}{\partial z}f_3, \quad (A.17)$$

$$\frac{\partial^2 v}{\partial z^2} = x\frac{\partial^3\lambda}{\partial y\partial z^2}f_1 + x\left(\frac{\partial^2\lambda}{\partial z^2}\frac{\partial\lambda}{\partial y} + 2\frac{\partial\lambda}{\partial z}\frac{\partial^2\lambda}{\partial y\partial z}\right)f_2 + x\frac{\partial\lambda}{\partial y}\left(\frac{\partial\lambda}{\partial z}\right)^2 f_3, \quad (A.18)$$

$$\frac{\partial^2 w}{\partial z^2} = x\frac{\partial^3\lambda}{\partial z^3}f_1 + 3x\frac{\partial^2\lambda}{\partial z^2}\frac{\partial\lambda}{\partial z}f_2 + x\left(\frac{\partial\lambda}{\partial z}\right)^3 f_3, \quad (A.19)$$

with

$$\frac{\partial\lambda}{\partial x},\quad \frac{\partial\lambda}{\partial y},\quad \frac{\partial\lambda}{\partial z},$$

$$\frac{\partial^2\lambda}{\partial x^2},\quad \frac{\partial^2\lambda}{\partial x\partial y},\quad \frac{\partial^2\lambda}{\partial y^2},\quad \frac{\partial^2\lambda}{\partial y\partial z},\quad \frac{\partial^2\lambda}{\partial z^2},\quad \frac{\partial^2\lambda}{\partial z\partial x},$$

$$\frac{\partial^3\lambda}{\partial x^3},\quad \frac{\partial^3\lambda}{\partial x^2\partial y},\quad \frac{\partial^3\lambda}{\partial x^2\partial z},\quad \frac{\partial^3\lambda}{\partial x\partial y^2},\quad \frac{\partial^3\lambda}{\partial x\partial z^2},\quad \frac{\partial^3\lambda}{\partial x\partial y\partial z},\quad \frac{\partial^3\lambda}{\partial y^3},\quad \frac{\partial^3\lambda}{\partial y^2\partial z},\quad \frac{\partial^3\lambda}{\partial y\partial z^2},\quad \frac{\partial^3\lambda}{\partial z^3}.$$

A.1 Geometry

The ellipsoid with a papameter θ is expressed as

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 1, \quad (A.20)$$

which then gives

$$\begin{aligned}x^2\left(b^2+\theta\right)\left(c^2+\theta\right) + y^2\left(a^2+\theta\right)\left(c^2+\theta\right) + z^2\left(a^2+\theta\right)\left(b^2+\theta\right) \\ -\left(a^2+\theta\right)\left(b^2+\theta\right)\left(c^2+\theta\right) = -\left(\theta-\lambda\right)\left(\theta-\mu\right)\left(\theta-\nu\right) \equiv f(\theta), \quad (A.21)\end{aligned}$$

thus

$$f(\lambda) = f(\mu) = f(\nu) = 0, \quad (A.22)$$

$$\begin{aligned} f(\lambda) &= -\lambda^3 - \left(a^2+b^2+c^2-x^2-y^2-z^2\right)\lambda^2 \\ &\quad - \left(a^2b^2+b^2c^2+c^2a^2-x^2\left(b^2+c^2\right)-y^2\left(c^2+a^2\right)-z^2\left(a^2+b^2\right)\right)\lambda \\ &\quad -a^2b^2c^2+x^2b^2c^2+y^2c^2a^2+z^2a^2b^2=0 \\ &= -\lambda^3-A_2\lambda^2-A_1\lambda-A_0=0, \end{aligned} \tag{A.23}$$

with

$$A_2 = a^2+b^2+c^2-x^2-y^2-z^2, \tag{A.24}$$

$$A_1 = a^2b^2+b^2c^2+c^2a^2-x^2\left(b^2+c^2\right)-y^2\left(c^2+a^2\right)-z^2\left(a^2+b^2\right), \tag{A.25}$$

$$A_0 = a^2b^2c^2-x^2b^2c^2-y^2c^2a^2-z^2a^2b^2. \tag{A.26}$$

It is noted that λ is a solution of the cubic equation $f(\lambda)=0$ which satisfies $a^2+\lambda>0$, $b^2+\lambda>0$ and $c^2+\lambda>0$.

For $i=1,\,2,\,3$ ($(x_1,x_2,x_3)=(x,y,z)$), we have the first derivatives

$$\left(3\lambda^2+2A_2\lambda+A_1\right)\frac{\partial\lambda}{\partial x_i}+\frac{\partial A_2}{\partial x_i}\lambda^2+\frac{\partial A_1}{\partial x_i}\lambda+\frac{\partial A_0}{\partial x_i}=0, \tag{A.27}$$

$$\left(3\lambda^2+2A_2\lambda+A_1\right)\frac{\partial\lambda}{\partial x}+\frac{\partial A_2}{\partial x}\lambda^2+\frac{\partial A_1}{\partial x}\lambda+\frac{\partial A_0}{\partial x}=0, \tag{A.28}$$

$$\left(3\lambda^2+2A_2\lambda+A_1\right)\frac{\partial\lambda}{\partial y}+\frac{\partial A_2}{\partial y}\lambda^2+\frac{\partial A_1}{\partial y}\lambda+\frac{\partial A_0}{\partial y}=0, \tag{A.29}$$

$$\left(3\lambda^2+2A_2\lambda+A_1\right)\frac{\partial\lambda}{\partial z}+\frac{\partial A_2}{\partial z}\lambda^2+\frac{\partial A_1}{\partial z}\lambda+\frac{\partial A_0}{\partial z}=0, \tag{A.30}$$

and the second derivatives

$$\left(3\lambda^2+2A_2\lambda+A_1\right)\frac{\partial^2\lambda}{\partial x_i^2}+(6\lambda+2A_2)\left(\frac{\partial\lambda}{\partial x_i}\right)^2+\left(4\lambda\frac{\partial A_2}{\partial x_i}+2\frac{\partial A_1}{\partial x_i}\right)\frac{\partial\lambda}{\partial x_i}+\frac{\partial^2 A_2}{\partial x_i^2}\lambda^2+\frac{\partial^2 A_1}{\partial x_i^2}\lambda+\frac{\partial^2 A_0}{\partial x_i^2}=0, \tag{A.31}$$

$$\left(3\lambda^2+2A_2\lambda+A_1\right)\frac{\partial^2\lambda}{\partial x^2}+(6\lambda+2A_2)\left(\frac{\partial\lambda}{\partial x}\right)^2+\left(4\lambda\frac{\partial A_2}{\partial x}+2\frac{\partial A_1}{\partial x}\right)\frac{\partial\lambda}{\partial x}+\frac{\partial^2 A_2}{\partial x^2}\lambda^2+\frac{\partial^2 A_1}{\partial x^2}\lambda+\frac{\partial^2 A_0}{\partial x^2}=0, \tag{A.32}$$

$$\left(3\lambda^2+2A_2\lambda+A_1\right)\frac{\partial^2\lambda}{\partial y^2}+(6\lambda+2A_2)\left(\frac{\partial\lambda}{\partial y}\right)^2+\left(4\lambda\frac{\partial A_2}{\partial y}+2\frac{\partial A_1}{\partial y}\right)\frac{\partial\lambda}{\partial y}+\frac{\partial^2 A_2}{\partial y^2}\lambda^2+\frac{\partial^2 A_1}{\partial y^2}\lambda+\frac{\partial^2 A_0}{\partial y^2}=0, \tag{A.33}$$

$$\left(3\lambda^2+2A_2\lambda+A_1\right)\frac{\partial^2\lambda}{\partial z^2}+(6\lambda+2A_2)\left(\frac{\partial\lambda}{\partial z}\right)^2+\left(4\lambda\frac{\partial A_2}{\partial z}+2\frac{\partial A_1}{\partial z}\right)\frac{\partial\lambda}{\partial z}+\frac{\partial^2 A_2}{\partial z^2}\lambda^2+\frac{\partial^2 A_1}{\partial z^2}\lambda+\frac{\partial^2 A_0}{\partial z^2}=0. \tag{A.34}$$

For $i\neq j$, the second derivatives are given by

$$\begin{aligned} &\left(3\lambda^2+2A_2\lambda+A_1\right)\frac{\partial^2\lambda}{\partial x_i\partial x_j}+(6\lambda+2A_2)\left(\frac{\partial\lambda}{\partial x_i}\frac{\partial\lambda}{\partial x_j}\right)+2\lambda\left(\frac{\partial A_2}{\partial x_i}\frac{\partial\lambda}{\partial x_j}+\frac{\partial\lambda}{\partial x_i}\frac{\partial A_2}{\partial x_j}\right) \\ &\qquad +\frac{\partial A_1}{\partial x_i}\frac{\partial\lambda}{\partial x_j}+\frac{\partial\lambda}{\partial x_i}\frac{\partial A_1}{\partial x_j}+\frac{\partial^2 A_2}{\partial x_i\partial x_j}\lambda^2+\frac{\partial^2 A_1}{\partial x_i\partial x_j}\lambda+\frac{\partial^2 A_0}{\partial x_i\partial x_j}=0 \\ \to\quad &\left(3\lambda^2+2A_2\lambda+A_1\right)\frac{\partial^2\lambda}{\partial x_i\partial x_j}+(6\lambda+2A_2)\left(\frac{\partial\lambda}{\partial x_i}\frac{\partial\lambda}{\partial x_j}\right)+2\lambda\left(\frac{\partial A_2}{\partial x_i}\frac{\partial\lambda}{\partial x_j}+\frac{\partial\lambda}{\partial x_i}\frac{\partial A_2}{\partial x_j}\right) \\ &\qquad +\frac{\partial A_1}{\partial x_i}\frac{\partial\lambda}{\partial x_j}+\frac{\partial\lambda}{\partial x_i}\frac{\partial A_1}{\partial x_j}=0, \end{aligned} \tag{A.35}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^2\lambda}{\partial x\partial y}+\left(6\lambda\frac{\partial\lambda}{\partial y}+2\frac{\partial\lambda}{\partial y}A_2+2\lambda\frac{\partial A_2}{\partial y}+\frac{\partial A_1}{\partial y}\right)\frac{\partial\lambda}{\partial x}+2\lambda\frac{\partial A_2}{\partial x}\frac{\partial\lambda}{\partial y}+\frac{\partial A_1}{\partial x}\frac{\partial\lambda}{\partial y}=0, \tag{A.36}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^2\lambda}{\partial x\partial z}+\left(6\lambda\frac{\partial\lambda}{\partial z}+2\frac{\partial\lambda}{\partial z}A_2+2\lambda\frac{\partial A_2}{\partial z}+\frac{\partial A_1}{\partial z}\right)\frac{\partial\lambda}{\partial x}+2\lambda\frac{\partial A_2}{\partial x}\frac{\partial\lambda}{\partial z}+\frac{\partial A_1}{\partial x}\frac{\partial\lambda}{\partial z}=0, \tag{A.37}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^2\lambda}{\partial y\partial z}+\left(6\lambda\frac{\partial\lambda}{\partial z}+2\frac{\partial\lambda}{\partial z}A_2+2\lambda\frac{\partial A_2}{\partial z}+\frac{\partial A_1}{\partial z}\right)\frac{\partial\lambda}{\partial y}+2\lambda\frac{\partial A_2}{\partial y}\frac{\partial\lambda}{\partial z}+\frac{\partial A_1}{\partial y}\frac{\partial\lambda}{\partial z}=0. \tag{A.38}$$

For i, j, k, we have the third derivatives

$$\left(3\lambda^2+2A_2\lambda+A_1\right)\frac{\partial^3\lambda}{\partial x_i\partial x_j\partial x_k}+(6\lambda+2A_2)\left(\frac{\partial^2\lambda}{\partial x_i\partial x_j}\frac{\partial\lambda}{\partial x_k}+\frac{\partial^2\lambda}{\partial x_i\partial x_k}\frac{\partial\lambda}{\partial x_j}+\frac{\partial^2\lambda}{\partial x_j\partial x_k}\frac{\partial\lambda}{\partial x_i}\right)$$
$$+\frac{\partial^2\lambda}{\partial x_i\partial x_j}\left(2\lambda\frac{\partial A_2}{\partial x_k}+\frac{\partial A_1}{\partial x_k}\right)+\frac{\partial^2\lambda}{\partial x_j\partial x_k}\left(2\lambda\frac{\partial A_2}{\partial x_i}+\frac{\partial A_1}{\partial x_i}\right)+\frac{\partial^2\lambda}{\partial x_i\partial x_k}\left(2\lambda\frac{\partial A_2}{\partial x_j}+\frac{\partial A_1}{\partial x_j}\right)+6\frac{\partial\lambda}{\partial x_i}\frac{\partial\lambda}{\partial x_j}\frac{\partial\lambda}{\partial x_k}$$
$$+2\frac{\partial A_2}{\partial x_i}\frac{\partial\lambda}{\partial x_j}\frac{\partial\lambda}{\partial x_k}+2\frac{\partial A_2}{\partial x_j}\frac{\partial\lambda}{\partial x_k}\frac{\partial\lambda}{\partial x_i}+2\frac{\partial A_2}{\partial x_k}\frac{\partial\lambda}{\partial x_i}\frac{\partial\lambda}{\partial x_j}+2\lambda\left(\frac{\partial^2A_2}{\partial x_i\partial x_k}\frac{\partial\lambda}{\partial x_j}+\frac{\partial^2A_2}{\partial x_j\partial x_k}\frac{\partial\lambda}{\partial x_i}+\frac{\partial^2A_2}{\partial x_i\partial x_j}\frac{\partial\lambda}{\partial x_k}\right)$$
$$+\frac{\partial^2A_1}{\partial x_i\partial x_k}\frac{\partial\lambda}{\partial x_j}+\frac{\partial^2A_1}{\partial x_i\partial x_j}\frac{\partial\lambda}{\partial x_k}+\frac{\partial^2A_1}{\partial x_j\partial x_k}\frac{\partial\lambda}{\partial x_i}=0, \tag{A.39}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^3\lambda}{\partial x^3}+(18\lambda+6A_2)\frac{\partial^2\lambda}{\partial x^2}\frac{\partial\lambda}{\partial x}+\frac{\partial^2\lambda}{\partial x^2}\left(6\lambda\frac{\partial A_2}{\partial x}+3\frac{\partial A_1}{\partial x}\right)+6\left(\frac{\partial\lambda}{\partial x}\right)^3$$
$$+6\frac{\partial A_2}{\partial x}\left(\frac{\partial\lambda}{\partial x}\right)^2+6\lambda\frac{\partial^2A_2}{\partial x^2}\frac{\partial\lambda}{\partial x}+3\frac{\partial^2A_1}{\partial x^2}\frac{\partial\lambda}{\partial x}=0, \tag{A.40}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^3\lambda}{\partial x^2\partial y}+(6\lambda+2A_2)\left(\frac{\partial^2\lambda}{\partial x^2}\frac{\partial\lambda}{\partial y}+2\frac{\partial^2\lambda}{\partial x\partial y}\frac{\partial\lambda}{\partial x}\right)+\frac{\partial^2\lambda}{\partial x^2}\left(2\lambda\frac{\partial A_2}{\partial y}+\frac{\partial A_1}{\partial y}\right)$$
$$+\frac{\partial^2\lambda}{\partial x\partial y}\left(4\lambda\frac{\partial A_2}{\partial x}+2\frac{\partial A_1}{\partial x}\right)+6\left(\frac{\partial\lambda}{\partial x}\right)^2\frac{\partial\lambda}{\partial y}+4\frac{\partial A_2}{\partial x}\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}+2\frac{\partial A_2}{\partial y}\left(\frac{\partial\lambda}{\partial x}\right)^2$$
$$+2\lambda\left(\frac{\partial^2A_2}{\partial x^2}\frac{\partial\lambda}{\partial y}+2\frac{\partial^2A_2}{\partial x\partial y}\frac{\partial\lambda}{\partial x}\right)+\frac{\partial^2A_1}{\partial x^2}\frac{\partial\lambda}{\partial y}+2\frac{\partial^2A_1}{\partial x\partial y}\frac{\partial\lambda}{\partial x}=0, \tag{A.41}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^3\lambda}{\partial x^2\partial z}+(6\lambda+2A_2)\left(\frac{\partial^2\lambda}{\partial x^2}\frac{\partial\lambda}{\partial z}+2\frac{\partial^2\lambda}{\partial x\partial z}\frac{\partial\lambda}{\partial x}\right)+\frac{\partial^2\lambda}{\partial x^2}\left(2\lambda\frac{\partial A_2}{\partial z}+\frac{\partial A_1}{\partial z}\right)$$
$$+\frac{\partial^2\lambda}{\partial x\partial z}\left(4\lambda\frac{\partial A_2}{\partial x}+2\frac{\partial A_1}{\partial x}\right)+6\left(\frac{\partial\lambda}{\partial x}\right)^2\frac{\partial\lambda}{\partial z}+4\frac{\partial A_2}{\partial x}\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial z}+2\frac{\partial A_2}{\partial z}\left(\frac{\partial\lambda}{\partial x}\right)^2$$
$$+2\lambda\left(\frac{\partial^2A_2}{\partial x^2}\frac{\partial\lambda}{\partial z}+2\frac{\partial^2A_2}{\partial x\partial z}\frac{\partial\lambda}{\partial x}\right)+\frac{\partial^2A_1}{\partial x^2}\frac{\partial\lambda}{\partial z}+2\frac{\partial^2A_1}{\partial x\partial z}\frac{\partial\lambda}{\partial x}=0, \tag{A.42}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^3\lambda}{\partial x\partial y^2}+(6\lambda+2A_2)\left(\frac{\partial^2\lambda}{\partial y^2}\frac{\partial\lambda}{\partial x}+2\frac{\partial^2\lambda}{\partial x\partial y}\frac{\partial\lambda}{\partial y}\right)+\frac{\partial^2\lambda}{\partial x\partial y}\left(4\lambda\frac{\partial A_2}{\partial y}+2\frac{\partial A_1}{\partial y}\right)$$
$$+\frac{\partial^2\lambda}{\partial y^2}\left(2\lambda\frac{\partial A_2}{\partial x}+\frac{\partial A_1}{\partial x}\right)+6\frac{\partial\lambda}{\partial x}\left(\frac{\partial\lambda}{\partial y}\right)^2+2\frac{\partial A_2}{\partial x}\left(\frac{\partial\lambda}{\partial y}\right)^2+4\frac{\partial A_2}{\partial y}\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}+2\lambda\left(\frac{\partial^2A_2}{\partial y^2}\frac{\partial\lambda}{\partial x}+2\frac{\partial^2A_2}{\partial x\partial y}\frac{\partial\lambda}{\partial y}\right)$$
$$+2\frac{\partial^2A_1}{\partial x\partial y}\frac{\partial\lambda}{\partial y}+\frac{\partial^2A_1}{\partial y^2}\frac{\partial\lambda}{\partial x}=0, \tag{A.43}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^3\lambda}{\partial x\partial z^2}+(6\lambda+2A_2)\left(\frac{\partial^2\lambda}{\partial z^2}\frac{\partial\lambda}{\partial x}+2\frac{\partial^2\lambda}{\partial x\partial z}\frac{\partial\lambda}{\partial z}\right)+\frac{\partial^2\lambda}{\partial x\partial z}\left(4\lambda\frac{\partial A_2}{\partial z}+2\frac{\partial A_1}{\partial z}\right)$$
$$+\frac{\partial^2\lambda}{\partial z^2}\left(2\lambda\frac{\partial A_2}{\partial x}+\frac{\partial A_1}{\partial x}\right)+6\frac{\partial\lambda}{\partial x}\left(\frac{\partial\lambda}{\partial z}\right)^2+2\frac{\partial A_2}{\partial x}\left(\frac{\partial\lambda}{\partial z}\right)^2+4\frac{\partial A_2}{\partial z}\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial z}+2\lambda\left(\frac{\partial^2A_2}{\partial z^2}\frac{\partial\lambda}{\partial x}+2\frac{\partial^2A_2}{\partial x\partial z}\frac{\partial\lambda}{\partial z}\right)$$
$$+2\frac{\partial^2A_1}{\partial x\partial z}\frac{\partial\lambda}{\partial z}+\frac{\partial^2A_1}{\partial z^2}\frac{\partial\lambda}{\partial x}=0, \tag{A.44}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^3\lambda}{\partial x\partial y\partial z}+(6\lambda+2A_2)\left(\frac{\partial^2\lambda}{\partial x\partial y}\frac{\partial\lambda}{\partial z}+\frac{\partial^2\lambda}{\partial x\partial z}\frac{\partial\lambda}{\partial y}+\frac{\partial^2\lambda}{\partial y\partial z}\frac{\partial\lambda}{\partial x}\right)$$
$$+\frac{\partial^2\lambda}{\partial x\partial y}\left(2\lambda\frac{\partial A_2}{\partial z}+\frac{\partial A_1}{\partial z}\right)+\frac{\partial^2\lambda}{\partial y\partial z}\left(2\lambda\frac{\partial A_2}{\partial x}+\frac{\partial A_1}{\partial x}\right)+\frac{\partial^2\lambda}{\partial x\partial z}\left(2\lambda\frac{\partial A_2}{\partial y}+\frac{\partial A_1}{\partial y}\right)+6\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}\frac{\partial\lambda}{\partial z}$$
$$+2\frac{\partial A_2}{\partial x}\frac{\partial\lambda}{\partial y}\frac{\partial\lambda}{\partial z}+2\frac{\partial A_2}{\partial y}\frac{\partial\lambda}{\partial z}\frac{\partial\lambda}{\partial x}+2\frac{\partial A_2}{\partial z}\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}+2\lambda\left(\frac{\partial^2A_2}{\partial x\partial z}\frac{\partial\lambda}{\partial y}+\frac{\partial^2A_2}{\partial y\partial z}\frac{\partial\lambda}{\partial x}+\frac{\partial^2A_2}{\partial x\partial y}\frac{\partial\lambda}{\partial z}\right)$$
$$+\frac{\partial^2A_1}{\partial x\partial z}\frac{\partial\lambda}{\partial y}+\frac{\partial^2A_1}{\partial x\partial y}\frac{\partial\lambda}{\partial z}+\frac{\partial^2A_1}{\partial y\partial z}\frac{\partial\lambda}{\partial x}=0, \tag{A.45}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^3\lambda}{\partial y^3}+(18\lambda+6A_2)\frac{\partial^2\lambda}{\partial y^2}\frac{\partial\lambda}{\partial y}+\frac{\partial^2\lambda}{\partial y^2}\left(6\lambda\frac{\partial A_2}{\partial y}+3\frac{\partial A_1}{\partial y}\right)+6\left(\frac{\partial\lambda}{\partial y}\right)^3$$
$$+6\frac{\partial A_2}{\partial y}\left(\frac{\partial\lambda}{\partial y}\right)^2+6\lambda\frac{\partial^2A_2}{\partial y^2}\frac{\partial\lambda}{\partial y}+3\frac{\partial^2A_1}{\partial y^2}\frac{\partial\lambda}{\partial y}=0, \tag{A.46}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^3\lambda}{\partial y^2\partial z}+(6\lambda+2A_2)\left(\frac{\partial^2\lambda}{\partial y^2}\frac{\partial\lambda}{\partial z}+2\frac{\partial^2\lambda}{\partial y\partial z}\frac{\partial\lambda}{\partial y}\right)+\frac{\partial^2\lambda}{\partial y^2}\left(2\lambda\frac{\partial A_2}{\partial z}+\frac{\partial A_1}{\partial z}\right)$$
$$+\frac{\partial^2\lambda}{\partial y\partial z}\left(4\lambda\frac{\partial A_2}{\partial y}+2\frac{\partial A_1}{\partial y}\right)+6\left(\frac{\partial\lambda}{\partial y}\right)^2\frac{\partial\lambda}{\partial z}+4\frac{\partial A_2}{\partial y}\frac{\partial\lambda}{\partial y}\frac{\partial\lambda}{\partial z}+2\frac{\partial A_2}{\partial z}\left(\frac{\partial\lambda}{\partial y}\right)^2$$
$$+2\lambda\left(\frac{\partial^2A_2}{\partial y^2}\frac{\partial\lambda}{\partial z}+2\frac{\partial^2A_2}{\partial y\partial z}\frac{\partial\lambda}{\partial y}\right)+\frac{\partial^2A_1}{\partial y^2}\frac{\partial\lambda}{\partial z}+2\frac{\partial^2A_1}{\partial y\partial z}\frac{\partial\lambda}{\partial y}=0, \tag{A.47}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^3\lambda}{\partial y\partial z^2}+(6\lambda+2A_2)\left(\frac{\partial^2\lambda}{\partial z^2}\frac{\partial\lambda}{\partial y}+2\frac{\partial^2\lambda}{\partial y\partial z}\frac{\partial\lambda}{\partial z}\right)+\frac{\partial^2\lambda}{\partial y\partial z}\left(4\lambda\frac{\partial A_2}{\partial z}+2\frac{\partial A_1}{\partial z}\right)$$
$$+\frac{\partial^2\lambda}{\partial z^2}\left(2\lambda\frac{\partial A_2}{\partial y}+\frac{\partial A_1}{\partial y}\right)+6\frac{\partial\lambda}{\partial y}\left(\frac{\partial\lambda}{\partial z}\right)^2+2\frac{\partial A_2}{\partial y}\left(\frac{\partial\lambda}{\partial z}\right)^2+4\frac{\partial A_2}{\partial z}\frac{\partial\lambda}{\partial y}\frac{\partial\lambda}{\partial z}+2\lambda\left(\frac{\partial^2A_2}{\partial z^2}\frac{\partial\lambda}{\partial y}+2\frac{\partial^2A_2}{\partial y\partial z}\frac{\partial\lambda}{\partial z}\right)$$
$$+2\frac{\partial^2A_1}{\partial y\partial z}\frac{\partial\lambda}{\partial z}+\frac{\partial^2A_1}{\partial z^2}\frac{\partial\lambda}{\partial y}=0, \tag{A.48}$$

$$\left(3\lambda^2+2\lambda A_2+A_1\right)\frac{\partial^3\lambda}{\partial z^3}+(18\lambda+6A_2)\frac{\partial^2\lambda}{\partial z^2}\frac{\partial\lambda}{\partial z}+\frac{\partial^2\lambda}{\partial z^2}\left(6\lambda\frac{\partial A_2}{\partial z}+3\frac{\partial A_1}{\partial z}\right)+6\left(\frac{\partial\lambda}{\partial z}\right)^3$$
$$+6\frac{\partial A_2}{\partial z}\left(\frac{\partial\lambda}{\partial z}\right)^2+6\lambda\frac{\partial^2A_2}{\partial z^2}\frac{\partial\lambda}{\partial z}+3\frac{\partial^2A_1}{\partial z^2}\frac{\partial\lambda}{\partial z}=0. \tag{A.49}$$

The derivatives of the coefficients are given by

$$A_2=a^2+b^2+c^2-x^2-y^2-z^2, \tag{A.50}$$

$$\frac{\partial A_2}{\partial x}=-2x,\quad \frac{\partial A_2}{\partial y}=-2y,\quad \frac{\partial A_2}{\partial z}=-2z, \tag{A.51}$$
$$\frac{\partial^2A_2}{\partial x^2}=-2,\quad \frac{\partial^2A_2}{\partial y^2}=-2,\quad \frac{\partial^2A_2}{\partial z^2}=-2, \tag{A.52}$$
$$\frac{\partial^2A_2}{\partial x\partial y}=\frac{\partial^2A_2}{\partial y\partial z}=\frac{\partial^2A_2}{\partial z\partial x}=0, \tag{A.53}$$

$$A_1=a^2b^2+b^2c^2+c^2a^2-x^2\left(b^2+c^2\right)-y^2\left(c^2+a^2\right)-z^2\left(a^2+b^2\right), \tag{A.54}$$

$$\frac{\partial A_1}{\partial x} = -2x\left(b^2+c^2\right), \quad \frac{\partial A_1}{\partial y} = -2y\left(c^2+a^2\right), \quad \frac{\partial A_1}{\partial z} = -2z\left(a^2+b^2\right), \tag{A.55}$$

$$\frac{\partial^2 A_1}{\partial x^2} = -2\left(b^2+c^2\right), \quad \frac{\partial^2 A_1}{\partial y^2} = -2\left(c^2+a^2\right), \quad \frac{\partial^2 A_1}{\partial z^2} = -2\left(a^2+b^2\right), \tag{A.56}$$

$$\frac{\partial^2 A_1}{\partial x \partial y} = \frac{\partial^2 A_1}{\partial y \partial z} = \frac{\partial^2 A_1}{\partial z \partial x} = 0, \tag{A.57}$$

$$A_0 = a^2b^2c^2 - x^2b^2c^2 - y^2c^2a^2 - z^2a^2b^2, \tag{A.58}$$

$$\frac{\partial A_0}{\partial x} = -2xb^2c^2, \quad \frac{\partial A_0}{\partial y} = -2yc^2a^2, \quad \frac{\partial A_0}{\partial z} = -2za^2b^2, \tag{A.59}$$

$$\frac{\partial^2 A_0}{\partial x^2} = -2b^2c^2, \quad \frac{\partial^2 A_0}{\partial y^2} = -2c^2a^2, \quad \frac{\partial^2 A_0}{\partial z^2} = -2a^2b^2, \tag{A.60}$$

$$\frac{\partial^2 A_0}{\partial x \partial y} = \frac{\partial^2 A_0}{\partial y \partial z} = \frac{\partial^2 A_0}{\partial z \partial x} = 0. \tag{A.61}$$

A note on flows of a second order fluid around an ellipsoid of revolution

T.Funada, June 5, 2004 / revolution-june05a.tex / printed July 16, 2004

1 “17.52 Translational motion of an ellipsoid” Milne-Thomson (1974)

Consider the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \tag{1}$$

moving in the negative direction of the x-axis with velocity U in a fluid at rest, whence the velocity potential ϕ for which $\boldsymbol{v}=\nabla\phi$ is given by

$$\phi=\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}=xf_0, \tag{2}$$

where

$$\alpha_0=abc\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}, \tag{3}$$

$$f_0\equiv f_0(\lambda)=\frac{abcU}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}},$$

with the ellipsoidal coordinates $(\lambda,\,\mu,\,\nu)=(\lambda_1,\,\lambda_2,\,\lambda_3)=\lambda_1\boldsymbol{e}_1+\lambda_2\boldsymbol{e}_2+\lambda_3\boldsymbol{e}_3$ which are three solutions of the following equation through cartesian coordinates $(x,\,y,\,z)=x\boldsymbol{e}_x+y\boldsymbol{e}_y+z\boldsymbol{e}_z$

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1. \tag{4}$$

In the cartesian coordinates (x,y,z) fixed on the ellipsoid, the potential is given by

$$\phi=Ux+xf_0. \tag{1.1}$$

The velocity $\boldsymbol{v}=(u,v,w)=\nabla\phi$ in the frame (x,y,z) is given by

$$u=U+\frac{\partial(xf_0)}{\partial x}=U+f_0+x\frac{\partial\lambda}{\partial x}\frac{\partial f_0}{\partial\lambda}=U+f_0+x\frac{\partial\lambda}{\partial x}f_1, \tag{1.2}$$

$$v=\frac{\partial(xf_0)}{\partial y}=x\frac{\partial\lambda}{\partial y}\frac{\partial f_0}{\partial\lambda}=x\frac{\partial\lambda}{\partial y}f_1,\quad w=\frac{\partial(xf_0)}{\partial z}=x\frac{\partial\lambda}{\partial z}\frac{\partial f_0}{\partial\lambda}=x\frac{\partial\lambda}{\partial z}f_1, \tag{1.3}$$

where $f_1\equiv\partial f_0/\partial\lambda$ and the relevant expressions will be shown in Appendix.

2 Ellipsoid of revolution

For an ellipsoid of revolution with $a\neq b$, $b=c$, we have the velocity potential ϕ

$$\begin{aligned}\phi=\frac{ab^2Ux}{2-\alpha_0}\int_\lambda^\infty\left(a^2+\lambda\right)^{-\frac{3}{2}}\left(b^2+\lambda\right)^{-1}\mathrm{d}\lambda&=\frac{ab^2Ux}{2-\alpha_0}\frac{1}{a^2-b^2}\int_\lambda^\infty\frac{1}{\sqrt{a^2+\lambda}}\left(\frac{1}{b^2+\lambda}-\frac{1}{a^2+\lambda}\right)\mathrm{d}\lambda\\&=\frac{ab^2Ux}{2-\alpha_0}\frac{1}{a^2-b^2}\left[\int_\lambda^\infty\frac{\mathrm{d}\lambda}{\sqrt{a^2+\lambda}\,(b^2+\lambda)}+2\left(a^2+\lambda\right)^{-\frac{1}{2}}\Big|_\lambda^\infty\right]\\&=\frac{ab^2Ux}{2-\alpha_0}\frac{1}{a^2-b^2}\left[\int_\xi^\infty\frac{2\sqrt{a^2+\lambda}\,\mathrm{d}\xi}{\sqrt{a^2+\lambda}\,(b^2-a^2+\xi^2)}-2\left(a^2+\lambda\right)^{-\frac{1}{2}}\right]\\&=\frac{ab^2Ux}{2-\alpha_0}\frac{1}{a^2-b^2}\left[\int_{\sqrt{a^2+\lambda}}^\infty\frac{2\mathrm{d}\xi}{b^2-a^2+\xi^2}-2\left(a^2+\lambda\right)^{-\frac{1}{2}}\right],\end{aligned} \tag{2.1}$$

with

$$\alpha_0 = ab^2 \int_0^\infty \left(a^2+\lambda\right)^{-\frac{3}{2}} \left(b^2+\lambda\right)^{-1} \mathrm{d}\lambda, \quad \xi = \sqrt{a^2+\lambda}, \quad \mathrm{d}\xi = \frac{1}{\sqrt{a^2+\lambda}} \frac{1}{2} \mathrm{d}\lambda. \tag{2.2}$$

When $b^2 - a^2 > 0$, $\xi = \sqrt{b^2-a^2}\tan\theta$, $\mathrm{d}\xi = \sqrt{b^2-a^2}\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)\mathrm{d}\theta$ and

$$\begin{aligned} \int_{\sqrt{a^2+\lambda}}^\infty \frac{2\mathrm{d}\xi}{b^2-a^2+\xi^2} &= \int_\theta^{\pi/2} \frac{2\sqrt{b^2-a^2}\frac{1}{\cos^2\theta}\mathrm{d}\theta}{\left(b^2-a^2\right)\left(1+\tan^2\theta\right)} = \frac{2}{\sqrt{b^2-a^2}}\theta\Big|_\theta^{\pi/2} \\ &= \frac{2}{\sqrt{b^2-a^2}}\left(\frac{\pi}{2}-\theta\right) = \frac{2}{\sqrt{b^2-a^2}}\left[\frac{\pi}{2} - \arctan\left(\frac{\sqrt{a^2+\lambda}}{\sqrt{b^2-a^2}}\right)\right], \end{aligned} \tag{2.3}$$

then the potential is expressed explicitly as

$$\phi = \frac{ab^2Ux}{2-\alpha_0}\frac{1}{a^2-b^2}\left[\frac{2}{\sqrt{b^2-a^2}}\left(\frac{\pi}{2} - \arctan\sqrt{\frac{a^2+\lambda}{b^2-a^2}}\right) - 2\left(a^2+\lambda\right)^{-\frac{1}{2}}\right], \tag{2.4}$$

with

$$\alpha_0 = \frac{ab^2}{a^2-b^2}\left[\frac{2}{\sqrt{b^2-a^2}}\left(\frac{\pi}{2} - \arctan\sqrt{\frac{a^2+\lambda}{b^2-a^2}}\right) - 2\left(a^2+\lambda\right)^{-\frac{1}{2}}\right]_{\lambda=0}. \tag{2.5}$$

When $b^2 - a^2 < 0$, $\xi = \sqrt{a^2-b^2}\coth\theta$, $\mathrm{d}\xi = \sqrt{a^2-b^2}\left(1-\frac{\cosh^2\theta}{\sinh^2\theta}\right)\mathrm{d}\theta = \sqrt{a^2-b^2}\frac{-1}{\sinh^2\theta}\mathrm{d}\theta$, and

$$\begin{aligned} \int_{\sqrt{a^2+\lambda}}^\infty \frac{2\mathrm{d}\xi}{b^2-a^2+\xi^2} = \int_\theta^0 \frac{2\sqrt{a^2-b^2}\frac{-1}{\sinh^2\theta}\mathrm{d}\theta}{b^2-a^2+\left(a^2-b^2\right)\coth^2\theta} = \frac{-2}{\sqrt{a^2-b^2}}\theta\Big|_\theta^0 = \frac{2}{\sqrt{a^2-b^2}}\theta \\ = \frac{2}{\sqrt{a^2-b^2}}\coth^{-1}\left(\frac{\sqrt{a^2+\lambda}}{\sqrt{a^2-b^2}}\right). \end{aligned} \tag{2.6}$$

Using the relations:

$$\coth\theta = \sqrt{\frac{a^2+\lambda}{a^2-b^2}} = \frac{X+1/X}{X-1/X}, \quad \left(X^2-1\right)\sqrt{\frac{a^2+\lambda}{a^2-b^2}} = X^2+1 \quad \rightarrow \quad X^2 = \frac{\sqrt{a^2+\lambda}+\sqrt{a^2-b^2}}{\sqrt{a^2+\lambda}-\sqrt{a^2-b^2}} = \exp(2\theta),$$

$$\theta = \ln\left[\frac{\sqrt{a^2+\lambda}+\sqrt{a^2-b^2}}{\sqrt{a^2+\lambda}-\sqrt{a^2-b^2}}\right]^{\frac{1}{2}}, \tag{2.7}$$

the potential is expressed as

$$\phi = \frac{ab^2Ux}{2-\alpha_0}\frac{1}{a^2-b^2}\left[\frac{1}{\sqrt{a^2-b^2}}\ln\left[\frac{\sqrt{a^2+\lambda}+\sqrt{a^2-b^2}}{\sqrt{a^2+\lambda}-\sqrt{a^2-b^2}}\right] - 2\left(a^2+\lambda\right)^{-\frac{1}{2}}\right], \tag{2.8}$$

with

$$\alpha_0 = \frac{ab^2}{a^2-b^2}\left[\frac{1}{\sqrt{a^2-b^2}}\ln\left[\frac{\sqrt{a^2+\lambda}+\sqrt{a^2-b^2}}{\sqrt{a^2+\lambda}-\sqrt{a^2-b^2}}\right] - 2\left(a^2+\lambda\right)^{-\frac{1}{2}}\right]_{\lambda=0}. \tag{2.9}$$

2.1 Geometry

From the equation of an ellipsoid of revolution with $a \neq b$ and $b = c$:

$$\frac{x^2}{a^2+\lambda} + \frac{y^2+z^2}{b^2+\lambda} = 1, \tag{2.10}$$

we have the equation of λ

$$\begin{aligned} x^2\left(b^2+\lambda\right) + \left(y^2+z^2\right)\left(a^2+\lambda\right) - \left(\mathrm{a}^2+\lambda\right)\left(b^2+\lambda\right) = 0, \\ -\lambda^2 + \left(-a^2-b^2+x^2+y^2+z^2\right)\lambda - a^2b^2 + x^2b^2 + \left(y^2+z^2\right)a^2 = 0, \end{aligned} \tag{2.11}$$

which gives the solution λ

$$\lambda = -\frac{a^2+b^2-x^2-y^2-z^2}{2} \pm \sqrt{\left(\frac{a^2+b^2-x^2-y^2-z^2}{2}\right)^2 - a^2b^2 + x^2b^2 + \left(y^2+z^2\right)a^2} = \lambda_1,\ \lambda_2 \text{ say.} \tag{2.12}$$

For an ellipsoid with $a = 2$ and $b = 1$, we have $\lambda_1 > 0$ and $\lambda_2 < -1$; the former gives an ellipsoid and the latter gives a hyperboloid. Therefore we use $\lambda = \lambda_1$

$$\lambda = -\frac{a^2+b^2-x^2-y^2-z^2}{2} + \sqrt{\left(\frac{a^2+b^2-x^2-y^2-z^2}{2}\right)^2 - a^2b^2 + x^2b^2 + \left(y^2+z^2\right)a^2}, \tag{2.13}$$

$$\lambda = -\frac{a^2+b^2-x^2-y^2-z^2}{2} + D^{1/2}, \quad D = \frac{1}{4}\left(a^2+b^2-x^2-y^2-z^2\right)^2 - a^2b^2 + x^2b^2 + \left(y^2+z^2\right)a^2, \tag{2.14}$$

whence

$$\frac{\partial\lambda}{\partial x} = x + \frac{1}{2}\frac{\partial D}{\partial x}D^{-1/2}, \quad \frac{\partial\lambda}{\partial y} = y + \frac{1}{2}\frac{\partial D}{\partial y}D^{-1/2}, \quad \frac{\partial\lambda}{\partial z} = z + \frac{1}{2}\frac{\partial D}{\partial z}D^{-1/2}, \tag{2.15}$$

$$\frac{\partial^2\lambda}{\partial x^2} = 1 + \frac{1}{2}\frac{\partial^2 D}{\partial x^2}D^{-1/2} - \frac{1}{4}\left(\frac{\partial D}{\partial x}\right)^2 D^{-3/2}, \tag{2.16}$$

$$\frac{\partial^2\lambda}{\partial y^2} = 1 + \frac{1}{2}\frac{\partial^2 D}{\partial y^2}D^{-1/2} - \frac{1}{4}\left(\frac{\partial D}{\partial y}\right)^2 D^{-3/2}, \tag{2.17}$$

$$\frac{\partial^2\lambda}{\partial x\partial y} = \frac{1}{2}\frac{\partial^2 D}{\partial x\partial y}D^{-1/2} - \frac{1}{4}\left(\frac{\partial D}{\partial x}\frac{\partial D}{\partial y}\right) D^{-3/2}, \tag{2.18}$$

$$\frac{\partial^2\lambda}{\partial y\partial z} = \frac{1}{2}\frac{\partial^2 D}{\partial y\partial z}D^{-1/2} - \frac{1}{4}\left(\frac{\partial D}{\partial y}\frac{\partial D}{\partial z}\right) D^{-3/2}, \tag{2.19}$$

$$\frac{\partial^2\lambda}{\partial z\partial x} = \frac{1}{2}\frac{\partial^2 D}{\partial z\partial x}D^{-1/2} - \frac{1}{4}\left(\frac{\partial D}{\partial z}\frac{\partial D}{\partial x}\right) D^{-3/2}, \tag{2.20}$$

$$\frac{\partial^2\lambda}{\partial z^2} = 1 + \frac{1}{2}\frac{\partial^2 D}{\partial z^2}D^{-1/2} - \frac{1}{4}\left(\frac{\partial D}{\partial z}\right)^2 D^{-3/2}, \tag{2.21}$$

$$\frac{\partial^3\lambda}{\partial x^3} = \frac{1}{2}\frac{\partial^3 D}{\partial x^3}D^{-1/2} - \frac{3}{4}\left(\frac{\partial D}{\partial x}\frac{\partial^2 D}{\partial x^2}\right) D^{-3/2} + \frac{3}{8}\left(\frac{\partial D}{\partial x}\right)^3 D^{-5/2}, \tag{2.22}$$

$$\frac{\partial^3\lambda}{\partial x^2\partial y} = \frac{1}{2}\frac{\partial^3 D}{\partial x^2\partial y}D^{-1/2} - \frac{1}{4}\frac{\partial^2 D}{\partial x^2}\frac{\partial D}{\partial y}D^{-3/2} - \frac{2}{4}\frac{\partial D}{\partial x}\frac{\partial^2 D}{\partial x\partial y}D^{-3/2} + \frac{3}{8}\left(\frac{\partial D}{\partial x}\right)^2\frac{\partial D}{\partial y}D^{-5/2}, \tag{2.23}$$

$$\frac{\partial^3\lambda}{\partial x^2\partial z} = \frac{1}{2}\frac{\partial^3 D}{\partial x^2\partial z}D^{-1/2} - \frac{1}{4}\frac{\partial^2 D}{\partial x^2}\frac{\partial D}{\partial z}D^{-3/2} - \frac{2}{4}\frac{\partial D}{\partial x}\frac{\partial^2 D}{\partial x\partial z}D^{-3/2} + \frac{3}{8}\left(\frac{\partial D}{\partial x}\right)^2\frac{\partial D}{\partial z}D^{-5/2}, \tag{2.24}$$

$$\frac{\partial^3\lambda}{\partial x\partial y^2} = \frac{1}{2}\frac{\partial^3 D}{\partial x\partial y^2}D^{-1/2} - \frac{1}{4}\frac{\partial^2 D}{\partial y^2}\frac{\partial D}{\partial x}D^{-3/2} - \frac{2}{4}\left(\frac{\partial D}{\partial y}\frac{\partial^2 D}{\partial x\partial y}\right) D^{-3/2} + \frac{3}{8}\frac{\partial D}{\partial x}\left(\frac{\partial D}{\partial y}\right)^2 D^{-5/2}, \tag{2.25}$$

$$\frac{\partial^3\lambda}{\partial x\partial z^2} = \frac{1}{2}\frac{\partial^3 D}{\partial x\partial z^2}D^{-1/2} - \frac{1}{4}\frac{\partial^2 D}{\partial z^2}\frac{\partial D}{\partial x}D^{-3/2} - \frac{2}{4}\left(\frac{\partial D}{\partial z}\frac{\partial^2 D}{\partial x\partial z}\right) D^{-3/2} + \frac{3}{8}\frac{\partial D}{\partial x}\left(\frac{\partial D}{\partial z}\right)^2 D^{-5/2}, \tag{2.26}$$

$$\frac{\partial^3\lambda}{\partial x\partial y\partial z}=\frac{1}{2}\frac{\partial^3 D}{\partial x\partial y\partial z}D^{-1/2}-\frac{1}{4}\left(\frac{\partial^2 D}{\partial x\partial y}\frac{\partial D}{\partial z}+\frac{\partial^2 D}{\partial x\partial z}\frac{\partial D}{\partial y}+\frac{\partial^2 D}{\partial y\partial z}\frac{\partial D}{\partial x}\right)D^{-3/2}+\frac{3}{8}\frac{\partial D}{\partial x}\frac{\partial D}{\partial y}\frac{\partial D}{\partial z}D^{-5/2}, \quad (2.27)$$

$$\frac{\partial^3\lambda}{\partial y^3}=\frac{1}{2}\frac{\partial^3 D}{\partial y^3}D^{-1/2}-\frac{3}{4}\left(\frac{\partial D}{\partial y}\frac{\partial^2 D}{\partial y^2}\right)D^{-3/2}+\frac{3}{8}\left(\frac{\partial D}{\partial y}\right)^3 D^{-5/2}, \quad (2.28)$$

$$\frac{\partial^3\lambda}{\partial y^2\partial z}=\frac{1}{2}\frac{\partial^3 D}{\partial y^2\partial z}D^{-1/2}-\frac{1}{4}\frac{\partial^2 D}{\partial y^2}\frac{\partial D}{\partial z}D^{-3/2}-\frac{2}{4}\left(\frac{\partial D}{\partial y}\frac{\partial^2 D}{\partial y\partial z}\right)D^{-3/2}+\frac{3}{8}\frac{\partial D}{\partial z}\left(\frac{\partial D}{\partial y}\right)^2 D^{-5/2} \quad (2.29)$$

$$\frac{\partial^3\lambda}{\partial y\partial z^2}=\frac{1}{2}\frac{\partial^3 D}{\partial y\partial z^2}D^{-1/2}-\frac{1}{4}\frac{\partial^2 D}{\partial z^2}\frac{\partial D}{\partial y}D^{-3/2}-\frac{2}{4}\left(\frac{\partial D}{\partial z}\frac{\partial^2 D}{\partial y\partial z}\right)D^{-3/2}+\frac{3}{8}\frac{\partial D}{\partial y}\left(\frac{\partial D}{\partial z}\right)^2 D^{-5/2} \quad (2.30)$$

$$\frac{\partial^3\lambda}{\partial z^3}=\frac{1}{2}\frac{\partial^3 D}{\partial z^3}D^{-1/2}-\frac{3}{4}\left(\frac{\partial D}{\partial z}\frac{\partial^2 D}{\partial z^2}\right)D^{-3/2}+\frac{3}{8}\left(\frac{\partial D}{\partial z}\right)^3 D^{-5/2}, \quad (2.31)$$

$$\frac{\partial D}{\partial x}=-x\left(a^2+b^2-x^2-y^2-z^2\right)+2b^2x,\quad \frac{\partial D}{\partial y}=-y\left(a^2+b^2-x^2-y^2-z^2\right)+2a^2y, \quad (2.32)$$

$$\frac{\partial D}{\partial z}=-z\left(a^2+b^2-x^2-y^2-z^2\right)+2a^2z, \quad (2.33)$$

$$\frac{\partial^2 D}{\partial x^2}=-\left(a^2+b^2-3x^2-y^2-z^2\right)+2b^2,\quad \frac{\partial^2 D}{\partial y^2}=-\left(a^2+b^2-x^2-3y^2-z^2\right)+2a^2, \quad (2.34)$$

$$\frac{\partial^2 D}{\partial x\partial y}=2xy,\quad \frac{\partial^2 D}{\partial x\partial z}=2xz,\quad \frac{\partial^2 D}{\partial y\partial z}=2yz, \quad (2.35)$$

$$\frac{\partial^2 D}{\partial z^2}=-\left(a^2+b^2-x^2-y^2-3z^2\right)+2a^2, \quad (2.36)$$

$$\frac{\partial^3 D}{\partial x^3}=6x,\quad \frac{\partial^3 D}{\partial y^3}=6y,\quad \frac{\partial^3 D}{\partial x^2\partial y}=2y,\quad \frac{\partial^3 D}{\partial x^2\partial z}=2z,\quad \frac{\partial^3 D}{\partial x\partial y^2}=2x,\quad \frac{\partial^3 D}{\partial x\partial z^2}=2x, \quad (2.37)$$

$$\frac{\partial^3 D}{\partial x\partial y\partial z}=0,\quad \frac{\partial^3 D}{\partial y^2\partial z}=2z,\quad \frac{\partial^3 D}{\partial y\partial z^2}=2y,\quad \frac{\partial^3 D}{\partial z^3}=6z, \quad (2.38)$$

On the ellipsoid surface, $x=a\cos\theta$, $y=b\sin\theta\cos\varphi$, $z=b\sin\theta\sin\varphi$, $\lambda=0$,

$$\sqrt{D}=\frac{a^2+b^2-x^2-y^2-z^2}{2}=\frac{a^2+b^2-a^2\cos^2\theta-b^2\sin^2\theta}{2}=\frac{a^2\sin^2\theta+b^2\cos^2\theta}{2}, \quad (2.39)$$

$$\frac{\partial\lambda}{\partial x}=x+\frac{1}{2}\frac{\partial D}{\partial x}D^{-1/2}=\frac{b^2x}{\sqrt{D}}=\frac{2b^2x}{a^2\sin^2\theta+b^2\cos^2\theta}, \quad (2.40)$$

$$\frac{\partial\lambda}{\partial y}=\frac{a^2y}{\sqrt{D}}=\frac{2a^2y}{a^2\sin^2\theta+b^2\cos^2\theta},\quad \frac{\partial\lambda}{\partial z}=\frac{a^2z}{\sqrt{D}}=\frac{2a^2z}{a^2\sin^2\theta+b^2\cos^2\theta}. \quad (2.41)$$

Then we have the expression

$$x^2\left(b^2+\lambda\right)+\left(y^2+z^2\right)\left(a^2+\lambda\right)-\left(\mathrm{a}^2+\lambda\right)\left(b^2+\lambda\right)=-\left(\lambda-\lambda_1\right)\left(\lambda-\lambda_2\right), \quad (2.42)$$

which gives

$$x^2=\frac{\left(a^2+\lambda_1\right)\left(a^2+\lambda_2\right)}{a^2-b^2}, \quad (2.43)$$

when $\lambda=-a^2$, and

$$y^2+z^2=-\frac{\left(b^2+\lambda_1\right)\left(b^2+\lambda_2\right)}{a^2-b^2}, \quad (2.44)$$

when $\lambda = -b^2$

The gradient of the equation of ellipsoid is

$$\nabla\left(\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}-1\right)=\left(\frac{2x}{a^2}\boldsymbol{e}_x+\frac{2y}{b^2}\boldsymbol{e}_y+\frac{2z}{b^2}\boldsymbol{e}_z\right), \tag{2.45}$$

thus the normal vector is given by

$$\boldsymbol{n}=(n_x,n_y,n_z)=\left(\frac{x}{a^2}\boldsymbol{e}_x+\frac{y}{b^2}\boldsymbol{e}_y+\frac{z}{b^2}\boldsymbol{e}_z\right)/\sqrt{\left(\frac{x}{a^2}\right)^2+\left(\frac{y}{b^2}\right)^2+\left(\frac{z}{b^2}\right)^2}. \tag{2.46}$$

It is shown here that the velocity (u,v,w) satisfies the kinematic condition at the ellipsoid surface ($x=a\cos\theta$, $y=b\sin\theta\cos\varphi$, $z=b\sin\theta\sin\varphi$, $\lambda=0$):

$$n_jv_j=0 \quad\rightarrow\quad n_xu+n_yv+n_zw=0(\text{to be written later}). \tag{2.47}$$

The normal stress at the ellipsoid surface ($x=a\cos\theta$, $y=b\sin\theta\cos\varphi$, $z=b\sin\theta\sin\varphi$, $\lambda=0$) is

$$n_in_jT_{ij}=\gamma\nabla\cdot\boldsymbol{n}. \tag{2.48}$$

The stress tensor for Newtonian fluid is given by

$$\begin{aligned}T_{nn}&=n_in_jT_{ij}=n_in_j\left(-p\delta_{ij}+\mu A_{ij}\right)\\&=-p+\mu\left(n_x^2A_{11}+2n_xn_yA_{12}+2n_xn_zA_{13}+n_y^2A_{22}+2n_yn_zA_{23}+n_z^2A_{33}\right).\end{aligned} \tag{2.49}$$

The stress tensor for the second order fluid is given by

$$B_{ij}=\frac{\partial A_{ij}}{\partial t}+v_k\frac{\partial A_{ij}}{\partial x_k}+A_{ik}L_{kj}+L_{ik}A_{kj}=v_k\frac{\partial A_{ij}}{\partial x_k}+A_{ik}L_{kj}+L_{ik}A_{kj}, \tag{2.50}$$

$$T_{nn}=n_in_jT_{ij}=n_in_j\left(-p\delta_{ij}+\mu A_{ij}+\alpha_1B_{ij}+\alpha_2A_{ik}A_{kj}\right). \tag{2.51}$$

The Bernoulli function is given by

$$\rho\frac{\partial\phi}{\partial t}+\frac{\rho}{2}\left|\nabla\phi\right|^2+p-\hat{\beta}\chi=C=\frac{\rho}{2}U^2+p_\infty \tag{2.52}$$

The normal stress of the second order fluid on the ellipsoid surface is given by

$$\begin{aligned}T_{nn}\quad&=-p_\infty-\tfrac{\rho}{2}U^2+\tfrac{\rho}{2}\left(u^2+v^2+w^2\right)+\mu\left(n_x^2A_{11}+2n_xn_yA_{12}+2n_xn_zA_{13}+n_y^2A_{22}+2n_yn_zA_{23}+n_z^2A_{33}\right)\\&\quad-\left(3\alpha_1+2\alpha_2\right)\chi+\alpha_1n_in_jB_{ij}+\alpha_2n_in_jA_{ik}A_{kj}.\end{aligned} \tag{2.53}$$

with

$$\begin{aligned}\chi\equiv\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}&=\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+2\left(\frac{\partial^2\phi}{\partial x\partial z}\right)^2+\left(\frac{\partial^2\phi}{\partial y^2}\right)^2+2\left(\frac{\partial^2\phi}{\partial y\partial z}\right)^2+\left(\frac{\partial^2\phi}{\partial z^2}\right)^2\\&=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\\&=\frac{1}{4}\text{trace}\left(A_{ik}A_{kj}\right)=\frac{1}{4}\text{trace}\left(C_{ij}\right)\end{aligned} \tag{2.54}$$

$$\begin{aligned}n_in_jA_{ij}&=n_x^2A_{11}+2n_xn_yA_{12}+2n_xn_zA_{13}+n_y^2A_{22}+2n_yn_zA_{23}+n_z^2A_{33}\\&=2\left[n_x^2\frac{\partial u}{\partial x}+2n_xn_y\frac{\partial u}{\partial y}+2n_xn_z\frac{\partial u}{\partial z}+n_y^2\frac{\partial v}{\partial y}+2n_yn_z\frac{\partial v}{\partial z}+n_z^2\frac{\partial w}{\partial z}\right]\end{aligned} \tag{2.55}$$

The term of $n_i n_j v_k \dfrac{\partial A_{ij}}{\partial x_k}$ is expressed as

$$u\frac{\partial A_{11}}{\partial x} + v\frac{\partial A_{11}}{\partial y} + w\frac{\partial A_{11}}{\partial z} = 2u\frac{\partial^2 u}{\partial x^2} + 2v\frac{\partial^2 u}{\partial x\partial y} + 2w\frac{\partial^2 u}{\partial x\partial z}, \tag{2.56}$$

$$u\frac{\partial A_{12}}{\partial x} + v\frac{\partial A_{12}}{\partial y} + w\frac{\partial A_{12}}{\partial z} = 2u\frac{\partial^2 u}{\partial x\partial y} + 2v\frac{\partial^2 u}{\partial y^2} + 2w\frac{\partial^2 u}{\partial y\partial z}, \tag{2.57}$$

$$u\frac{\partial A_{13}}{\partial x} + v\frac{\partial A_{13}}{\partial y} + w\frac{\partial A_{13}}{\partial z} = 2u\frac{\partial^2 u}{\partial x\partial z} + 2v\frac{\partial^2 u}{\partial y\partial z} + 2w\frac{\partial^2 u}{\partial z^2}, \tag{2.58}$$

$$u\frac{\partial A_{22}}{\partial x} + v\frac{\partial A_{22}}{\partial y} + w\frac{\partial A_{22}}{\partial z} = 2u\frac{\partial v}{\partial x\partial y} + 2v\frac{\partial^2 v}{\partial y^2} + 2w\frac{\partial^2 v}{\partial y\partial z}, \tag{2.59}$$

$$u\frac{\partial A_{23}}{\partial x} + v\frac{\partial A_{23}}{\partial y} + w\frac{\partial A_{23}}{\partial z} = 2u\frac{\partial^2 v}{\partial x\partial z} + 2v\frac{\partial^2 v}{\partial y\partial z} + 2w\frac{\partial^2 v}{\partial z^2}, \tag{2.60}$$

$$u\frac{\partial A_{33}}{\partial x} + v\frac{\partial A_{33}}{\partial y} + w\frac{\partial A_{33}}{\partial z} = 2u\frac{\partial^2 w}{\partial x\partial z} + 2v\frac{\partial^2 w}{\partial y\partial z} + 2w\frac{\partial^2 w}{\partial z^2}. \tag{2.61}$$

$$\begin{aligned}
n_i n_j \frac{\partial \phi}{\partial x_k}\frac{\partial A_{ij}}{\partial x_k} &= n_x^2\left(u\frac{\partial A_{11}}{\partial x} + v\frac{\partial A_{11}}{\partial y} + w\frac{\partial A_{11}}{\partial z}\right) + 2n_x n_y\left(u\frac{\partial A_{12}}{\partial x} + v\frac{\partial A_{12}}{\partial y} + w\frac{\partial A_{12}}{\partial z}\right) \\
&+2n_x n_z\left(u\frac{\partial A_{13}}{\partial x} + v\frac{\partial A_{13}}{\partial y} + w\frac{\partial A_{13}}{\partial z}\right) + n_y^2\left(u\frac{\partial A_{22}}{\partial x} + v\frac{\partial A_{22}}{\partial y} + w\frac{\partial A_{22}}{\partial z}\right) \\
&+2n_y n_z\left(u\frac{\partial A_{23}}{\partial x} + v\frac{\partial A_{23}}{\partial y} + w\frac{\partial A_{23}}{\partial z}\right) + n_z^2\left(u\frac{\partial A_{33}}{\partial x} + v\frac{\partial A_{33}}{\partial y} + w\frac{\partial A_{33}}{\partial z}\right) \\
&= 2n_x^2\left(u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial x\partial y} + w\frac{\partial^2 u}{\partial x\partial z}\right) + 4n_x n_y\left(u\frac{\partial^2 u}{\partial x\partial y} + v\frac{\partial^2 u}{\partial y^2} + w\frac{\partial^2 u}{\partial y\partial z}\right) \\
&+4n_x n_z\left(u\frac{\partial^2 u}{\partial x\partial z} + v\frac{\partial^2 u}{\partial y\partial z} + w\frac{\partial^2 u}{\partial z^2}\right) + 2n_y^2\left(u\frac{\partial v}{\partial x\partial y} + v\frac{\partial^2 v}{\partial y^2} + w\frac{\partial^2 v}{\partial y\partial z}\right) \\
&+4n_y n_z\left(u\frac{\partial^2 v}{\partial x\partial z} + v\frac{\partial^2 v}{\partial y\partial z} + w\frac{\partial^2 v}{\partial z^2}\right) + 2n_z^2\left(u\frac{\partial^2 w}{\partial x\partial z} + v\frac{\partial^2 w}{\partial y\partial z} + w\frac{\partial^2 w}{\partial z^2}\right)
\end{aligned} \tag{2.62}$$

$$\begin{aligned}
n_i n_j T_{ij} &= -\left[C + \hat{\beta}\chi - \frac{\rho}{2}\left|\nabla\phi\right|^2\right] + \mu n_i n_j A_{ij} + \alpha_1 n_i n_j \frac{\partial \phi}{\partial x_k}\frac{\partial A_{ij}}{\partial x_k} + (\alpha_1+\alpha_2)\, n_i n_j A_{ik} A_{kj} \\
&= -\left[p_\infty + \frac{\rho}{2}U^2 + \hat{\beta}\chi - \frac{\rho}{2}\left|\nabla\phi\right|^2\right] + \mu n_i n_j A_{ij} + \alpha_1 n_i n_j \frac{\partial \phi}{\partial x_k}\frac{\partial A_{ij}}{\partial x_k} + (\alpha_1+\alpha_2)\, n_i n_j C_{ij} \\
&= -\left[p_\infty + \frac{\rho}{2}U^2 + (3\alpha_1 + 2\alpha_2)\,\chi - \frac{\rho}{2}\left(u^2+v^2+w^2\right)\right] \\
&+2\mu\left(n_x^2\frac{\partial u}{\partial x} + 2n_x n_y\frac{\partial u}{\partial y} + 2n_x n_z\frac{\partial u}{\partial z} + n_y^2\frac{\partial v}{\partial y} + 2n_y n_z\frac{\partial v}{\partial z} + n_z^2\frac{\partial w}{\partial z}\right) \\
&+2\alpha_1\left[n_x^2\left(u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial x\partial y} + w\frac{\partial^2 u}{\partial x\partial z}\right) + 2n_x n_y\left(u\frac{\partial^2 u}{\partial x\partial y} + v\frac{\partial^2 u}{\partial y^2} + w\frac{\partial^2 u}{\partial y\partial z}\right)\right. \\
&+2n_x n_z\left(u\frac{\partial^2 u}{\partial x\partial z} + v\frac{\partial^2 u}{\partial y\partial z} + w\frac{\partial^2 u}{\partial z^2}\right) + n_y^2\left(u\frac{\partial v}{\partial x\partial y} + v\frac{\partial^2 v}{\partial y^2} + w\frac{\partial^2 v}{\partial y\partial z}\right) \\
&\left.+2n_y n_z\left(u\frac{\partial^2 v}{\partial x\partial z} + v\frac{\partial^2 v}{\partial y\partial z} + w\frac{\partial^2 v}{\partial z^2}\right) + n_z^2\left(u\frac{\partial^2 w}{\partial x\partial z} + v\frac{\partial^2 w}{\partial y\partial z} + w\frac{\partial^2 w}{\partial z^2}\right)\right] \\
&+(\alpha_1+\alpha_2)\left[n_x^2 C_{11} + 2n_x n_y C_{12} + 2n_x n_z C_{13} + n_y^2 C_{22} + 2n_y n_z C_{23} + n_z^2 C_{33}\right].
\end{aligned} \tag{2.63}$$

Normalized form:

$$T_{nn}^* = ???\text{(to be written later)} \tag{2.64}$$

2.2 Cartesian coordinates (x, y, z) fixed on the ellipsoid

In the cartesian coordinates (x, y, z) fixed on the ellipsoid, the potential is given by

$$\phi = Ux + xf_0. \tag{2.65}$$

The velocity $\boldsymbol{v} = (u, v, w) = \nabla\phi$ in the frame (x, y, z) is given by

$$u = U + \frac{\partial(xf_0)}{\partial x} = U + f_0 + x\frac{\partial\lambda}{\partial x}\frac{\partial f_0}{\partial\lambda} = U + f_0 + x\frac{\partial\lambda}{\partial x}f_1, \tag{2.66}$$

$$v = \frac{\partial(xf_0)}{\partial y} = x\frac{\partial\lambda}{\partial y}\frac{\partial f_0}{\partial\lambda} = x\frac{\partial\lambda}{\partial y}f_1, \quad w = \frac{\partial(xf_0)}{\partial z} = x\frac{\partial\lambda}{\partial z}\frac{\partial f_0}{\partial\lambda} = x\frac{\partial\lambda}{\partial z}f_1, \tag{2.67}$$

where $f_1 \equiv \partial f_0/\partial\lambda$ and the relevant expressions will be shown in Appendix.

2.3 Newtonian fluid

For irrotational flows of Newtonian fluid, we have

$$\nabla\times\boldsymbol{v} = \nabla\times\nabla\phi = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0, \tag{2.68}$$

$$(L_{ij}) = \left(\frac{\partial^2\phi}{\partial x_i\partial x_j}\right) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix}, \tag{2.69}$$

$$(A_{ij}) = (L_{ij} + L_{ji}) = 2\left(\frac{\partial^2\phi}{\partial x_i\partial x_j}\right) = \begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{pmatrix} = 2\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix}. \tag{2.70}$$

On the boundary of the ellipsoid, the normal vector $\boldsymbol{n}$ is defined as

$$\nabla\left(\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} - 1\right) = 2\left(\frac{x}{a^2}\boldsymbol{e}_x + \frac{y}{b^2}\boldsymbol{e}_y + \frac{z}{b^2}\boldsymbol{e}_z\right), \tag{2.71}$$

$$\boldsymbol{n} = n_x\boldsymbol{e}_x + n_y\boldsymbol{e}_y + n_z\boldsymbol{e}_z = \left(\frac{x}{a^2}\boldsymbol{e}_x + \frac{y}{b^2}\boldsymbol{e}_y + \frac{z}{b^2}\boldsymbol{e}_z\right)/\sqrt{\left(\frac{x}{a^2}\right)^2 + \left(\frac{y}{b^2}\right)^2 + \left(\frac{z}{b^2}\right)^2}. \tag{2.72}$$

The boundary conditions at the boundary are given by the kinematic condition

$$n_i v_i = 0, \tag{2.73}$$

and by the normal stress balance for the stress tensor $T_{ij} = -p\delta_{ij} + \mu A_{ij}$

$$n_i T_{ij} n_j = \gamma\nabla\cdot\boldsymbol{n}, \tag{2.74}$$

where γ is the surface tension coefficient.

3 Second order fluid

For the second order fluid, the gradient of velocity, the lower convective derivative and the stress tensor are given by

$$L_{ij}=\frac{\partial v_i}{\partial x_j},\quad A_{ij}=L_{ij}+L_{ji},\quad B_{ij}=\frac{\partial A_{ij}}{\partial t}+v_k\frac{\partial A_{ij}}{\partial x_k}+A_{ik}L_{kj}+L_{ik}A_{kj}, \tag{3.1}$$

$$T_{ij}=-p\delta_{ij}+\mu A_{ij}+\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj}. \tag{3.2}$$

Using the velocity potential for which

$$\boldsymbol{v}=\nabla\phi,\quad v_i=\frac{\partial\phi}{\partial x_i},\quad \nabla\cdot\boldsymbol{v}=\nabla^2\phi=0,\quad L_{ij}=\frac{\partial^2\phi}{\partial x_i\partial x_j},\quad A_{ij}=2\frac{\partial^2\phi}{\partial x_i\partial x_j},\quad C_{ij}=A_{ik}A_{kj}, \tag{3.3}$$

we have the following expressions

$$\begin{aligned}B_{ij}&=2\frac{\partial^3\phi}{\partial t\partial x_i\partial x_j}+2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}+2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}+2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\\&=2\frac{\partial^3\phi}{\partial t\partial x_i\partial x_j}+2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}+4\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j},\end{aligned} \tag{3.4}$$

$$\frac{\partial B_{ij}}{\partial x_j}=\frac{\partial}{\partial x_j}\left[2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}+4\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right],\quad \frac{\partial T_{ij}}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial A_{ij}}{\partial x_j}+\alpha_1\frac{\partial B_{ij}}{\partial x_j}+\alpha_2\frac{\partial\left(A_{ik}A_{kj}\right)}{\partial x_j}. \tag{3.5}$$

Thus we have the following expressions:

$$\chi\equiv\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k},\quad \frac{\partial}{\partial x_j}\left(v_k\frac{\partial A_{ij}}{\partial x_k}\right)=\frac{\partial\chi}{\partial x_i},\quad \frac{\partial}{\partial x_j}\left(A_{ik}L_{kj}\right)=\frac{\partial\chi}{\partial x_i},\quad \frac{\partial}{\partial x_j}A_{ik}A_{kj}=2\frac{\partial\chi}{\partial x_i}, \tag{3.6}$$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_k}2\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial^2\phi}{\partial x_k\partial x_j}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right),\quad \frac{\partial}{\partial x_j}\left(2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}\right),$$

$$\frac{\partial}{\partial x_j}\left(2\frac{\partial^2\phi}{\partial x_i\partial x_k}2\frac{\partial^2\phi}{\partial x_k\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(2\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}\right),\quad \frac{\partial}{\partial x_j}\left(\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj}\right)=\frac{\partial}{\partial x_i}\left[\left(3\alpha_1+2\alpha_2\right)\chi\right].$$

The equation of motion is given by

$$\rho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\nabla^2 v_i+\frac{\partial}{\partial x_j}\left(\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj}\right), \tag{3.7}$$

whence the Bernoulli function is expressed as

$$\frac{\partial}{\partial x_i}\left[\rho\frac{\partial\phi}{\partial t}+\frac{\rho}{2}\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_j}+p-\hat\beta\chi\right]=0\ \text{ with }\hat\beta=3\alpha_1+2\alpha_2\geq 0, \tag{3.8}$$

$$\rho\frac{\partial\phi}{\partial t}+\frac{\rho}{2}\left|\nabla\phi\right|^2+p-\hat\beta\chi=C(t)=\frac{\rho}{2}U^2+p_\infty. \tag{3.9}$$

The stress tensor is given by

$$T_{ij}=-\left[C+\hat\beta\chi-\frac{\rho}{2}\left|\nabla\phi\right|^2-\rho\frac{\partial\phi}{\partial t}\right]\delta_{ij}+\left[\mu+\alpha_1\left(\frac{\partial}{\partial t}+\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\right)\right]A_{ij}+\left(\alpha_1+\alpha_2\right)A_{ik}A_{kj}, \tag{3.10}$$

which has the normal component at the boundary

$$n_in_jT_{ij}=-\left[C+\hat\beta\chi-\frac{\rho}{2}\left|\nabla\phi\right|^2\right]+\mu n_in_jA_{ij}+\alpha_1 n_in_j\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}A_{ij}+\left(\alpha_1+\alpha_2\right)n_in_jA_{ik}A_{kj}. \tag{3.11}$$

The term is expressed as

$$\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}A_{ij} = u\frac{\partial A_{ij}}{\partial x} + v\frac{\partial A_{ij}}{\partial y} + w\frac{\partial A_{ij}}{\partial z}, \tag{3.12}$$

for which

$$u\frac{\partial A_{11}}{\partial x} + v\frac{\partial A_{11}}{\partial y} + w\frac{\partial A_{11}}{\partial z} = 2u\frac{\partial^2 u}{\partial x^2} + 2v\frac{\partial^2 u}{\partial x\partial y} + 2w\frac{\partial^2 u}{\partial x\partial z}, \tag{3.13}$$

$$u\frac{\partial A_{12}}{\partial x} + v\frac{\partial A_{12}}{\partial y} + w\frac{\partial A_{12}}{\partial z} = 2u\frac{\partial^2 u}{\partial x\partial y} + 2v\frac{\partial^2 u}{\partial y^2} + 2w\frac{\partial^2 u}{\partial y\partial z}, \tag{3.14}$$

$$u\frac{\partial A_{13}}{\partial x} + v\frac{\partial A_{13}}{\partial y} + w\frac{\partial A_{13}}{\partial z} = 2u\frac{\partial^2 u}{\partial x\partial z} + 2v\frac{\partial^2 u}{\partial y\partial z} + 2w\frac{\partial^2 u}{\partial z^2}, \tag{3.15}$$

$$u\frac{\partial A_{22}}{\partial x} + v\frac{\partial A_{22}}{\partial y} + w\frac{\partial A_{22}}{\partial z} = 2u\frac{\partial v}{\partial x\partial y} + 2v\frac{\partial^2 v}{\partial y^2} + 2w\frac{\partial^2 v}{\partial y\partial z}, \tag{3.16}$$

$$u\frac{\partial A_{23}}{\partial x} + v\frac{\partial A_{23}}{\partial y} + w\frac{\partial A_{23}}{\partial z} = 2u\frac{\partial^2 v}{\partial x\partial z} + 2v\frac{\partial^2 v}{\partial y\partial z} + 2w\frac{\partial^2 v}{\partial z^2}, \tag{3.17}$$

$$u\frac{\partial A_{33}}{\partial x} + v\frac{\partial A_{33}}{\partial y} + w\frac{\partial A_{33}}{\partial z} = 2u\frac{\partial^2 w}{\partial x\partial z} + 2v\frac{\partial^2 w}{\partial y\partial z} + 2w\frac{\partial^2 w}{\partial z^2}. \tag{3.18}$$

The stress tensor and the second order term are

$$(A_{ij}) = \begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{pmatrix} = 2\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} = 2\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix}, \tag{3.19}$$

$$(A_{ik}A_{kj}) = (A_{ik}A_{kj})^T = (C_{ij}) = 4\left(\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right) = 4\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$

$$= 4\begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}\right) & \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}\right) & \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}\right) + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}\right) & \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}\right) & \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial x}\right) + \left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial x}\right) & \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial y}\right) & \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial z}\right) \end{pmatrix}$$

$$= 4\begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 & \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) & \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) & \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 & \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right) & \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\right) & \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \end{pmatrix}$$

$$= 4\begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 & \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 & \frac{\partial v}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} & \frac{\partial v}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} & \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \end{pmatrix}. \tag{3.20}$$

A Expressions

Consider the ellipsoid of revolution with $a \neq b$ and $b = c$

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1, \tag{1}$$

moving in the negative direction of the x-axis with velocity U, whence the velocity potential ϕ for which $\boldsymbol{v} = \nabla\phi$ is given by

$$\phi = \frac{ab^2 U x}{2 - \alpha_0} \int_\lambda^\infty \left(a^2 + \lambda\right)^{-3/2} \left(b^2 + \lambda\right)^{-1} \mathrm{d}\lambda = x f_0, \tag{2}$$

where

$$\alpha_0 = ab^2 \int_0^\infty \left(a^2 + \lambda\right)^{-3/2} \left(b^2 + \lambda\right)^{-1} \mathrm{d}\lambda, \tag{3}$$

$$f_0 \equiv f_0(\lambda) = \frac{ab^2 U}{2 - \alpha_0} \int_\lambda^\infty \left(a^2 + \lambda\right)^{-3/2} \left(b^2 + \lambda\right)^{-1} \mathrm{d}\lambda,$$

then

$$f_1 = \frac{\partial f_0}{\partial \lambda} = -\frac{ab^2 U}{2 - \alpha_0} \left(a^2 + \lambda\right)^{-3/2} \left(b^2 + \lambda\right)^{-1},$$

$$f_2 = \frac{\partial^2 f_0}{\partial \lambda^2} = \frac{\partial f_1}{\partial \lambda} = -f_1 \left[\frac{3}{2}\left(a^2 + \lambda\right)^{-1} + \left(b^2 + \lambda\right)^{-1}\right],$$

$$f_3 = \frac{\partial^3 f_0}{\partial \lambda^3} = \frac{\partial f_2}{\partial \lambda} = f_1 \left[\frac{3}{2}\left(a^2 + \lambda\right)^{-1} + \left(b^2 + \lambda\right)^{-1}\right]^2 + f_1 \left[\frac{3}{2}\left(a^2 + \lambda\right)^{-2} + \left(b^2 + \lambda\right)^{-2}\right].$$

In the cartesian coordinates (x, y, z) fixed on the ellipsoid, the potential is given by

$$\phi = Ux + x f_0. \tag{A.1}$$

The velocity $\boldsymbol{v} = (u, v, w) = \nabla\phi$ in the frame (x, y, z) is given by

$$u = U + \frac{\partial (x f_0)}{\partial x} = U + f_0 + x \frac{\partial \lambda}{\partial x}\frac{\partial f_0}{\partial \lambda} = U + f_0 + x \frac{\partial \lambda}{\partial x} f_1, \tag{A.2}$$

$$v = \frac{\partial (x f_0)}{\partial y} = x \frac{\partial \lambda}{\partial y}\frac{\partial f_0}{\partial \lambda} = x \frac{\partial \lambda}{\partial y} f_1, \quad w = \frac{\partial (x f_0)}{\partial z} = x \frac{\partial \lambda}{\partial z}\frac{\partial f_0}{\partial \lambda} = x \frac{\partial \lambda}{\partial z} f_1. \tag{A.3}$$

The components of the strain tensor are expressed as

$$\frac{\partial u}{\partial x} = 2 \frac{\partial \lambda}{\partial x}\frac{\partial f_0}{\partial \lambda} + x \frac{\partial^2 \lambda}{\partial x^2}\frac{\partial f_0}{\partial \lambda} + x \left(\frac{\partial \lambda}{\partial x}\right)^2 \frac{\partial^2 f_0}{\partial \lambda^2} = 2 \frac{\partial \lambda}{\partial x} f_1 + x \frac{\partial^2 \lambda}{\partial x^2} f_1 + x \left(\frac{\partial \lambda}{\partial x}\right)^2 f_2, \tag{A.4}$$

$$\frac{\partial v}{\partial x} = \frac{\partial \lambda}{\partial y}\frac{\partial f_0}{\partial \lambda} + x \frac{\partial^2 \lambda}{\partial x \partial y}\frac{\partial f_0}{\partial \lambda} + x \left(\frac{\partial \lambda}{\partial x}\frac{\partial \lambda}{\partial y}\right) \frac{\partial^2 f_0}{\partial \lambda^2} = \frac{\partial \lambda}{\partial y} f_1 + x \frac{\partial^2 \lambda}{\partial x \partial y} f_1 + x \left(\frac{\partial \lambda}{\partial x}\frac{\partial \lambda}{\partial y}\right) f_2 = \frac{\partial u}{\partial y}, \tag{A.5}$$

$$\frac{\partial w}{\partial x} = \frac{\partial \lambda}{\partial z}\frac{\partial f_0}{\partial \lambda} + x \frac{\partial^2 \lambda}{\partial x \partial z}\frac{\partial f_0}{\partial \lambda} + x \left(\frac{\partial \lambda}{\partial x}\frac{\partial \lambda}{\partial z}\right) \frac{\partial^2 f_0}{\partial \lambda^2} = \frac{\partial \lambda}{\partial z} f_1 + x \frac{\partial^2 \lambda}{\partial x \partial z} f_1 + x \left(\frac{\partial \lambda}{\partial x}\frac{\partial \lambda}{\partial z}\right) f_2 = \frac{\partial u}{\partial z}, \tag{A.6}$$

$$\frac{\partial v}{\partial y} = x \frac{\partial^2 \lambda}{\partial y^2}\frac{\partial f_0}{\partial \lambda} + x \left(\frac{\partial \lambda}{\partial y}\right)^2 \frac{\partial^2 f_0}{\partial \lambda^2} = x \frac{\partial^2 \lambda}{\partial y^2} f_1 + x \left(\frac{\partial \lambda}{\partial y}\right)^2 f_2, \tag{A.7}$$

$$\frac{\partial w}{\partial y} = x \frac{\partial^2 \lambda}{\partial y \partial z}\frac{\partial f_0}{\partial \lambda} + x \left(\frac{\partial \lambda}{\partial y}\frac{\partial \lambda}{\partial z}\right) \frac{\partial^2 f_0}{\partial \lambda^2} = x \frac{\partial^2 \lambda}{\partial y \partial z} f_1 + x \left(\frac{\partial \lambda}{\partial y}\frac{\partial \lambda}{\partial z}\right) f_2 = \frac{\partial v}{\partial z}, \tag{A.8}$$

$$\frac{\partial w}{\partial z} = x \frac{\partial^2 \lambda}{\partial z^2}\frac{\partial f_0}{\partial \lambda} + x \left(\frac{\partial \lambda}{\partial z}\right)^2 \frac{\partial^2 f_0}{\partial \lambda^2} = x \frac{\partial^2 \lambda}{\partial z^2} f_1 + x \left(\frac{\partial \lambda}{\partial z}\right)^2 f_2. \tag{A.9}$$

For the second order fluid, we then need the following expressions

$$\frac{\partial^2 u}{\partial x^2} = 3\frac{\partial^2\lambda}{\partial x^2}f_1 + 3\left(\frac{\partial\lambda}{\partial x}\right)^2 f_2 + x\frac{\partial^3\lambda}{\partial x^3}f_1 + 3x\frac{\partial^2\lambda}{\partial x^2}\frac{\partial\lambda}{\partial x}f_2 + x\left(\frac{\partial\lambda}{\partial x}\right)^3 f_3, \quad (A.10)$$

$$\frac{\partial^2 u}{\partial x\partial y} = 2\frac{\partial^2\lambda}{\partial x\partial y}f_1 + 2\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}f_2 + x\frac{\partial^3\lambda}{\partial x^2\partial y}f_1 + x\frac{\partial^2\lambda}{\partial x^2}\frac{\partial\lambda}{\partial y}f_2 + 2x\frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial x\partial y}f_2 + x\left(\frac{\partial\lambda}{\partial x}\right)^2\frac{\partial\lambda}{\partial y}f_3, \quad (A.11)$$

$$\frac{\partial^2 u}{\partial x\partial z} = 2\frac{\partial^2\lambda}{\partial x\partial z}f_1 + 2\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial z}f_2 + x\frac{\partial^3\lambda}{\partial x^2\partial z}f_1 + x\frac{\partial^2\lambda}{\partial x^2}\frac{\partial\lambda}{\partial z}f_2 + 2x\frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial x\partial z}f_2 + x\left(\frac{\partial\lambda}{\partial x}\right)^2\frac{\partial\lambda}{\partial z}f_3, \quad (A.12)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2\lambda}{\partial y^2}f_1 + \left(\frac{\partial\lambda}{\partial y}\right)^2 f_2 + x\frac{\partial^3\lambda}{\partial x\partial y^2}f_1 + x\left(2\frac{\partial^2\lambda}{\partial x\partial y}\frac{\partial\lambda}{\partial y} + \frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial y^2}\right)f_2 + x\frac{\partial\lambda}{\partial x}\left(\frac{\partial\lambda}{\partial y}\right)^2 f_3 = \frac{\partial^2 u}{\partial y^2}, \quad (A.13)$$

$$\frac{\partial^2 u}{\partial y\partial z} = \frac{\partial^2\lambda}{\partial y\partial z}f_1 + \frac{\partial\lambda}{\partial y}\frac{\partial\lambda}{\partial z}f_2 + x\frac{\partial^3\lambda}{\partial x\partial y\partial z}f_1 + x\left(\frac{\partial^2\lambda}{\partial x\partial y}\frac{\partial\lambda}{\partial z} + \frac{\partial^2\lambda}{\partial x\partial z}\frac{\partial\lambda}{\partial y} + \frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial y\partial z}\right)f_2$$
$$+x\left(\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}\frac{\partial\lambda}{\partial z}\right)f_3 = \frac{\partial^2 v}{\partial x\partial z}, \quad (A.14)$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2\lambda}{\partial z^2}f_1 + \left(\frac{\partial\lambda}{\partial z}\right)^2 f_2 + x\frac{\partial^3\lambda}{\partial x\partial z^2}f_1 + x\left(2\frac{\partial^2\lambda}{\partial x\partial z}\frac{\partial\lambda}{\partial z} + \frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial z^2}\right)f_2 + x\frac{\partial\lambda}{\partial x}\left(\frac{\partial\lambda}{\partial z}\right)^2 f_3, \quad (A.15)$$

$$\frac{\partial^2 v}{\partial y^2} = x\frac{\partial^3\lambda}{\partial y^3}f_1 + 3x\frac{\partial\lambda}{\partial y}\frac{\partial^2\lambda}{\partial y^2}f_2 + x\left(\frac{\partial\lambda}{\partial y}\right)^3 f_3, \quad (A.16)$$

$$\frac{\partial^2 v}{\partial y\partial z} = x\frac{\partial^3\lambda}{\partial y^2\partial z}f_1 + x\frac{\partial^2\lambda}{\partial y^2}\frac{\partial\lambda}{\partial z}f_2 + 2x\frac{\partial\lambda}{\partial y}\frac{\partial^2\lambda}{\partial y\partial z}f_2 + x\left(\frac{\partial\lambda}{\partial y}\right)^2\frac{\partial\lambda}{\partial z}f_3, \quad (A.17)$$

$$\frac{\partial^2 v}{\partial z^2} = x\frac{\partial^3\lambda}{\partial y\partial z^2}f_1 + x\left(\frac{\partial^2\lambda}{\partial z^2}\frac{\partial\lambda}{\partial y} + 2\frac{\partial\lambda}{\partial z}\frac{\partial^2\lambda}{\partial y\partial z}\right)f_2 + x\frac{\partial\lambda}{\partial y}\left(\frac{\partial\lambda}{\partial z}\right)^2 f_3, \quad (A.18)$$

$$\frac{\partial^2 w}{\partial z^2} = x\frac{\partial^3\lambda}{\partial z^3}f_1 + 3x\frac{\partial^2\lambda}{\partial z^2}\frac{\partial\lambda}{\partial z}f_2 + x\left(\frac{\partial\lambda}{\partial z}\right)^3 f_3, \quad (A.19)$$

with

$$\frac{\partial\lambda}{\partial x},\quad \frac{\partial\lambda}{\partial y},\quad \frac{\partial\lambda}{\partial z},$$

$$\frac{\partial^2\lambda}{\partial x^2},\quad \frac{\partial^2\lambda}{\partial x\partial y},\quad \frac{\partial^2\lambda}{\partial y^2},\quad \frac{\partial^2\lambda}{\partial y\partial z},\quad \frac{\partial^2\lambda}{\partial z^2},\quad \frac{\partial^2\lambda}{\partial z\partial x},$$

$$\frac{\partial^3\lambda}{\partial x^3},\quad \frac{\partial^3\lambda}{\partial x^2\partial y},\quad \frac{\partial^3\lambda}{\partial x^2\partial z},\quad \frac{\partial^3\lambda}{\partial x\partial y^2},\quad \frac{\partial^3\lambda}{\partial x\partial z^2},\quad \frac{\partial^3\lambda}{\partial x\partial y\partial z},\quad \frac{\partial^3\lambda}{\partial y^3},\quad \frac{\partial^3\lambda}{\partial y^2\partial z},\quad \frac{\partial^3\lambda}{\partial y\partial z^2},\quad \frac{\partial^3\lambda}{\partial z^3}.$$

VPF around a Sphere in Cartesian coordinates

T.Funada, June 15, 2004 / sphere-june15.tex / printed July 16, 2004

1 Sphere

For a sphere for which $a^2 = b^2 = c^2$, we use the spherical coordinates $(r,\, \theta,\, \varphi)$ together with the cartesian coordinates $(x,\, y,\, z) = (r\cos\theta,\, r\sin\theta\cos\varphi,\, r\sin\theta\sin\varphi)$ to give the equation of λ

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1 \quad \rightarrow \quad x^2+y^2+z^2 = a^2+\lambda \equiv r^2. \tag{1.1}$$

The velocity potential ϕ for which $\boldsymbol{v} = \nabla\phi$ is given by

$$\phi = -\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}$$

$$\rightarrow \quad \phi = -\frac{a^3Ux}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)^3}} = \frac{-a^3Ux}{2-\alpha_0}\frac{2}{3}\left(a^2+\lambda\right)^{-3/2} = \frac{-Uxa^3}{2\sqrt{(a^2+\lambda)^3}} = -\frac{Uxa^3}{2r^3}, \tag{1.2}$$

with α_0 defined as

$$\alpha_0 = abc\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad \rightarrow \quad \alpha_0 = a^3\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)^3}} = \frac{2}{3}. \tag{1.3}$$

The velocity $\boldsymbol{v} = (u,\, v,\, w)$ in the $(x,\, y,\, z)$ coordinates or $\boldsymbol{v} = (v_r,\, v_\theta,\, v_\varphi)$ in the $(r,\, \theta,\, \varphi)$ coordinates are given by

$$\begin{aligned}\boldsymbol{v} = (u,\, v,\, w) = \nabla\phi &= -\boldsymbol{e}_x\frac{\partial}{\partial x}\left(Ux\frac{a^3}{2r^3}\right) - \boldsymbol{e}_y\frac{\partial}{\partial y}\left(Ux\frac{a^3}{2r^3}\right) - \boldsymbol{e}_z\frac{\partial}{\partial z}\left(Ux\frac{a^3}{2r^3}\right)\\ &= -\boldsymbol{e}_xU\left(\frac{a^3}{2r^3} - \frac{3a^3}{2r^4}\frac{x^2}{r}\right) + \boldsymbol{e}_yU\left(\frac{3a^3}{2r^4}\frac{xy}{r}\right) + \boldsymbol{e}_zU\left(\frac{3a^3}{2r^4}\frac{xz}{r}\right),\end{aligned} \tag{1.4}$$

$$\begin{aligned}\boldsymbol{v} = (v_r,\, v_\theta,\, v_\varphi) = \nabla\phi = -\boldsymbol{e}_r\frac{\partial}{\partial r}\left(Ux\frac{a^3}{2r^3}\right) - \boldsymbol{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left(Ux\frac{a^3}{2r^3}\right) - \boldsymbol{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(Ux\frac{a^3}{2r^3}\right)\\ = \boldsymbol{e}_rU\cos\theta\frac{a^3}{r^3} + \boldsymbol{e}_\theta U\sin\theta\frac{a^3}{2r^3}.\end{aligned} \tag{1.5}$$

If the frame is taken on the ellipsoid, the velocity potential is modified as

$$\phi = -Ux\left(1 + \frac{a^3}{2r^3}\right). \tag{1.6}$$

The unit vectors in the spherical coordinates (r, θ, φ) are given by

$$\boldsymbol{e} = \cos\varphi\boldsymbol{e}_1 + \sin\varphi\boldsymbol{e}_2, \quad \boldsymbol{e}_r = \sin\theta\cos\varphi\boldsymbol{e}_1 + \sin\theta\sin\varphi\boldsymbol{e}_2 + \cos\theta\boldsymbol{e}_3 = \sin\theta\boldsymbol{e} + \cos\theta\boldsymbol{e}_3, \tag{1.7}$$

$$\frac{\partial\boldsymbol{e}_r}{\partial\theta} = \cos\theta\cos\varphi\boldsymbol{e}_1 + \cos\theta\sin\varphi\boldsymbol{e}_2 - \sin\theta\boldsymbol{e}_3 = \boldsymbol{e}_\theta, \tag{1.8}$$

$$\frac{\partial\boldsymbol{e}_r}{\partial\varphi} = -\sin\theta\sin\varphi\boldsymbol{e}_1 + \sin\theta\cos\varphi\boldsymbol{e}_2 = \sin\theta\frac{\partial\boldsymbol{e}}{\partial\varphi}, \quad \frac{\partial\boldsymbol{e}_\theta}{\partial\theta} = -\boldsymbol{e}_r. \tag{1.9}$$

1.1 Normal stress on the sphere

The potential is given by

$$\phi = -U\cos\theta\left(r + \frac{a^3}{2r^2}\right), \tag{1.10}$$

whence the velocity $\boldsymbol{v} = \nabla\phi$

$$(v_r,\, v_\theta,\, v_\varphi) = \left(\frac{\partial\phi}{\partial r},\; \frac{1}{r}\frac{\partial\phi}{\partial\theta},\; \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right) = (v_r,\, v_\theta,\, 0)\,, \tag{1.11}$$

$$v_r = \frac{\partial\phi}{\partial r} = -U\cos\theta\left(1 - \frac{a^3}{r^3}\right), \quad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\sin\theta\left(1 + \frac{a^3}{2r^3}\right), \tag{1.12}$$

$$\frac{\partial v_r}{\partial r} = -U\cos\theta\frac{3a^3}{r^4}, \quad \frac{1}{r}\frac{\partial v_r}{\partial\theta} = U\sin\theta\left(\frac{1}{r} - \frac{a^3}{r^4}\right), \tag{1.13}$$

$$\frac{\partial v_\theta}{\partial r} = -U\sin\theta\frac{3a^3}{2r^4}, \quad \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} = U\cos\theta\left(\frac{1}{r} + \frac{a^3}{2r^4}\right), \tag{1.14}$$

$$-\frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial\theta} = -U\sin\theta\left(\frac{1}{r} + \frac{a^3}{2r^4}\right) + U\sin\theta\left(\frac{1}{r} - \frac{a^3}{r^4}\right) = -U\sin\theta\frac{3a^3}{2r^4}, \tag{1.15}$$

$$\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r} = U\cos\theta\left(\frac{1}{r} + \frac{a^3}{2r^4}\right) - U\cos\theta\left(\frac{1}{r} - \frac{a^3}{r^4}\right) = U\cos\theta\frac{3a^3}{2r^4}, \tag{1.16}$$

$$\frac{v_r}{r} + \frac{v_\theta}{r}\cot\theta = -U\cos\theta\left(\frac{1}{r} - \frac{a^3}{r^4}\right) + U\sin\theta\left(\frac{1}{r} + \frac{a^3}{2r^4}\right)\cot\theta = U\cos\theta\frac{3a^3}{2r^4}. \tag{1.17}$$

The gradient of the velocity is given by

$$(L_{ij}) = \begin{pmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{\partial v_\theta}{\partial r} & \dfrac{\partial v_\varphi}{\partial r} \\ -\dfrac{v_\theta}{r} + \dfrac{1}{r}\dfrac{\partial v_r}{\partial\theta} & \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial\theta} + \dfrac{v_r}{r} & \dfrac{1}{r}\dfrac{\partial v_\varphi}{\partial\theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial v_r}{\partial\varphi} - \dfrac{v_\varphi}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial v_\theta}{\partial\varphi} - \dfrac{v_\varphi}{r}\cot\theta & \dfrac{1}{r\sin\theta}\dfrac{\partial v_\varphi}{\partial\varphi} + \dfrac{v_r}{r} + \dfrac{v_\theta}{r}\cot\theta \end{pmatrix} \tag{1.18}$$

$$(L_{ij}) = \frac{3a^3U}{2r^4}\begin{pmatrix} -2\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix}. \tag{1.19}$$

The strain tensor is given by

$$A_{ij} = L_{ij} + L_{ji} = 2L_{ij},$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

$$c_{11} = \frac{9a^6U^2}{r^8}\left(4\cos^2\theta + \sin^2\theta\right), \quad c_{12} = \frac{9a^6U^2}{r^8}\sin\theta\cos\theta, \quad c_{13} = 0,$$

$$c_{21} = \frac{9a^6U^2}{r^8}\sin\theta\cos\theta, \quad c_{22} = \frac{9a^6U^2}{r^8}, \quad c_{23} = 0,$$

$$c_{31} = c_{32} = 0, \quad c_{33} = \frac{9a^6U^2}{r^8}\cos^2\theta,$$

$$\boldsymbol{A}^2 = (A_{ik}A_{kj}) = \frac{9a^6U^2}{r^8}\begin{pmatrix} 4\cos^2\theta + \sin^2\theta & \sin\theta\cos\theta & 0 \\ \sin\theta\cos\theta & 1 & 0 \\ 0 & 0 & \cos^2\theta \end{pmatrix} \tag{1.20}$$

$$\frac{1}{4}\mathrm{tr}\boldsymbol{A}^2 = \frac{9U^2}{4a^2}\left(5\cos^2\theta + \sin^2\theta + 1\right) = \frac{9U^2}{4a^2}\left(4\cos^2\theta + 2\right) = \frac{9U^2}{2a^2}\left(1 + 2\cos^2\theta\right) \tag{1.21}$$

$$(3\alpha_1 + 2\alpha_2)\,\frac{9}{2} = \left(\frac{27}{2}\alpha_1 + 9\alpha_2\right) \tag{1.22}$$

$$p = p_\infty + \frac{\rho}{2}U^2 - \rho\frac{\partial\phi}{\partial t} - \frac{1}{2}\rho\,|\nabla\phi|^2 + \hat{\beta}\chi = p_\infty + \frac{\rho}{2}U^2\left(1 - \frac{9}{4}\sin^2\theta\right) + \frac{U^2}{a^2}\left(\frac{27}{2}\alpha_1 + 9\alpha_2\right)\left(1 + 2\cos^2\theta\right) \tag{1.23}$$

The normal component of $(\boldsymbol{v}\cdot\nabla)A_{ij}$ is

$$\boldsymbol{e}_1\boldsymbol{e}_1\left(v_r\frac{\partial c_{11}}{\partial r}+\frac{v_\theta}{r}\frac{\partial c_{11}}{\partial\theta}\right)+c_{12}\frac{v_\theta}{r}\boldsymbol{e}_1\frac{\partial\boldsymbol{e}_2}{\partial\theta}+c_{21}\frac{v_\theta}{r}\boldsymbol{e}_1\frac{\partial\boldsymbol{e}_2}{\partial\theta}=\boldsymbol{e}_1\boldsymbol{e}_1\left[v_r\frac{\partial c_{11}}{\partial r}+\frac{v_\theta}{r}\frac{\partial c_{11}}{\partial\theta}-\frac{v_\theta}{r}\left(c_{12}+c_{21}\right)\right]$$
$$=\boldsymbol{e}_1\boldsymbol{e}_1\left[v_r\frac{\partial^2 v_r}{\partial r^2}+\frac{v_\theta}{r}\frac{\partial^2 v_r}{\partial\theta\partial r}-\frac{v_\theta}{r}\left(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}+\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\right]=\boldsymbol{e}_1\boldsymbol{e}_1\frac{9U^2}{a^2}\sin^2\theta \quad (1.24)$$

$$n_in_j\left(\boldsymbol{v}\cdot\nabla\right)A_{ij}=\frac{18U^2}{a^2}\sin^2\theta \quad (1.25)$$

$$\begin{aligned}n_iT_{ij}n_j &= -p+n_in_j\left[\left[\mu+\alpha_1\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\right]A_{ij}+\left(\alpha_1+\alpha_2\right)A_{ik}A_{kj}\right]\\ &= -p+\mu n_in_jA_{ij}+n_in_j\alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij}+\left(\alpha_1+\alpha_2\right)n_in_jA_{ik}A_{kj}\\ &= -p+\mu\left(-\frac{6a^3U}{r^4}\cos\theta\right)+n_in_j\alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij}+\left(\alpha_1+\alpha_2\right)\frac{9a^6U^2}{r^8}\left(4\cos^2\theta+\sin^2\theta\right)\end{aligned}$$

$$\begin{aligned}n_in_j\left(T_{ij}\right)_{r=a} &= -p_\infty-\left(\frac{27}{2}\alpha_1+9\alpha_2\right)\frac{U^2}{a^2}\left(1+2\cos^2\theta\right)-\frac{\rho}{2}U^2\left(1-\frac{9}{4}\sin^2\theta\right)\\ &\quad -\frac{6U}{a}\mu\cos\theta+n_in_j\alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij}+\left(\alpha_1+\alpha_2\right)\frac{9U^2}{a^2}\left(4\cos^2\theta+\sin^2\theta\right)\\ &= -p_\infty-\frac{\rho}{2}U^2\left(1-\frac{9}{4}\sin^2\theta\right)-\frac{6U}{a}\mu\cos\theta+n_in_j\alpha_1\left(\boldsymbol{v}\cdot\nabla\right)A_{ij}\\ &\quad +\frac{U^2}{a^2}\alpha_1\left(36\cos^2\theta+9\sin^2\theta-\frac{27}{2}-27\cos^2\theta\right)+\frac{U^2}{a^2}\alpha_2\left(36\cos^2\theta+9\sin^2\theta-9-18\cos^2\theta\right)\\ &= -p_\infty-\frac{\rho}{2}U^2\left(1-\frac{9}{4}\sin^2\theta\right)-\frac{6U}{a}\mu\cos\theta+\frac{18U^2}{a^2}\alpha_1\sin^2\theta-\frac{U^2}{a^2}\frac{9}{2}\alpha_1+\frac{U^2}{a^2}\alpha_2 9\cos^2\theta\end{aligned}$$

Therefore the normalised normal stress on the sphere is now expressed as

$$T^*_{rr}=\frac{T_{rr}+p_\infty}{\rho U^2/2}=-\left(1-\frac{9}{4}\sin^2\theta\right)-\frac{12}{R}\cos\theta+\frac{\alpha_1}{\rho a^2}\left(36\sin^2\theta-9\right)+\frac{\alpha_2}{\rho a^2}18\cos^2\theta. \quad (1.26)$$

2 Sphere in Cartesian coordinates

The potential is given by

$$\phi=-Ux\left(1+\frac{a^3}{2r^3}\right), \quad (2.1)$$

whence the velocity $\boldsymbol{v}=(u,v,w)=\nabla\phi$ in the cartesian coordinates (x,y,z)

$$(u,\,v,\,w)=\left(\frac{\partial\phi}{\partial x},\,\frac{\partial\phi}{\partial y},\,\frac{\partial\phi}{\partial z}\right), \quad (2.2)$$

$$u=\frac{\partial\phi}{\partial x}=-U\left(1+\frac{a^3}{2r^3}-\frac{3a^3x^2}{2r^5}\right),\quad v=\frac{\partial\phi}{\partial y}=U\frac{3a^3xy}{2r^5},\quad w=\frac{\partial\phi}{\partial z}=U\frac{3a^3xz}{2r^5}. \quad (2.3)$$

$$\nabla\cdot\boldsymbol{v}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \quad (2.4)$$

$$\nabla\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u & v & w\end{vmatrix}=\boldsymbol{e}_x\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+\boldsymbol{e}_y\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+\boldsymbol{e}_z\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=0 \quad (2.5)$$

$$\frac{\partial u}{\partial x}=Ua^3\left(\frac{9x}{2r^5}-\frac{15x^3}{2r^7}\right),\quad \frac{\partial u}{\partial y}=Ua^3\left(\frac{3y}{2r^5}-\frac{15x^2y}{2r^7}\right),\quad \frac{\partial u}{\partial z}=Ua^3\left(\frac{3z}{2r^5}-\frac{15x^2z}{2r^7}\right), \tag{2.6}$$

$$\frac{\partial v}{\partial y}=Ua^3\left(\frac{3x}{2r^5}-\frac{15xy^2}{2r^7}\right),\quad \frac{\partial v}{\partial z}=-Ua^3\frac{15xyz}{2r^7},\quad \frac{\partial w}{\partial z}=Ua^3\left(\frac{3x}{2r^5}-\frac{15xz^2}{2r^7}\right), \tag{2.7}$$

$$\frac{\partial^2 u}{\partial x^2}=Ua^3\left(\frac{9}{2r^5}-\frac{45x^2}{r^7}+\frac{105x^4}{2r^9}\right),\quad \frac{\partial^2 u}{\partial x\partial y}=Ua^3\left(\frac{-45xy}{2r^7}+\frac{105x^3y}{2r^9}\right), \tag{2.8}$$

$$\frac{\partial^2 u}{\partial z\partial x}=Ua^3\left(\frac{-45xz}{2r^7}+\frac{105x^3z}{2r^9}\right),\quad \frac{\partial^2 u}{\partial y^2}=Ua^3\left(\frac{3}{2r^5}-\frac{15\left(y^2+x^2\right)}{2r^7}+\frac{105x^2y^2}{2r^9}\right), \tag{2.9}$$

$$\frac{\partial^2 u}{\partial y\partial z}=Ua^3\left(\frac{-15yz}{2r^7}+\frac{105x^2yz}{2r^9}\right),\quad \frac{\partial^2 u}{\partial z^2}=Ua^3\left(\frac{3}{2r^5}-\frac{15\left(z^2+x^2\right)}{2r^7}+\frac{105x^2z^2}{2r^9}\right), \tag{2.10}$$

$$\frac{\partial^2 v}{\partial y^2}=Ua^3\left(\frac{-45xy}{2r^7}+\frac{105xy^3}{2r^9}\right),\quad \frac{\partial^2 v}{\partial y\partial z}=Ua^3\left(\frac{-15xz}{2r^7}+\frac{105xy^2z}{2r^9}\right), \tag{2.11}$$

$$\frac{\partial^2 v}{\partial z^2}=-Ua^3\left(\frac{15xy}{2r^7}-\frac{105xyz^2}{2r^9}\right),\quad \frac{\partial^2 w}{\partial z^2}=Ua^3\left(\frac{-45xz}{2r^7}+\frac{105xz^3}{2r^9}\right). \tag{2.12}$$

A Second order fluid

For the second order fluid, the gradient of velocity, the lower convective derivative and the stress tensor are given by

$$L_{ij}=\frac{\partial v_i}{\partial x_j},\quad A_{ij}=L_{ij}+L_{ji},\quad B_{ij}=\frac{\partial A_{ij}}{\partial t}+v_k\frac{\partial A_{ij}}{\partial x_k}+A_{ik}L_{kj}+L_{ik}A_{kj}, \tag{A.1}$$

$$T_{ij}=-p\delta_{ij}+\mu A_{ij}+\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj}. \tag{A.2}$$

Using the velocity potential for which

$$\boldsymbol{v}=\nabla\phi,\quad v_i=\frac{\partial\phi}{\partial x_i},\quad \nabla\cdot\boldsymbol{v}=\nabla^2\phi=0,\quad L_{ij}=\frac{\partial^2\phi}{\partial x_i\partial x_j},\quad A_{ij}=2\frac{\partial^2\phi}{\partial x_i\partial x_j}, \tag{A.3}$$

we have the following expressions

$$\begin{aligned} B_{ij}&=2\frac{\partial^3\phi}{\partial t\partial x_i\partial x_j}+2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}+2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}+2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\\ &=2\frac{\partial^3\phi}{\partial t\partial x_i\partial x_j}+2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}+4\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}, \end{aligned} \tag{A.4}$$

$$\frac{\partial B_{ij}}{\partial x_j}=\frac{\partial}{\partial x_j}\left[2\frac{\partial\phi}{\partial x_k}\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}+4\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right],\quad \frac{\partial T_{ij}}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial A_{ij}}{\partial x_j}+\alpha_1\frac{\partial B_{ij}}{\partial x_j}+\alpha_2\frac{\partial\left(A_{ik}A_{kj}\right)}{\partial x_j}. \tag{A.5}$$

Thus we have the following expressions:

$$\chi\equiv\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k},\quad \frac{\partial}{\partial x_j}\left(v_k\frac{\partial A_{ij}}{\partial x_k}\right)=\frac{\partial\chi}{\partial x_i},\quad \frac{\partial}{\partial x_j}\left(A_{ik}L_{kj}\right)=\frac{\partial\chi}{\partial x_i},\quad \frac{\partial}{\partial x_j}\left(A_{ik}A_{kj}\right)=2\frac{\partial\chi}{\partial x_i}, \tag{A.6}$$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_k}2\frac{\partial^3\phi}{\partial x_k\partial x_i\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial^2\phi}{\partial x_k\partial x_j}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right),\quad \frac{\partial}{\partial x_j}\left(2\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}\right),$$

$$\frac{\partial}{\partial x_j}\left(2\frac{\partial^2\phi}{\partial x_i\partial x_k}2\frac{\partial^2\phi}{\partial x_k\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(2\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial^2\phi}{\partial x_j\partial x_k}\right),\quad \frac{\partial}{\partial x_j}\left(\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj}\right)=\frac{\partial}{\partial x_i}\left[\left(3\alpha_1+2\alpha_2\right)\chi\right].$$

The equation of motion is given by

$$\rho\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\nabla^2 v_i+\frac{\partial}{\partial x_j}\left(\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj}\right), \tag{A.7}$$

whence the Bernoulli function is expressed as

$$\frac{\partial}{\partial x_i}\left[\rho\frac{\partial\phi}{\partial t}+\frac{\rho}{2}\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_j}+p-\hat\beta\chi\right]=0\ \text{ with } \hat\beta=3\alpha_1+2\alpha_2\geq 0, \tag{A.8}$$

$$\rho\frac{\partial\phi}{\partial t}+\frac{\rho}{2}\left|\nabla\phi\right|^2+p-\hat\beta\chi=C(t). \tag{A.9}$$

The stress tensor is given by

$$T_{ij}=-\left[C+\hat\beta\chi-\frac{\rho}{2}\left|\nabla\phi\right|^2-\rho\frac{\partial\phi}{\partial t}\right]\delta_{ij}+\left[\mu+\alpha_1\left(\frac{\partial}{\partial t}+\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}\right)\right]A_{ij}+\left(\alpha_1+\alpha_2\right)A_{ik}A_{kj}, \tag{A.10}$$

which has the normal component at the boundary

$$n_in_jT_{ij}=-\left[C+\hat\beta\chi-\frac{\rho}{2}\left|\nabla\phi\right|^2\right]+\mu n_in_jA_{ij}+\alpha_1 n_in_j\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}A_{ij}+\left(\alpha_1+\alpha_2\right)n_in_jA_{ik}A_{kj}. \tag{A.11}$$

The term is expressed as

$$\frac{\partial\phi}{\partial x_k}\frac{\partial}{\partial x_k}A_{ij}=u\frac{\partial A_{ij}}{\partial x}+v\frac{\partial A_{ij}}{\partial y}+w\frac{\partial A_{ij}}{\partial z}, \tag{A.12}$$

for which

$$u\frac{\partial A_{11}}{\partial x}+v\frac{\partial A_{11}}{\partial y}+w\frac{\partial A_{11}}{\partial z}=2u\frac{\partial^2 u}{\partial x^2}+2v\frac{\partial^2 u}{\partial x\partial y}+2w\frac{\partial^2 u}{\partial x\partial z}, \tag{A.13}$$

$$u\frac{\partial A_{12}}{\partial x}+v\frac{\partial A_{12}}{\partial y}+w\frac{\partial A_{12}}{\partial z}=2u\frac{\partial^2 u}{\partial x\partial y}+2v\frac{\partial^2 u}{\partial y^2}+2w\frac{\partial^2 u}{\partial y\partial z}, \tag{A.14}$$

$$u\frac{\partial A_{13}}{\partial x}+v\frac{\partial A_{13}}{\partial y}+w\frac{\partial A_{13}}{\partial z}=2u\frac{\partial^2 u}{\partial x\partial z}+2v\frac{\partial^2 u}{\partial y\partial z}+2w\frac{\partial^2 u}{\partial z^2}, \tag{A.15}$$

$$u\frac{\partial A_{22}}{\partial x}+v\frac{\partial A_{22}}{\partial y}+w\frac{\partial A_{22}}{\partial z}=2u\frac{\partial v}{\partial x\partial y}+2v\frac{\partial^2 v}{\partial y^2}+2w\frac{\partial^2 v}{\partial y\partial z}, \tag{A.16}$$

$$u\frac{\partial A_{23}}{\partial x}+v\frac{\partial A_{23}}{\partial y}+w\frac{\partial A_{23}}{\partial z}=2u\frac{\partial^2 v}{\partial x\partial z}+2v\frac{\partial^2 v}{\partial y\partial z}+2w\frac{\partial^2 v}{\partial z^2}, \tag{A.17}$$

$$u\frac{\partial A_{33}}{\partial x}+v\frac{\partial A_{33}}{\partial y}+w\frac{\partial A_{33}}{\partial z}=2u\frac{\partial^2 w}{\partial x\partial z}+2v\frac{\partial^2 w}{\partial y\partial z}+2w\frac{\partial^2 w}{\partial z^2}. \tag{A.18}$$

The stress tensor and the second order term are

$$\left(A_{ij}\right)=\begin{pmatrix}2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z}\end{pmatrix}=2\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{pmatrix}=2\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}\end{pmatrix}, \tag{A.19}$$

$$\left(A_{ik}A_{kj}\right)=\left(A_{ik}A_{kj}\right)^T=4\left(\frac{\partial^2\phi}{\partial x_i\partial x_k}\frac{\partial^2\phi}{\partial x_k\partial x_j}\right)=4\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{pmatrix}$$

$$= 4\begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial x}\right) & \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}\right) & \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}\right) + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}\right) & \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}\right) & \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial x}\right) + \left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial x}\right) & \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial y}\right) & \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial w}{\partial z}\frac{\partial w}{\partial z}\right) \end{pmatrix}$$

$$= 4\begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 & \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) & \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) & \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 & \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\right) \\ \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right) & \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial v}{\partial z}\frac{\partial w}{\partial z}\right) & \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \end{pmatrix}$$

$$= 4\begin{pmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 & \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 & \frac{\partial v}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial z} & \frac{\partial v}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial z} & \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \end{pmatrix}. \tag{A.20}$$

The gradient of the equation of sphere is

$$\nabla\left(x^2+y^2+z^2-a^2\right) = \left(2x\boldsymbol{e}_x + 2y\boldsymbol{e}_y + 2z\boldsymbol{e}_z\right), \tag{A.21}$$

thus the normal vector is given by

$$\boldsymbol{n} = (n_x, n_y, n_z) = \frac{1}{a}\left(x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z\right). \tag{A.22}$$

It is shown here that the velocity (u, v, w) satisfies the kinematic condition at the ellipsoid surface ($x = a\cos\theta$, $y = b\sin\theta\cos\varphi$, $z = b\sin\theta\sin\varphi$, $\lambda = 0$):

$$n_j v_j = 0 \quad \rightarrow \quad n_x u + n_y v + n_z w = 0 (\text{to be written later}). \tag{A.23}$$

The normal stress at the ellipse surface ($x = a\cos\theta$, $y = b\sin\theta\cos\varphi$, $z = b\sin\theta\sin\varphi$, $\lambda = 0$) is

$$n_i n_j T_{ij} = \gamma\nabla\cdot\boldsymbol{n}. \tag{A.24}$$

The stress tensor for Newtonian is given by

$$\begin{aligned} T_{nn} = n_i n_j T_{ij} &= n_i n_j\left(-p\delta_{ij} + \mu A_{ij}\right) \\ = -p + \mu\left(n_x^2 A_{11} + 2n_x n_y A_{12} + 2n_x n_z A_{13} + n_y^2 A_{22} + 2n_y n_z A_{23} + n_z^2 A_{33}\right). \end{aligned} \tag{A.25}$$

The stress tensor for the second order fluid is given by

$$B_{ij} = \frac{\partial A_{ij}}{\partial t} + v_k\frac{\partial A_{ij}}{\partial x_k} + A_{ik}L_{kj} + L_{ik}A_{kj} = v_k\frac{\partial A_{ij}}{\partial x_k} + A_{ik}L_{kj} + L_{ik}A_{kj}, \tag{A.26}$$

$$T_{nn} = n_i n_j T_{ij} = n_i n_j\left(-p\delta_{ij} + \mu A_{ij} + \alpha_1 B_{ij} + \alpha_2 A_{ik}A_{kj}\right). \tag{A.27}$$

The term of $n_i n_j v_k \dfrac{\partial A_{ij}}{\partial x_k}$ is expressed as

$$n_i n_j v_k\frac{\partial A_{ij}}{\partial x_k} = n_x^2 v_k\frac{\partial A_{11}}{\partial x_k} + 2n_x n_y v_k\frac{\partial A_{12}}{\partial x_k} + 2n_x n_z v_k\frac{\partial A_{13}}{\partial x_k} + n_y^2 v_k\frac{\partial A_{22}}{\partial x_k} + 2n_y n_z v_k\frac{\partial A_{23}}{\partial x_k} + n_z^2 v_k\frac{\partial A_{33}}{\partial x_k}. \tag{A.28}$$

The normal stress of the second order fluid on the ellipse surface is given by

$$\begin{aligned} T_{nn} &= -p_\infty - \tfrac{\rho}{2}U^2 + \tfrac{\rho}{2}\left(u^2+v^2+w^2\right) + \mu\left(n_x^2 A_{11} + 2n_x n_y A_{12} + 2n_x n_z A_{13} + n_y^2 A_{22} + 2n_y n_z A_{23} + n_z^2 A_{33}\right) \\ &\quad - \left(3\alpha_1 + 2\alpha_2\right)\chi + \alpha_1 n_i n_j B_{ij} + \alpha_2 n_i n_j A_{ik}A_{kj}. \end{aligned} \tag{A.29}$$

with

$$\chi = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2. \tag{A.30}$$

Normalized form:

$$T^*_{nn} = ???\text{(to be written later)} \tag{A.31}$$

VPF around an elliptic cylinder

T.Funada, May 23, June 3, 2004 / may23a.tex / printed July 16, 2004

7 Elliptic cylinder

For an elliptic cylinder for which $a^2 \neq b^2$ and $c^2 \to \infty$, we have

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1 \quad \to \quad \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1. \tag{7.1}$$

The velocity potential ϕ (for which $\boldsymbol{v}=(u,v)=\nabla\phi$ in the cartesian coordinates (x,y)) is given for a flow around an elliptic cylinder moving with a uniform speed U in the negative x-direction

$$\phi=\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad \to \quad \phi=\frac{abUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)}}, \tag{7.2}$$

$$\alpha_0=abc\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad \to \quad \alpha_0=ab\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)}}. \tag{7.3}$$

The integral in the potential may be expressed as

$$\int \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)}}=\int \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{\left(\lambda+\frac{a^2+b^2}{2}\right)^2-\left(\frac{a^2-b^2}{2}\right)^2}}.$$

With the relations for $a>b$

$$\left(a^2+\lambda\right)\left(b^2+\lambda\right)=\left(\lambda+\frac{a^2+b^2}{2}\right)^2-\left(\frac{a^2+b^2}{2}\right)^2+a^2b^2, \quad \lambda+\frac{a^2+b^2}{2}=\frac{a^2-b^2}{2}\cosh\xi,$$

$$a^2+\lambda=\lambda+\frac{a^2+b^2}{2}+\frac{a^2-b^2}{2}=\frac{a^2-b^2}{2}(1+\cosh\xi), \quad \mathrm{d}\lambda=\frac{a^2-b^2}{2}\sinh\xi\mathrm{d}\xi.$$

it is then arranged as

$$\int \frac{\frac{a^2-b^2}{2}\sinh\xi\mathrm{d}\xi}{\frac{a^2-b^2}{2}(1+\cosh\xi)\left(\frac{a^2-b^2}{2}\right)\sqrt{\cosh^2\xi-1}}=\int\frac{\mathrm{d}\xi}{\frac{a^2-b^2}{2}(1+\cosh\xi)}=\int\frac{\mathrm{d}\xi}{\frac{a^2-b^2}{2}2\cosh^2\frac{\xi}{2}}=\frac{1}{\frac{a^2-b^2}{2}}\tanh\frac{\xi}{2}$$

$$=\frac{1}{\frac{a^2-b^2}{2}}\frac{2\sinh\frac{\xi}{2}\cosh\frac{\xi}{2}}{2\cosh^2\frac{\xi}{2}}=\frac{1}{\frac{a^2-b^2}{2}}\frac{\sinh\xi}{1+\cosh\xi}=\frac{2}{a^2-b^2}\frac{\sqrt{(\lambda+a^2)(\lambda+b^2)}}{\lambda+a^2}=\frac{2}{a^2-b^2}\sqrt{\frac{\lambda+b^2}{\lambda+a^2}}.$$

With the relations for $a<b$

$$\left(a^2+\lambda\right)\left(b^2+\lambda\right)=\left(\lambda+\frac{a^2+b^2}{2}\right)^2-\left(\frac{a^2+b^2}{2}\right)^2+a^2b^2, \quad \lambda+\frac{a^2+b^2}{2}=\frac{b^2-a^2}{2}\cosh\xi,$$

$$a^2+\lambda=\lambda+\frac{a^2+b^2}{2}+\frac{a^2-b^2}{2}=\frac{b^2-a^2}{2}(-1+\cosh\xi), \quad \mathrm{d}\lambda=\frac{b^2-a^2}{2}\sinh\xi\mathrm{d}\xi.$$

it is then arranged as

$$\int \frac{\frac{b^2-a^2}{2}\sinh\xi\mathrm{d}\xi}{\frac{b^2-a^2}{2}(-1+\cosh\xi)\left(\frac{b^2-a^2}{2}\right)\sqrt{\cosh^2\xi-1}}=\int\frac{\mathrm{d}\xi}{\frac{b^2-a^2}{2}(-1+\cosh\xi)}=\int\frac{\mathrm{d}\xi}{\frac{b^2-a^2}{2}2\sinh^2\frac{\xi}{2}}=\frac{-1}{\frac{b^2-a^2}{2}}\coth\frac{\xi}{2}$$

$$=\frac{1}{\frac{a^2-b^2}{2}}\frac{2\cosh^2\frac{\xi}{2}}{2\sinh\frac{\xi}{2}\cosh\frac{\xi}{2}}=\frac{1}{\frac{a^2-b^2}{2}}\frac{1+\cosh\xi}{\sinh\xi}=\frac{2}{a^2-b^2}\frac{\lambda+b^2}{\sqrt{(\lambda+a^2)(\lambda+b^2)}}=\frac{2}{a^2-b^2}\sqrt{\frac{\lambda+b^2}{\lambda+a^2}}.$$

Differentiation gives the integrand:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{2}{a^2-b^2}\sqrt{\frac{\lambda+b^2}{\lambda+a^2}}\right)=\frac{2}{a^2-b^2}\left\{\frac{-1}{2}\frac{\sqrt{(\lambda+b^2)}}{(\lambda+a^2)^{3/2}}+\frac{1}{2}\frac{1}{\sqrt{(\lambda+a^2)(\lambda+b^2)}}\right\}$$

$$=\frac{1}{a^2-b^2}\frac{-\left(\lambda+b^2\right)+\lambda+a^2}{(\lambda+a^2)\sqrt{(\lambda+a^2)(\lambda+b^2)}}=\frac{1}{(\lambda+a^2)\sqrt{(\lambda+a^2)(\lambda+b^2)}}.$$

Therefore we have α_0 and the velocity potential ϕ

$$\alpha_0=ab\int_0^\infty\frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)}}=\frac{2ab}{a^2-b^2}\left(1-\frac{b}{a}\right)=\frac{2b}{a+b},$$

$$\phi=\frac{abUx}{2-\alpha_0}\int_\lambda^\infty\frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)}}=\frac{abUx}{2-\alpha_0}\frac{2}{a^2-b^2}\left(1-\sqrt{\frac{\lambda+b^2}{\lambda+a^2}}\right)=\frac{bUx}{a-b}\left(1-\sqrt{\frac{\lambda+b^2}{\lambda+a^2}}\right),$$

where $\lambda\equiv\lambda(x,y)$.

The potential ϕ for a flow around an elliptic cylinder (which is of a uniform velocity U away from the cylinder) is now expressed as

$$\phi=Ux+\frac{bUx}{a-b}\left(1-\sqrt{\frac{\lambda+b^2}{\lambda+a^2}}\right)=Ux+\frac{bUx}{a-b}\left(1-f_0\right), \tag{7.4}$$

with

$$f_0=\sqrt{\frac{\lambda+b^2}{\lambda+a^2}},\quad f_1=\frac{\partial f_0}{\partial\lambda}=\frac{\left(a^2-b^2\right)/2}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)}}, \tag{7.5}$$

$$f_2=\frac{\partial^2 f_0}{\partial\lambda^2}=\frac{\left(a^2-b^2\right)}{2}\left[-\frac{3}{2}\left(a^2+\lambda\right)^{-5/2}\left(b^2+\lambda\right)^{-1/2}-\frac{1}{2}\left(a^2+\lambda\right)^{-3/2}\left(b^2+\lambda\right)^{-3/2}\right]$$

$$=\frac{\left(a^2-b^2\right)}{2}\frac{-\left(4\lambda+a^2+3b^2\right)/2}{(a^2+\lambda)^2(b^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)}}, \tag{7.6}$$

$$f_3=\frac{\partial^3 f_0}{\partial\lambda^3}=\frac{\left(a^2-b^2\right)}{2}\left[\frac{15}{4}\left(a^2+\lambda\right)^{-7/2}\left(b^2+\lambda\right)^{-1/2}+\frac{3}{2}\left(a^2+\lambda\right)^{-5/2}\left(b^2+\lambda\right)^{-3/2}\right.$$
$$\left.+\frac{3}{4}\left(a^2+\lambda\right)^{-3/2}\left(b^2+\lambda\right)^{-5/2}\right]. \tag{7.7}$$

The velocity (u,v) is given by

$$u=\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial x}+\frac{\partial\lambda}{\partial x}\frac{\partial\phi}{\partial\lambda}=U+\frac{bU}{a-b}\left(1-f_0\right)-\frac{bUx}{a-b}\frac{\partial\lambda}{\partial x}\frac{\partial f_0}{\partial\lambda}=U+\frac{bU}{a-b}\left(1-f_0\right)-\frac{bUx}{a-b}\frac{\partial\lambda}{\partial x}f_1, \tag{7.8}$$

$$v=\frac{\partial\phi}{\partial y}=\frac{\partial\lambda}{\partial y}\frac{\partial\phi}{\partial\lambda}=-\frac{bUx}{a-b}\frac{\partial\lambda}{\partial y}\frac{\partial f_0}{\partial\lambda}=-\frac{bUx}{a-b}\frac{\partial\lambda}{\partial y}f_1, \tag{7.9}$$

$$\frac{\partial u}{\partial x}=\frac{\partial^2\phi}{\partial x^2}=-\frac{2bU}{a-b}\frac{\partial\lambda}{\partial x}f_1-\frac{bUx}{a-b}\left[\frac{\partial^2\lambda}{\partial x^2}f_1+\left(\frac{\partial\lambda}{\partial x}\right)^2 f_2\right], \tag{7.10}$$

$$\frac{\partial u}{\partial y}=\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial v}{\partial x}=-\frac{bU}{a-b}\frac{\partial\lambda}{\partial y}f_1-\frac{bUx}{a-b}\left[\frac{\partial^2\lambda}{\partial x\partial y}f_1+\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}f_2\right], \tag{7.11}$$

$$\frac{\partial v}{\partial y}=\frac{\partial^2\phi}{\partial y^2}=-\frac{bUx}{a-b}\left[\frac{\partial^2\lambda}{\partial y^2}f_1+\left(\frac{\partial\lambda}{\partial y}\right)^2 f_2\right]. \tag{7.12}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^3\phi}{\partial x^3}=-\frac{3bU}{a-b}\left[\frac{\partial^2\lambda}{\partial x^2}f_1+\left(\frac{\partial\lambda}{\partial x}\right)^2 f_2\right]-\frac{bUx}{a-b}\left[\frac{\partial^3\lambda}{\partial x^3}f_1+3\frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial x^2}f_2+\left(\frac{\partial\lambda}{\partial x}\right)^3 f_3\right], \tag{7.13}$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^3\phi}{\partial x^2\partial y}=\frac{\partial^2 v}{\partial x^2}=-\frac{2bU}{a-b}\left[\frac{\partial^2\lambda}{\partial x\partial y}f_1+\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial y}f_2\right]$$
$$-\frac{bUx}{a-b}\left[\frac{\partial^3\lambda}{\partial x^2\partial y}f_1+2\frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial x\partial y}f_2+\frac{\partial^2\lambda}{\partial x^2}\frac{\partial\lambda}{\partial y}f_2+\left(\frac{\partial\lambda}{\partial x}\right)^2\frac{\partial\lambda}{\partial y}f_3\right], \tag{7.14}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^3\phi}{\partial x\partial y^2}=\frac{\partial^2 v}{\partial x\partial y}=-\frac{bU}{a-b}\left[\frac{\partial^2\lambda}{\partial y^2}f_1+\left(\frac{\partial\lambda}{\partial y}\right)^2 f_2\right]$$
$$-\frac{bUx}{a-b}\left[\frac{\partial^3\lambda}{\partial x\partial y^2}f_1+2\frac{\partial^2\lambda}{\partial x\partial y}\frac{\partial\lambda}{\partial y}f_2+\frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial y^2}f_2+\frac{\partial\lambda}{\partial x}\left(\frac{\partial\lambda}{\partial y}\right)^2 f_3\right], \tag{7.15}$$

$$(L_{ij})=\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}=\begin{pmatrix}\frac{\partial^2\phi}{\partial x^2} & \frac{\partial^2\phi}{\partial y\partial x}\\ \frac{\partial^2\phi}{\partial y\partial x} & \frac{\partial^2\phi}{\partial y^2}\end{pmatrix}=\begin{pmatrix}\frac{\partial^2\phi}{\partial x^2} & \frac{\partial^2\phi}{\partial y\partial x}\\ \frac{\partial^2\phi}{\partial y\partial x} & -\frac{\partial^2\phi}{\partial x^2}\end{pmatrix}, \tag{7.16}$$

$$(A_{ij})=(L_{ij}+L_{ji})=\begin{pmatrix}2\frac{\partial^2\phi}{\partial x^2} & 2\frac{\partial^2\phi}{\partial y\partial x}\\ 2\frac{\partial^2\phi}{\partial y\partial x} & 2\frac{\partial^2\phi}{\partial y^2}\end{pmatrix}=\begin{pmatrix}2\frac{\partial^2\phi}{\partial x^2} & 2\frac{\partial^2\phi}{\partial y\partial x}\\ 2\frac{\partial^2\phi}{\partial y\partial x} & -2\frac{\partial^2\phi}{\partial x^2}\end{pmatrix}, \tag{7.17}$$

$$(A_{ik}A_{kj})=\begin{pmatrix}4\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+4\left(\frac{\partial^2\phi}{\partial y\partial x}\right)^2 & 4\frac{\partial^2\phi}{\partial y\partial x}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)\\ 4\frac{\partial^2\phi}{\partial y\partial x}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right) & 4\left(\frac{\partial^2\phi}{\partial y\partial x}\right)^2+4\left(\frac{\partial^2\phi}{\partial y^2}\right)^2\end{pmatrix}$$
$$=4\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial y\partial x}\right)^2\right]\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=2\chi\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{7.18}$$

$$\chi=\frac{\partial^2\phi}{\partial x_i\partial x_j}\frac{\partial^2\phi}{\partial x_j\partial x_i}=\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\phi}{\partial y^2}\right)^2=2\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 \tag{7.19}$$

The stress tensor for Newtonian is given by

$$T_{ij}=-p\delta_{ij}+\mu A_{ij}, \tag{7.20}$$

and the Bernoulli function is given by

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left|\nabla\phi\right|^2+\frac{p}{\rho}=\frac{1}{2}U^2+\frac{p_\infty}{\rho}\quad\rightarrow\quad p=p_\infty+\frac{\rho}{2}U^2-\rho\frac{\partial\phi}{\partial t}-\frac{\rho}{2}\left(u^2+v^2\right). \tag{7.21}$$

From the paper of Wang & Joseph (2003), the stress tensor for the second order fluid is given by

$$B_{ij}=\frac{\partial A_{ij}}{\partial t}+v_k\frac{\partial A_{ij}}{\partial x_k}+A_{ik}L_{kj}+L_{ik}A_{kj}, \tag{7.22}$$

$$T_{ij}=-p\delta_{ij}+\mu A_{ij}+\alpha_1 B_{ij}+\alpha_2 A_{ik}A_{kj}, \tag{7.23}$$

and the Bernoulli function is given by

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left|\nabla\phi\right|^2+\frac{p}{\rho}-\frac{1}{\rho}\left(3\alpha_1+2\alpha_2\right)\chi=\frac{1}{2}U^2+\frac{p_\infty}{\rho}$$
$$\rightarrow\quad p=p_\infty+\frac{\rho}{2}U^2-\rho\frac{\partial\phi}{\partial t}-\frac{\rho}{2}\left(u^2+v^2\right)+\left(3\alpha_1+2\alpha_2\right)\chi. \tag{7.24}$$

7.1 Geometry

From the equation of an ellipse:

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \tag{7.25}$$

we have the equation of λ

$$\begin{aligned} x^2\left(b^2+\lambda\right)+y^2\left(a^2+\lambda\right)-\left(\mathrm{a}^2+\lambda\right)\left(b^2+\lambda\right)&=0,\\ -\lambda^2+\left(-a^2-b^2+x^2+y^2\right)\lambda-a^2b^2+x^2b^2+y^2a^2&=0, \end{aligned} \tag{7.26}$$

which gives the solution λ

$$\lambda=-\frac{a^2+b^2-x^2-y^2}{2}\pm\sqrt{\left(\frac{a^2+b^2-x^2-y^2}{2}\right)^2-a^2b^2+x^2b^2+y^2a^2}=\lambda_1,\ \lambda_2 \text{ say.} \tag{7.27}$$

For an ellipse with $a=2$ and $b=1$, we have $\lambda_1>0$ and $\lambda_2<-1$; the former gives an ellipse and the latter gives a hyperboloid. Therefore we use $\lambda=\lambda_1$

$$\lambda=-\frac{a^2+b^2-x^2-y^2}{2}+\sqrt{\left(\frac{a^2+b^2-x^2-y^2}{2}\right)^2-a^2b^2+x^2b^2+y^2a^2}, \tag{7.28}$$

$$\lambda=-\frac{a^2+b^2-x^2-y^2}{2}+D^{1/2},\quad D=\frac{1}{4}\left(a^2+b^2-x^2-y^2\right)^2-a^2b^2+x^2b^2+y^2a^2 \tag{7.29}$$

whence

$$\frac{\partial\lambda}{\partial x}=x+\frac{1}{2}\frac{\partial D}{\partial x}D^{-1/2},\quad \frac{\partial\lambda}{\partial y}=y+\frac{1}{2}\frac{\partial D}{\partial y}D^{-1/2} \tag{7.30}$$

$$\frac{\partial^2\lambda}{\partial x^2}=1+\frac{1}{2}\frac{\partial^2 D}{\partial x^2}D^{-1/2}-\frac{1}{4}\left(\frac{\partial D}{\partial x}\right)^2D^{-3/2}, \tag{7.31}$$

$$\frac{\partial^2\lambda}{\partial y^2}=1+\frac{1}{2}\frac{\partial^2 D}{\partial y^2}D^{-1/2}-\frac{1}{4}\left(\frac{\partial D}{\partial y}\right)^2D^{-3/2}, \tag{7.32}$$

$$\frac{\partial^2\lambda}{\partial x\partial y}=\frac{1}{2}\frac{\partial^2 D}{\partial x\partial y}D^{-1/2}-\frac{1}{4}\left(\frac{\partial D}{\partial x}\frac{\partial D}{\partial y}\right)D^{-3/2} \tag{7.33}$$

$$\frac{\partial^3\lambda}{\partial x^3}=\frac{1}{2}\frac{\partial^3 D}{\partial x^3}D^{-1/2}-\frac{3}{4}\left(\frac{\partial D}{\partial x}\frac{\partial^2 D}{\partial x^2}\right)D^{-3/2}+\frac{3}{8}\left(\frac{\partial D}{\partial x}\right)^3D^{-5/2}, \tag{7.34}$$

$$\frac{\partial^3\lambda}{\partial x\partial y^2}=\frac{1}{2}\frac{\partial^3 D}{\partial x\partial y^2}D^{-1/2}-\frac{1}{4}\frac{\partial^2 D}{\partial y^2}\frac{\partial D}{\partial x}D^{-3/2}-\frac{2}{4}\left(\frac{\partial D}{\partial y}\frac{\partial^2 D}{\partial x\partial y}\right)D^{-3/2}+\frac{3}{8}\frac{\partial D}{\partial x}\left(\frac{\partial D}{\partial y}\right)^2D^{-5/2}, \tag{7.35}$$

$$\frac{\partial^3\lambda}{\partial x^2\partial y}=\frac{1}{2}\frac{\partial^3 D}{\partial x^2\partial y}D^{-1/2}-\frac{1}{4}\frac{\partial^2 D}{\partial x^2}\frac{\partial D}{\partial y}D^{-3/2}-\frac{2}{4}\frac{\partial D}{\partial x}\frac{\partial^2 D}{\partial x\partial y}D^{-3/2}+\frac{3}{8}\left(\frac{\partial D}{\partial x}\right)^2\frac{\partial D}{\partial y}D^{-5/2}, \tag{7.36}$$

$$\frac{\partial^3\lambda}{\partial y^3}=\frac{1}{2}\frac{\partial^3 D}{\partial y^3}D^{-1/2}-\frac{3}{4}\left(\frac{\partial D}{\partial y}\frac{\partial^2 D}{\partial y^2}\right)D^{-3/2}+\frac{3}{8}\left(\frac{\partial D}{\partial y}\right)^3D^{-5/2}, \tag{7.37}$$

$$\frac{\partial D}{\partial x}=-x\left(a^2+b^2-x^2-y^2\right)+2b^2x,\quad \frac{\partial D}{\partial y}=-y\left(a^2+b^2-x^2-y^2\right)+2a^2y, \tag{7.38}$$

$$\frac{\partial^2 D}{\partial x^2}=-\left(a^2+b^2-3x^2-y^2\right)+2b^2,\quad \frac{\partial^2 D}{\partial y^2}=-\left(a^2+b^2-x^2-3y^2\right)+2a^2, \tag{7.39}$$

$$\frac{\partial^2 D}{\partial x\partial y}=2xy,\quad \frac{\partial^3 D}{\partial x^3}=6x,\quad \frac{\partial^3 D}{\partial y^3}=6y,\quad \frac{\partial^3 D}{\partial x^2\partial y}=2y,\quad \frac{\partial^3 D}{\partial x\partial y^2}=2x, \tag{7.40}$$

On the ellipse surface, $x = a\cos\theta$, $y = b\sin\theta$, $\lambda = 0$,

$$\sqrt{D} = \frac{a^2+b^2-x^2-y^2}{2} = \frac{a^2+b^2-a^2\cos^2\theta-b^2\sin^2\theta}{2} = \frac{a^2\sin^2\theta+b^2\cos^2\theta}{2}, \tag{7.41}$$

$$\frac{\partial\lambda}{\partial x} = \frac{b^2x}{\sqrt{D}} = \frac{2b^2x}{a^2\sin^2\theta+b^2\cos^2\theta}, \quad \frac{\partial\lambda}{\partial y} = \frac{a^2y}{\sqrt{D}} = \frac{2a^2y}{a^2\sin^2\theta+b^2\cos^2\theta}. \tag{7.42}$$

Then we have the expression

$$x^2\left(b^2+\lambda\right)+y^2\left(a^2+\lambda\right)-\left(\mathrm{a}^2+\lambda\right)\left(b^2+\lambda\right) = -\left(\lambda-\lambda_1\right)\left(\lambda-\lambda_2\right), \tag{7.43}$$

which gives

$$x^2 = \frac{\left(a^2+\lambda_1\right)\left(a^2+\lambda_2\right)}{a^2-b^2}, \tag{7.44}$$

when $\lambda = -a^2$, and

$$y^2 = -\frac{\left(b^2+\lambda_1\right)\left(b^2+\lambda_2\right)}{a^2-b^2}, \tag{7.45}$$

when $\lambda = -b^2$

The gradient of the equation of ellipse is

$$\nabla\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right) = \left(\frac{2x}{a^2}\boldsymbol{e}_x+\frac{2y}{b^2}\boldsymbol{e}_y\right), \tag{7.46}$$

thus the normal vector is given by

$$\boldsymbol{n} = (n_x, n_y) = \left(\frac{x}{a^2}\boldsymbol{e}_x+\frac{y}{b^2}\boldsymbol{e}_y\right)/\sqrt{\left(\frac{x}{a^2}\right)^2+\left(\frac{y}{b^2}\right)^2}. \tag{7.47}$$

It is shown here that the velocity (u, v) satisfies the kinematic condition at the ellipse surface ($x = a\cos\theta$, $y = b\sin\theta$, $\lambda = 0$):

$$n_jv_j = 0 \quad \rightarrow \quad n_xu+n_yv = \left[U+\frac{bU}{a-b}\left(1-f_0\right)-\frac{bUx}{a-b}\frac{\partial\lambda}{\partial x}f_1\right]n_x-\frac{bUx}{a-b}\frac{\partial\lambda}{\partial y}f_1n_y = 0, \tag{7.48}$$

$$\rightarrow \quad \left[U+\frac{bU}{a-b}\left(1-\frac{b}{a}\right)\right]\frac{x}{a^2}-\frac{bUx}{a-b}\frac{a^2-b^2}{2a^2ab}\left[\frac{\partial\lambda}{\partial x}\frac{x}{a^2}+\frac{\partial\lambda}{\partial y}\frac{y}{b^2}\right] = 0, \tag{7.49}$$

$$\rightarrow \quad U\frac{a+b}{a^2}\cos\theta-U\cos\theta\frac{a+b}{2a^2}\left[\frac{2b^2x}{a^2\sin^2\theta+b^2\cos^2\theta}\frac{x}{a^2}+\frac{2a^2y}{a^2\sin^2\theta+b^2\cos^2\theta}\frac{y}{b^2}\right] = 0. \tag{7.50}$$

The normal stress at the ellipse surface ($x = a\cos\theta$, $y = b\sin\theta$, $\lambda = 0$) is

$$n_in_jT_{ij} = \gamma\nabla\cdot\boldsymbol{n}. \tag{7.51}$$

The stress tensor for Newtonian is given by

$$\begin{aligned} T_{nn} = n_in_jT_{ij} = n_in_j\left(-p\delta_{ij}+\mu A_{ij}\right) &= -p+\mu\left(n_x^2A_{11}+2n_xn_yA_{12}+n_y^2A_{22}\right) \\ &= -p_\infty-\frac{\rho}{2}U^2+\frac{\rho}{2}\left(u^2+v^2\right)+\mu\left(\left(n_x^2-n_y^2\right)A_{11}+2n_xn_yA_{12}\right) \end{aligned} \tag{7.52}$$

The stress tensor for the second order fluid is given by

$$B_{ij} = \frac{\partial A_{ij}}{\partial t} + v_k \frac{\partial A_{ij}}{\partial x_k} + A_{ik}L_{kj} + L_{ik}A_{kj} = v_k \frac{\partial A_{ij}}{\partial x_k} + A_{ik}L_{kj} + L_{ik}A_{kj}, \tag{7.53}$$

$$\begin{aligned} T_{nn} &= n_i n_j T_{ij} = n_i n_j \left(-p\delta_{ij} + \mu A_{ij} + \alpha_1 B_{ij} + \alpha_2 A_{ik} A_{kj}\right) \\ &= -p + \mu \left(\left(n_x^2 - n_y^2\right) A_{11} + 2 n_x n_y A_{12} \right) + \alpha_1 n_i n_j B_{ij} + \alpha_2 n_i n_j A_{ik} A_{kj} \\ &= -p + \mu \left(\left(n_x^2 - n_y^2\right) A_{11} + 2 n_x n_y A_{12} \right) + \alpha_1 n_i n_j v_k \frac{\partial A_{ij}}{\partial x_k} + (\alpha_1 + \alpha_2)\, 2\chi. \end{aligned} \tag{7.54}$$

The term of $n_i n_j v_k \dfrac{\partial A_{ij}}{\partial x_k}$ is expressed as

$$\begin{aligned} n_i n_j v_k \frac{\partial A_{ij}}{\partial x_k} &= \left(n_x^2 - n_y^2\right) v_k \frac{\partial A_{11}}{\partial x_k} + 2 n_x n_y v_k \frac{\partial A_{12}}{\partial x_k} \\ &= \left(n_x^2 - n_y^2\right) \left(u \frac{\partial A_{11}}{\partial x} + v \frac{\partial A_{11}}{\partial y} \right) + 2 n_x n_y \left(u \frac{\partial A_{12}}{\partial x} + v \frac{\partial A_{12}}{\partial y} \right). \end{aligned} \tag{7.55}$$

The normal stress of the second order fluid on the ellipse surface is given by

$$\begin{aligned} T_{nn} &= -p_\infty - \frac{\rho}{2} U^2 + \frac{\rho}{2} \left(u^2 + v^2\right) + \mu \left(\left(n_x^2 - n_y^2\right) A_{11} + 2 n_x n_y A_{12} \right) - (3\alpha_1 + 2\alpha_2)\,\chi \\ &+ \alpha_1 \left[\left(n_x^2 - n_y^2\right) \left(u \frac{\partial A_{11}}{\partial x} + v \frac{\partial A_{11}}{\partial y} \right) + 2 n_x n_y \left(u \frac{\partial A_{12}}{\partial x} + v \frac{\partial A_{12}}{\partial y} \right) \right] + (\alpha_1 + \alpha_2)\, 2\chi, \end{aligned} \tag{7.56}$$

which is

$$\begin{aligned} T_{nn} &= -p_\infty - \frac{\rho}{2} U^2 + \frac{\rho}{2} \left(u^2 + v^2\right) + 2\mu \left(\left(n_x^2 - n_y^2\right) \frac{\partial u}{\partial x} + 2 n_x n_y \frac{\partial u}{\partial y} \right) - (3\alpha_1 + 2\alpha_2)\,\chi \\ &+ 2\alpha_1 \left[\left(n_x^2 - n_y^2\right) \left(u \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 u}{\partial x \partial y} \right) + 2 n_x n_y \left(u \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y^2} \right) \right] + (\alpha_1 + \alpha_2)\, 2\chi, \end{aligned} \tag{7.57}$$

with

$$\chi = 2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial u}{\partial y} \right)^2. \tag{7.58}$$

Normalized form:

$$\begin{aligned} T_{nn}^* &= \frac{T_{nn} + p_\infty}{\frac{\rho}{2} U^2} = -1 + \left(u^2 + v^2\right) + \frac{4}{R_e} \left(\left(n_x^2 - n_y^2\right) \frac{\partial u}{\partial x} + 2 n_x n_y \frac{\partial u}{\partial y} \right) \\ &+ 4 \frac{\alpha_1}{\rho a^2} \left[\left(n_x^2 - n_y^2\right) \left(u \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 u}{\partial x \partial y} \right) + 2 n_x n_y \left(u \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y^2} \right) \right] - 2 \frac{\alpha_1}{\rho a^2} \chi. \end{aligned} \tag{7.59}$$

We need the transformation $a \to 1$, $b \to b/a$ and $U \to -1$ for the dimensionless expressions used in the paper of Wang & Joseph 2003.

7.2 Newtonian (revised)

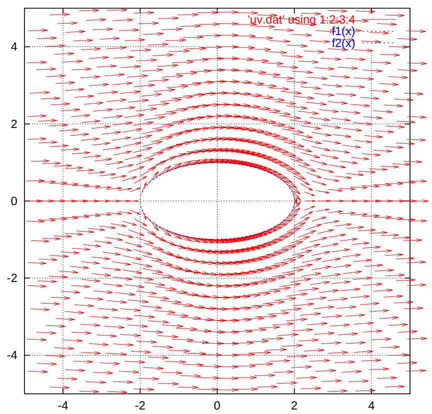

Fig.1(a). Velocity field (u, v) around an ellipse with $a = 2$ and $b = 1$ in the (x, y)-plane.

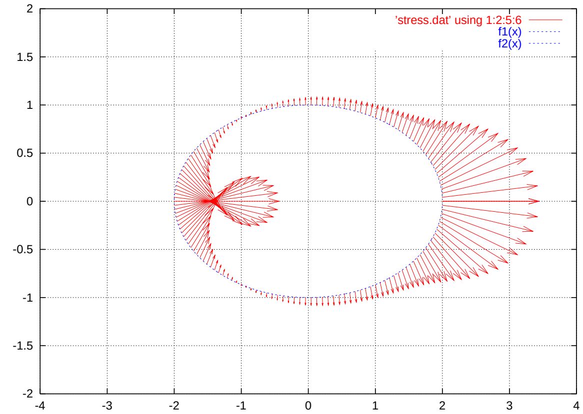

Fig.1(b). Normal stress $n_i T_{ij} n_j$ around an ellipse with $a = 2$ and $b = 1$ in the (x, y)-plane.

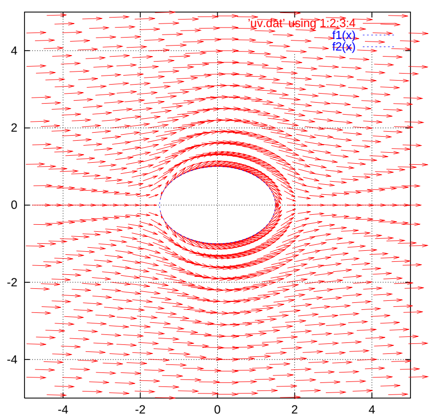

Fig.2(a). Velocity field (u, v) around an ellipse with $a = 1.5$ and $b = 1$ in the (x, y)-plane.

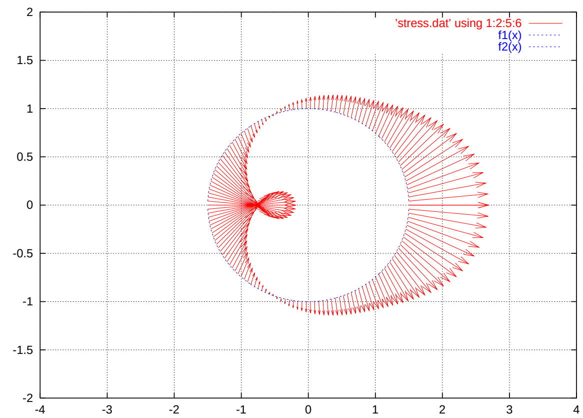

Fig.2(b). Normal stress $n_i T_{ij} n_j$ around an ellipse with $a = 1.5$ and $b = 1$ in the (x, y)-plane.

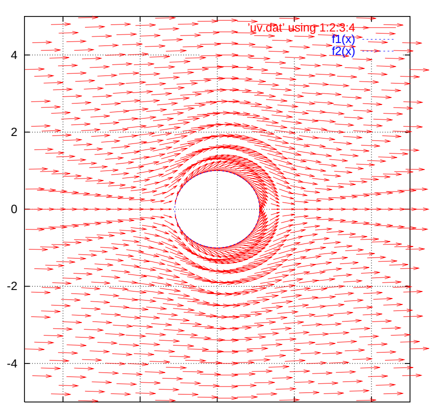

Fig.3(a). Velocity field (u, v) around an ellipse with $a = 1.1$ and $b = 1$ in the (x, y)-plane.

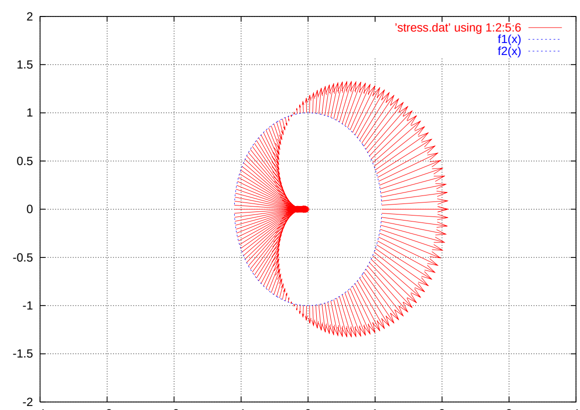

Fig.3(b). Normal stress $n_i T_{ij} n_j$ around an ellipse with $a = 1.1$ and $b = 1$ in the (x, y)-plane.

7.3 Normal stress on the ellipse surface in the second order fluid

In the following figures, red curves are due to the present theory, and green curves are due to the theory in Wang & Joseph 2003. It can be confirmed that the results are the same.

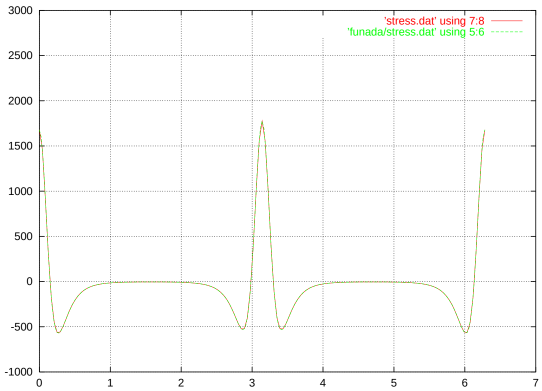

Fig.4(a). T^*_{nn} versus θ for a second order fluid flow around an ellipse with $a = 3$ and $b = 1$ in the (x, y)-plane; $R_e = 1$ and $-\alpha_1/(\rho a^2) = 3$.

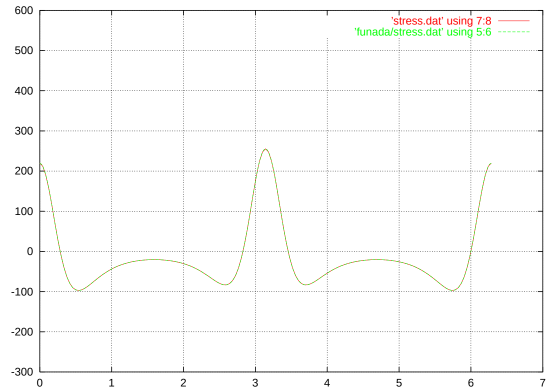

Fig.5(a). T^*_{nn} versus θ for a second order fluid flow around an ellipse with $a = 1.67$ and $b = 1$ in the (x, y)-plane; $R_e = 1$ and $-\alpha_1/(\rho a^2) = 3$.

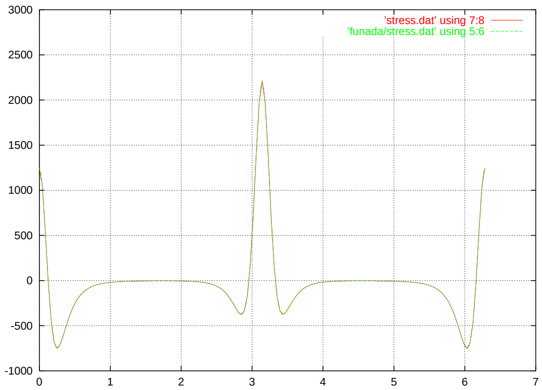

Fig.4(b). T^*_{nn} versus θ for a second order fluid flow around an ellipse with $a = 3$ and $b = 1$ in the (x, y)-plane; $R_e = 0.1$ and $-\alpha_1/(\rho a^2) = 3$.

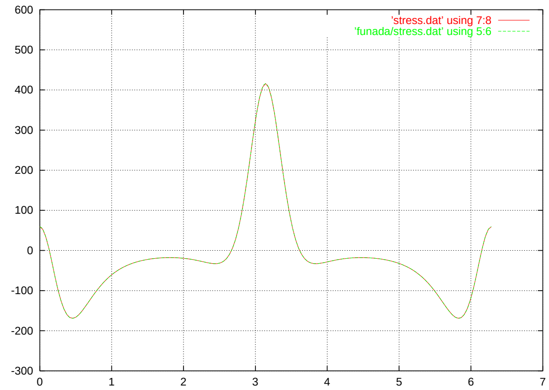

Fig.5(b). T^*_{nn} versus θ for a second order fluid flow around an ellipse with $a = 1.67$ and $b = 1$ in the (x, y)-plane; $R_e = 0.1$ and $-\alpha_1/(\rho a^2) = 3$.

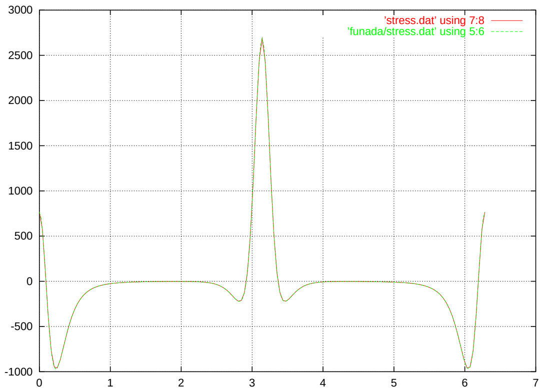

Fig.4(c). T^*_{nn} versus θ for a second order fluid flow around an ellipse with $a = 3$ and $b = 1$ in the (x, y)-plane; $R_e = 0.05$ and $-\alpha_1/(\rho a^2) = 3$.

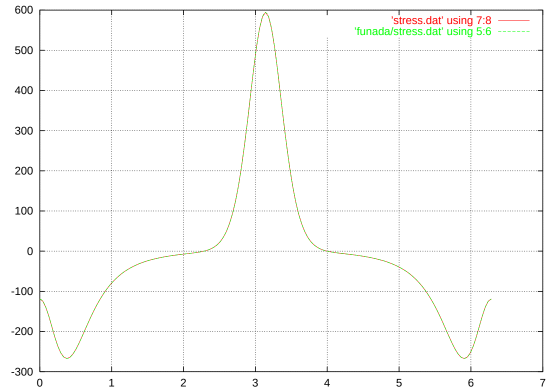

Fig.5(c). T^*_{nn} versus θ for a second order fluid flow around an ellipse with $a = 1.67$ and $b = 1$ in the (x, y)-plane; $R_e = 0.05$ and $-\alpha_1/(\rho a^2) = 3$.

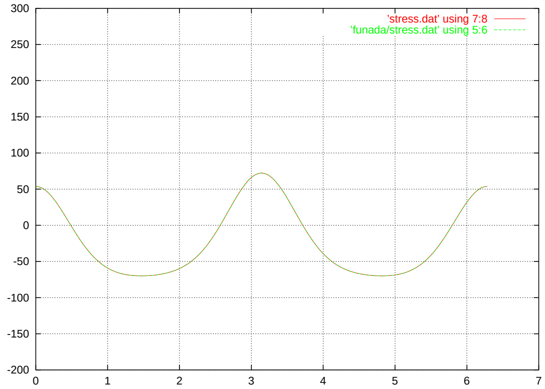

Fig.6(a). T^*_{nn} versus θ for a second order fluid flow around an ellipse with $a = 1.1$ and $b = 1$ in the (x, y)-plane; $R_e = 1$ and $-\alpha_1/(\rho a^2) = 3$.

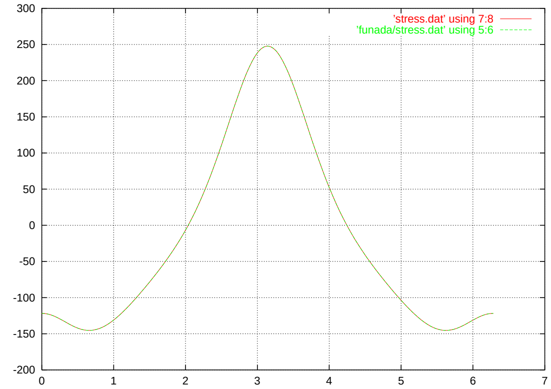

Fig.6(c). T^*_{nn} versus θ for a second order fluid flow around an ellipse with $a = 1.1$ and $b = 1$ in the (x, y)-plane; $R_e = 0.05$ and $-\alpha_1/(\rho a^2) = 3$.

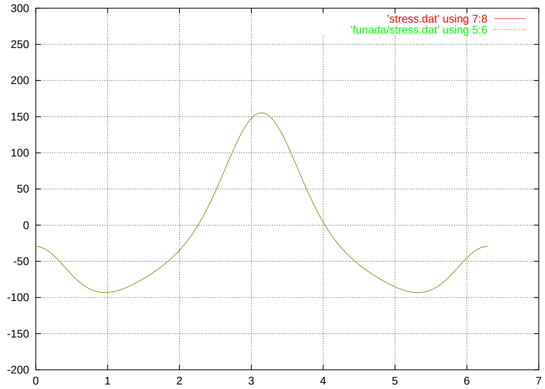

Fig.6(b). T^*_{nn} versus θ for a second order fluid flow around an ellipse with $a = 1.1$ and $b = 1$ in the (x, y)-plane; $R_e = 0.1$ and $-\alpha_1/(\rho a^2) = 3$.

7.4 Surface tension

With the normal vector $\boldsymbol{n}$

$$\boldsymbol{n} = (n_x, n_y) = \left(\frac{x}{a^2}\boldsymbol{e}_x + \frac{y}{b^2}\boldsymbol{e}_y\right) / \sqrt{\left(\frac{x}{a^2}\right)^2 + \left(\frac{y}{b^2}\right)^2}, \tag{7.60}$$

we have $\nabla \cdot \boldsymbol{n}$

$$\begin{aligned}\nabla \cdot \boldsymbol{n} = \frac{\partial n_x}{\partial x} + \frac{\partial n_y}{\partial y} &= \frac{1}{a^2}\left[\left(\frac{x}{a^2}\right)^2 + \left(\frac{y}{b^2}\right)^2\right]^{-1/2} - \frac{x^2}{a^6}\left[\left(\frac{x}{a^2}\right)^2 + \left(\frac{y}{b^2}\right)^2\right]^{-3/2} \\ &+ \frac{1}{b^2}\left[\left(\frac{x}{a^2}\right)^2 + \left(\frac{y}{b^2}\right)^2\right]^{-1/2} - \frac{y^2}{b^6}\left[\left(\frac{x}{a^2}\right)^2 + \left(\frac{y}{b^2}\right)^2\right]^{-3/2} \\ = \left(\frac{1}{a^2}\frac{y^2}{b^4} + \frac{1}{b^2}\frac{x^2}{a^4}\right)\left[\left(\frac{x}{a^2}\right)^2 + \left(\frac{y}{b^2}\right)^2\right]^{-3/2} &= \frac{1}{a^2 b^2}\left[\left(\frac{\cos\theta}{a}\right)^2 + \left(\frac{\sin\theta}{b}\right)^2\right]^{-3/2}.\end{aligned} \tag{7.61}$$

We can evaluate the surface tension term $\gamma \nabla \cdot \boldsymbol{n}$.

A note on flows of a second order fluid around an ellipsoid

T.Funada, May 4, 2004 / circular-june13.tex / printed July 16, 2004

6 Circular cylinder

For a circular cylinder for which $a^2 = b^2$ and $c^2 \to \infty$, we have

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1 \quad \to \quad \frac{x^2}{a^2+\lambda} + \frac{y^2}{a^2+\lambda} = 1 \quad \to \quad x^2 + y^2 = r^2 = a^2 + \lambda, \tag{6.1}$$

$$\phi = -\frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad \to \quad \phi = -\frac{a^2Ux}{2-\alpha_0}\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)^2} = -\frac{a^2Ux}{a^2+\lambda} = -\frac{a^2Ux}{r^2}, \tag{6.2}$$

$$\alpha_0 = abc\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \quad \to \quad \alpha_0 = a^2\int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)^2} = 1. \tag{6.3}$$

If the frame is taken on the ellipsoid, the velocity potential is modified as

$$\phi = -Ux\left(1 + \frac{a^2}{r^2}\right). \tag{6.4}$$

The unit vectors in the cylindrical coordinates (r, θ, z) are given by

$$\boldsymbol{e}_r = \cos\theta\boldsymbol{e}_1 + \sin\theta\boldsymbol{e}_2, \quad \frac{\partial\boldsymbol{e}_r}{\partial\theta} = -\sin\theta\boldsymbol{e}_1 + \cos\theta\boldsymbol{e}_2 = \boldsymbol{e}_\theta, \quad \frac{\partial\boldsymbol{e}_\theta}{\partial\theta} = -\boldsymbol{e}_r. \tag{6.5}$$

The relations between this coordinates and the cartesian coordinates (x, y) are given by

$$r^2 = x^2 + y^2, \quad 2r\mathrm{d}r = 2x\mathrm{d}x + 2y\mathrm{d}y, \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}. \tag{6.6}$$

The velocity (u, v) and the relevant relations are

$$u = -\frac{\partial\phi}{\partial x} = U\left(1 + \frac{a^2}{r^2}\right) - Ux\frac{2a^2}{r^3}\frac{\partial r}{\partial x} = U\left(1 + \frac{a^2}{r^2}\right) - 2U\frac{a^2x^2}{r^4}, \tag{6.7}$$

$$v = -\frac{\partial\phi}{\partial y} = -Ux\frac{2a^2}{r^3}\frac{y}{r} = -2U\frac{a^2xy}{r^4}, \tag{6.8}$$

$$\frac{\partial u}{\partial x} = U\frac{-2a^2}{r^3}\frac{\partial r}{\partial x} - \frac{4Ua^2x}{r^4} + \frac{8Ua^2x^2}{r^5}\frac{x}{r} = -\frac{6Ua^2x}{r^4} + \frac{8Ua^2x^3}{r^6}, \tag{6.9}$$

$$\frac{\partial u}{\partial y} = \frac{-2Ua^2}{r^3}\frac{y}{r} + \frac{8Ua^2x^2}{r^5}\frac{y}{r} = \frac{-2Ua^2y}{r^4} + \frac{8Ua^2x^2y}{r^6}, \tag{6.10}$$

$$\frac{\partial v}{\partial x} = -2U\frac{a^2y}{r^4} + \frac{8Ua^2xy}{r^5}\frac{x}{r} = -2U\frac{a^2y}{r^4} + \frac{8Ua^2x^2y}{r^6} = \frac{\partial u}{\partial y}, \tag{6.11}$$

$$\frac{\partial v}{\partial y} = -2U\frac{a^2x}{r^4} + \frac{8Ua^2xy}{r^5}\frac{y}{r} = -2U\frac{a^2x}{r^4} + \frac{8Ua^2xy^2}{r^6} = -\frac{\partial u}{\partial x}, \tag{6.12}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{6Ua^2}{r^4} + \frac{48Ua^2x^2}{r^6} - \frac{48Ua^2x^4}{r^8}, \quad \frac{\partial^2 u}{\partial x\partial y} = \frac{24Ua^2xy}{r^6} - \frac{48Ua^2x^3y}{r^8}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{6Ua^2}{r^4} - \frac{48Ua^2x^2y^2}{r^8}, \tag{6.13}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \to \boldsymbol{\omega} = \boldsymbol{0} \quad \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial/\partial x & \partial/\partial y & 0 \\ u & v & 0 \end{vmatrix} = \boldsymbol{0}, \tag{6.14}$$

$$(L_{ij}) = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial y} & -\dfrac{\partial u}{\partial x} \end{pmatrix}, \tag{6.15}$$

$$(A_{ij}) = (L_{ij} + L_{ji}) = \begin{pmatrix} 2\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} & 2\dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2\dfrac{\partial u}{\partial x} & 2\dfrac{\partial u}{\partial y} \\ 2\dfrac{\partial u}{\partial y} & -2\dfrac{\partial u}{\partial x} \end{pmatrix}, \tag{6.16}$$

$$\begin{aligned}(A_{ik}A_{kj}) &= \begin{pmatrix} 4\left(\dfrac{\partial^2\phi}{\partial x^2}\right)^2 + 4\left(\dfrac{\partial^2\phi}{\partial y\partial x}\right)^2 & 4\dfrac{\partial^2\phi}{\partial y\partial x}\left(\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right) \\ 4\dfrac{\partial^2\phi}{\partial y\partial x}\left(\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right) & 4\left(\dfrac{\partial^2\phi}{\partial y\partial x}\right)^2 + 4\left(\dfrac{\partial^2\phi}{\partial y^2}\right)^2 \end{pmatrix} \\ &= 4\left[\left(\dfrac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\dfrac{\partial^2\phi}{\partial y\partial x}\right)^2\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2\chi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\end{aligned} \tag{6.17}$$

with

$$\chi = \frac{\partial^2\phi}{\partial x_i\partial x_j}\frac{\partial^2\phi}{\partial x_j\partial x_i} = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2 = 2\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2. \tag{6.18}$$

The stress tensor for Newtonian is given by

$$T_{ij} = -p\delta_{ij} + \mu A_{ij}, \tag{6.19}$$

and the Bernoulli function is given by

$$-\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} = \frac{1}{2}U^2 + \frac{p_\infty}{\rho} \quad \rightarrow \quad p = p_\infty + \frac{\rho}{2}U^2 + \rho\frac{\partial\phi}{\partial t} - \frac{\rho}{2}\left(u^2 + v^2\right). \tag{6.20}$$

Using this, the normal stress on the cylinder surface ($r = a$) is given by

$$n_i n_j T_{ij} = -p + \mu n_i n_j A_{ij} = -\left[p_\infty + \frac{\rho}{2}U^2 - \frac{\rho}{2}\left(u^2 + v^2\right)\right] + \mu n_i n_j A_{ij}, \tag{6.21}$$

with the normal viscous stress specified as

$$n_i n_j A_{ij} = n_x^2 A_{11} + 2n_x n_y A_{12} + n_y^2 A_{22} = 2\left(n_x^2 - n_y^2\right)\frac{\partial u}{\partial x} + 4n_x n_y\frac{\partial u}{\partial y}. \tag{6.22}$$

The normalized form T^*_{nn} is given by

$$T^*_{nn} = \frac{n_i n_j T_{ij} + p_\infty}{\dfrac{\rho}{2}U^2} = -1 + \left(u^2 + v^2\right) + \frac{4}{R_e}\left[\left(n_x^2 - n_y^2\right)\frac{\partial u}{\partial x} + 2n_x n_y\frac{\partial u}{\partial y}\right], \tag{6.23}$$

where u, v, $\partial u/\partial x$, $\partial u/\partial y$ should be read as the normalized ones.

From the paper of Wang & Joseph (2003), the stress tensor for the second order fluid is given by

$$B_{ij} = \frac{\partial A_{ij}}{\partial t} + v_k\frac{\partial A_{ij}}{\partial x_k} + A_{ik}L_{kj} + L_{ik}A_{kj}, \tag{6.24}$$

$$T_{ij} = -p\delta_{ij} + \mu A_{ij} + \alpha_1 B_{ij} + \alpha_2 A_{ik}A_{kj}, \tag{6.25}$$

and the Bernoulli function is given by

$$\begin{aligned}&-\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} - \frac{1}{\rho}\left(3\alpha_1 + 2\alpha_2\right)\chi = \frac{1}{2}U^2 + \frac{p_\infty}{\rho} \\ &\rightarrow \quad p = p_\infty + \frac{\rho}{2}U^2 + \rho\frac{\partial\phi}{\partial t} - \frac{\rho}{2}\left(u^2 + v^2\right) + \left(3\alpha_1 + 2\alpha_2\right)\chi.\end{aligned} \tag{6.26}$$

Using these, the normal stress on the cylinder surface ($r = a$) is given by

$$\begin{aligned} n_i n_j T_{ij} &= -p + \mu n_i n_j A_{ij} + \alpha_1 n_i n_j B_{ij} + \alpha_2 n_i n_j A_{ik} A_{kj} \\ &= -\left[p_\infty + \frac{\rho}{2}U^2 - \frac{\rho}{2}\left(u^2+v^2\right) + \left(3\alpha_1 + 2\alpha_2\right)\chi\right] + \mu n_i n_j A_{ij} + \alpha_1 n_i n_j v_k \frac{\partial A_{ij}}{\partial x_k} + 2\left(\alpha_1+\alpha_2\right)\chi \\ &= -p_\infty - \frac{\rho}{2}U^2 + \frac{\rho}{2}\left(u^2+v^2\right) + \mu\left[2\left(n_x^2 - n_y^2\right)\frac{\partial u}{\partial x} + 4 n_x n_y \frac{\partial u}{\partial y}\right] + \alpha_1 n_i n_j v_k \frac{\partial A_{ij}}{\partial x_k} - \alpha_1 \chi, \end{aligned} \quad (6.27)$$

for which

$$\begin{aligned} n_i n_j v_k \frac{\partial A_{ij}}{\partial x_k} = n_x^2\left(u\frac{\partial A_{11}}{\partial x} + v\frac{\partial A_{11}}{\partial y}\right) + 2 n_x n_y\left(u\frac{\partial A_{12}}{\partial x} + v\frac{\partial A_{12}}{\partial y}\right) + n_y^2\left(u\frac{\partial A_{22}}{\partial x} + v\frac{\partial A_{22}}{\partial y}\right) \\ = 2\left(n_x^2 - n_y^2\right)\left(u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial x \partial y}\right) + 4 n_x n_y\left(u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2}\right). \end{aligned} \quad (6.28)$$

The normalized form T^*_{nn} is given by

$$\begin{aligned} T^*_{nn} = \frac{n_i n_j T_{ij} + p_\infty}{\frac{\rho}{2}U^2} = -1 + \left(u^2+v^2\right) + \frac{4}{R_e}\left[\left(n_x^2 - n_y^2\right)\frac{\partial u}{\partial x} + 2 n_x n_y \frac{\partial u}{\partial y}\right] \\ + \frac{4\alpha_1}{\rho a^2}\left[\left(n_x^2 - n_y^2\right)\left(u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial x \partial y}\right) + 2 n_x n_y\left(u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2}\right) - \chi/2\right], \end{aligned} \quad (6.29)$$

where u, v, $\partial u/\partial x$, $\partial u/\partial y$, $\chi = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial u}{\partial y}\right)^2$ should be read as the normalized ones.

7 Flows of Newtonian and a second order fluid around a cylinder with $a = 3$ cm

半径 $a = 3$ の静止円柱周りを Newtonian と second order fluid が流れている場合の円柱表面での応力について解析する. なお, 応力の分布図に関しては, 実際の値の 1/100 で表示させている. 一方, T^*_{nn} と θ の関係のグラフについては, 実際の値で表示させている.

7.1 Velocity field $(u,\, v)$

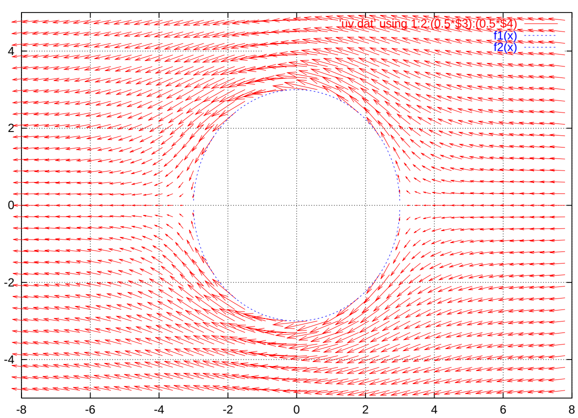

Fig.7.1 Velocity field in the $(x$-$y)$-plane with $a = 3$ cm.

7.2 Steady flow around a cylinder

7.2.1 Newtonian

レイノルズ数 Re を 0.05, 0.1, 1.0 と変化させ時の, Newtonian 中での静止円柱表面の応力の分布と, T^*_{nn} と θ の関係のグラフを Fig7.2 から Fig7.7 に示す.

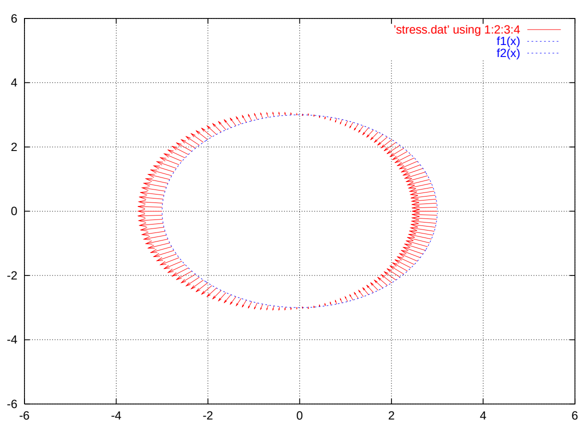

Fig.7.2 Distribution of $T^*_{nn}/100$ with $R_e = 0.05$.

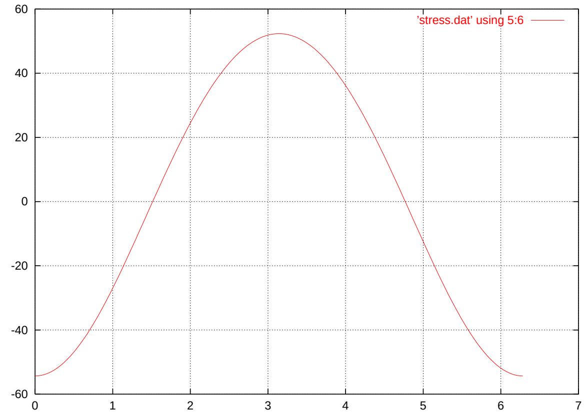

Fig.7.3 T^*_{nn} versus θ with $R_e = 0.05$.

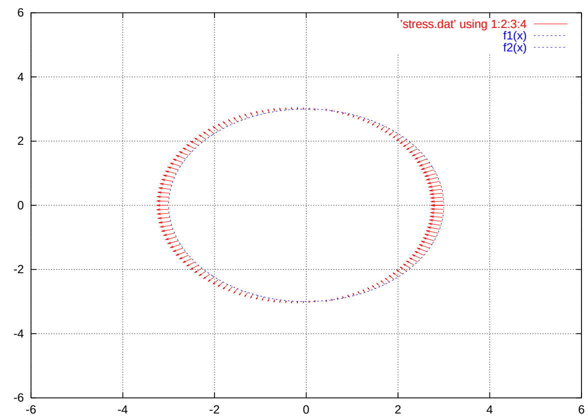

Fig.7.4 Distribution of $T^*_{nn}/100$ with $R_e = 0.1$.

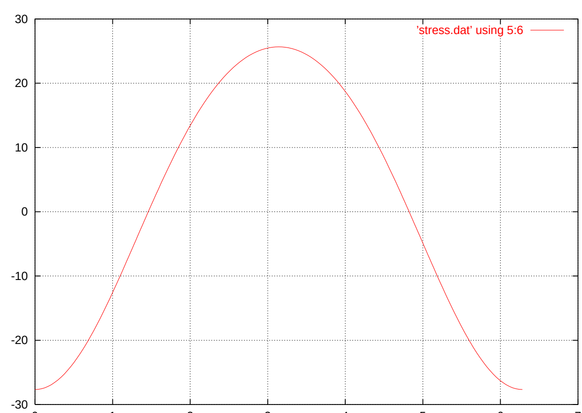

Fig.7.5 T^*_{nn} versus θ with $R_e = 0.1$.

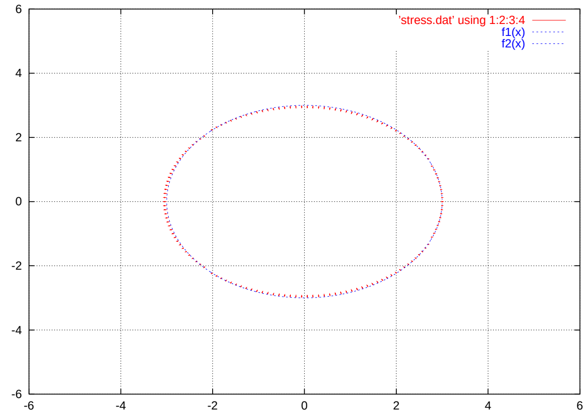

Fig.7.6 Distribution of $T^*_{nn}/100$ with $R_e = 1.0$.

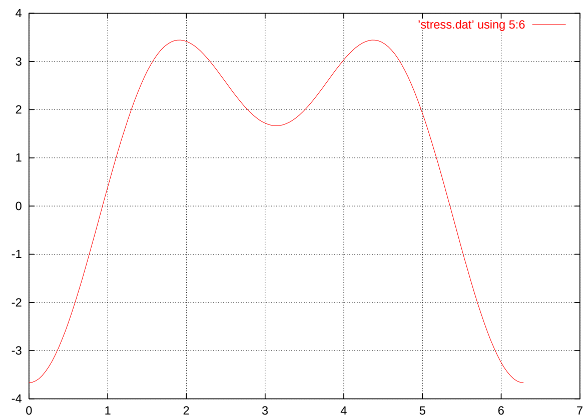

Fig.7.7 T^*_{nn} versus θ with $R_e = 1.0$.

7.2.2 Second order fluid ($-\frac{\alpha_1}{\rho a^2} = 3$)

レイノルズ数 Re を 0.05, 0.1, 1.0 と変化させ時の, second order fluid 中での静止円柱表面の応力の分布と, T^*_{nn} と θ の関係のグラフを Fig7.8 から Fig7.13 に示す.

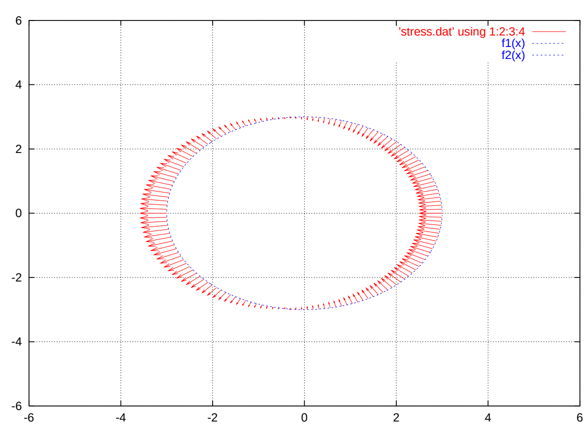

Fig.7.8 Distribution of $T^*_{nn}/100$ with $R_e = 0.05$.

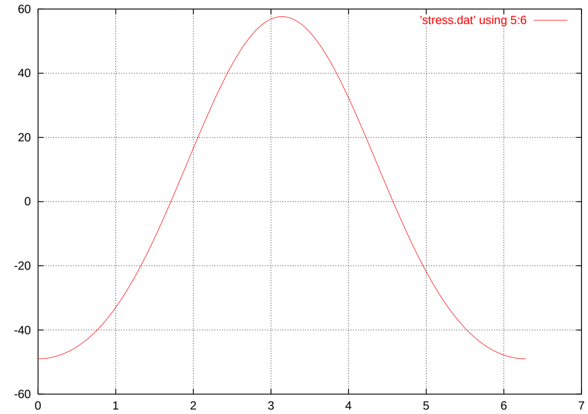

Fig.7.9 T^*_{nn} versus θ with $R_e = 0.05$.

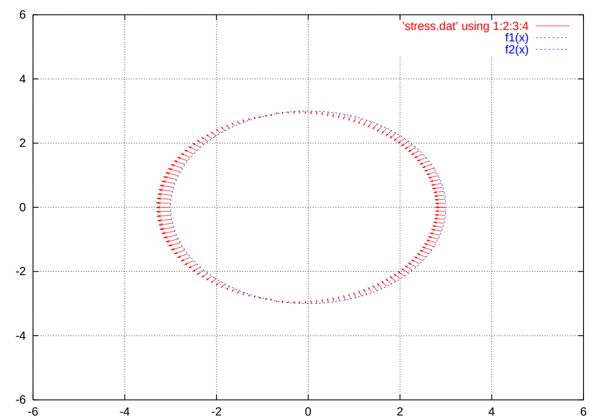

Fig.7.10 Distribution of $T^*_{nn}/100$ with $R_e = 0.1$.

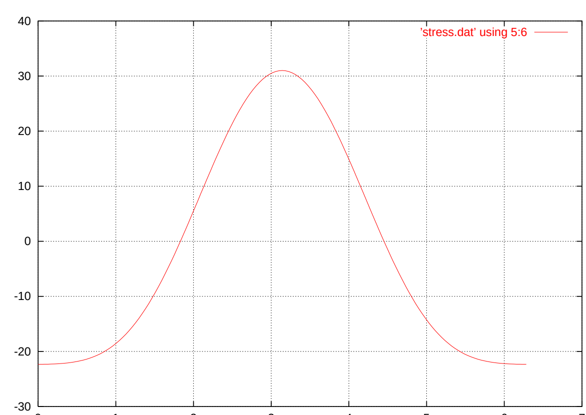

Fig.7.11 T^*_{nn} versus θ with $R_e = 0.1$.

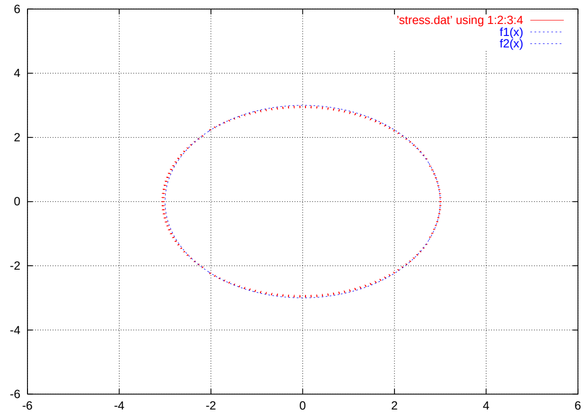

Fig.7.12 Distribution of $T^*_{nn}/100$ with $R_e = 1.0$.

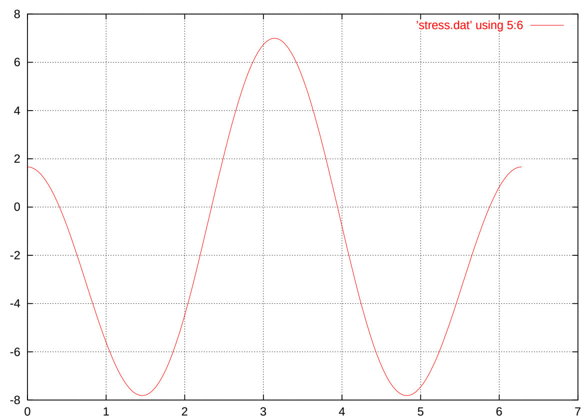

Fig.7.13 T^*_{nn} versus θ with $R_e = 1.0$.

8 Flows of Newtonian and a second order fluid around a cylinder with $a = 1$ cm

半径 $a = 1$ の静止円柱周りを Newtonian と second order fluid が流れている場合の円柱表面での応力について解析する. なお, 応力の分布図に関しては, 実際の値の 1/100 で表示させている. 一方, T^*_{nn} と θ の関係のグラフについては, 実際の値で表示させている.

8.1 Velocity field (u, v)

半径 $a = 1$ の円柱周りの速度場を Fig.8.1 に示す.

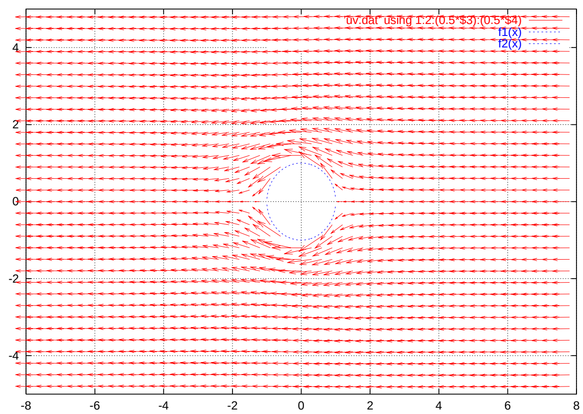

Fig.8.1 Velocity field in the $(x$-$y)$-plane $a = 1$ $U = -1.0$.

8.2 Steady flow around a cylinder

8.2.1 Newtonian

レイノルズ数 Re を 0.05, 0.1, 1.0 と変化させ時の, Newtonian 中での静止円柱表面の応力の分布と, T^*_{nn} と θ の関係のグラフを Fig.8.2 から Fig.8.7 に示す.

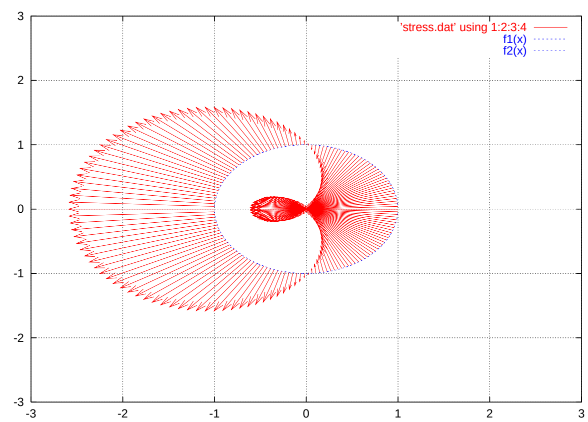

Fig.8.2 Distribution of $T^*_{nn}/100$ with $R_e = 0.05$.

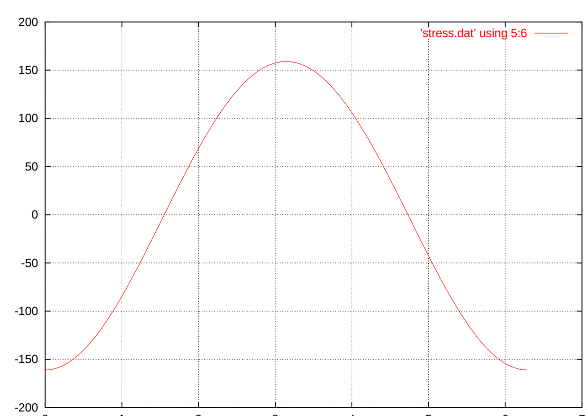

Fig.8.3 T^*_{nn} versus θ with $R_e = 0.05$.

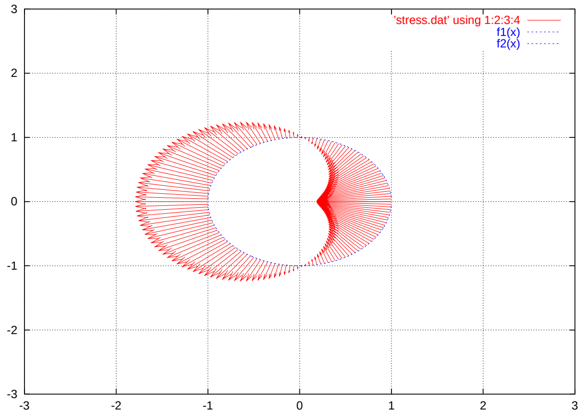

Fig.8.4 Distribution of $T^*_{nn}/100$ with $R_e = 0.1$.

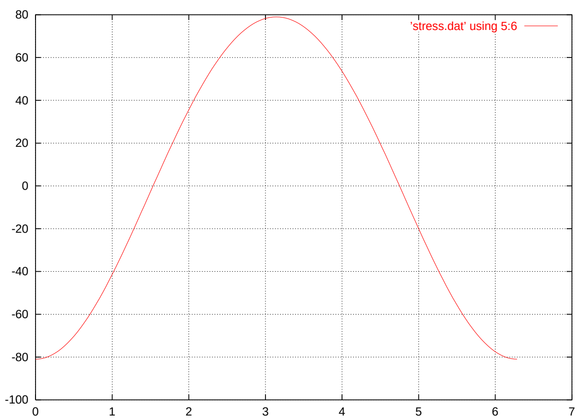

Fig.8.5 T^*_{nn} versus θ with $R_e = 0.1$.

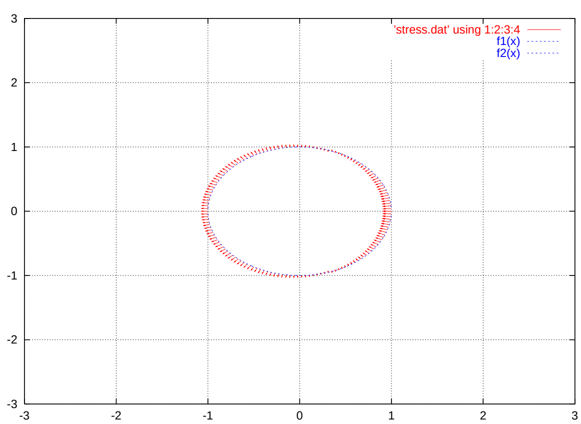

Fig.8.6 Distribution of $T^*_{nn}/100$ with $R_e = 1.0$.

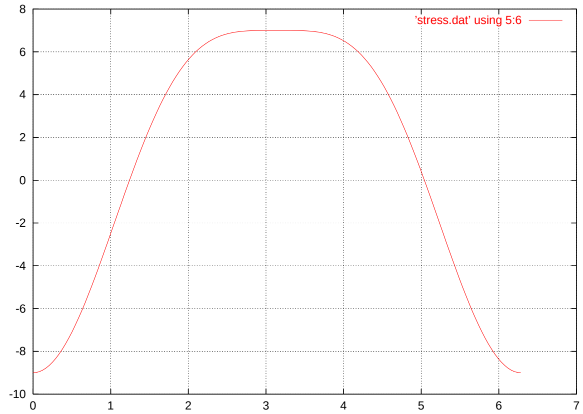

Fig.8.7 T^*_{nn} versus θ with $R_e = 1.0$.

8.2.2 Second order fluid ($-\frac{\alpha_1}{\rho a^2} = 3$)

レイノルズ数 Re を 0.05, 0.1, 1.0 と変化させ時の, second order fluid 中での静止円柱表面の応力の分布と, T^*_{nn} と θ の関係のグラフを Fig.8.8 から Fig.8.13 に示す.

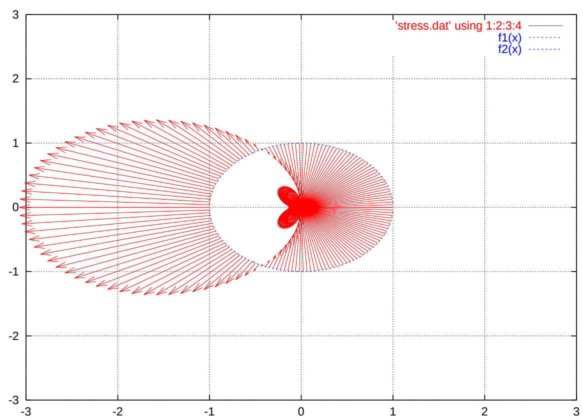

Fig.8.8 Distribution of $T^*_{nn}/100$ with $R_e = 0.05$.

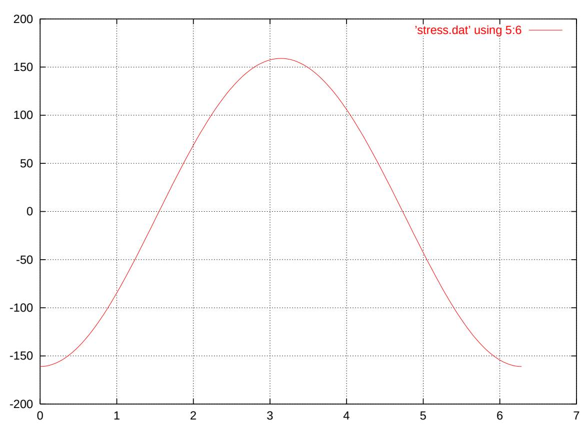

Fig.8.9 T^*_{nn} versus θ with $R_e = 0.05$.

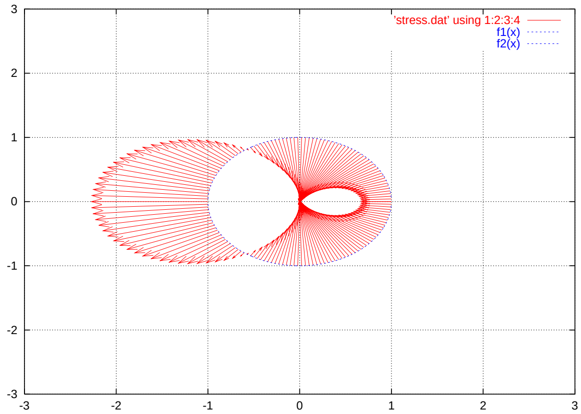

Fig.8.10 Distribution of $T^*_{nn}/100$ with $R_e = 0.1$.

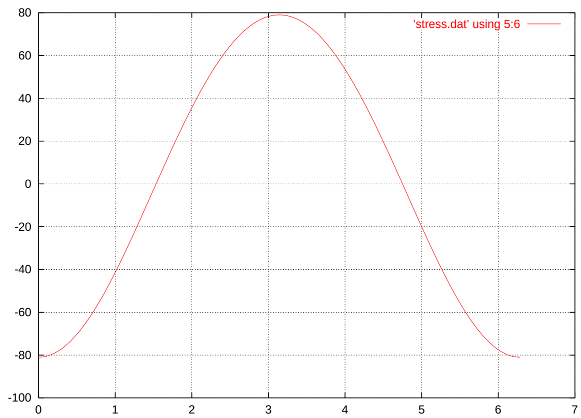

Fig.8.11 T^*_{nn} versus θ with $R_e = 0.1$.

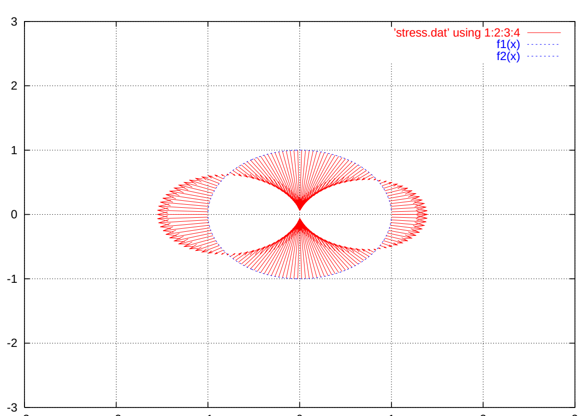

Fig.8.12 Distribution of $T^*_{nn}/100$ with $R_e = 1.0$.

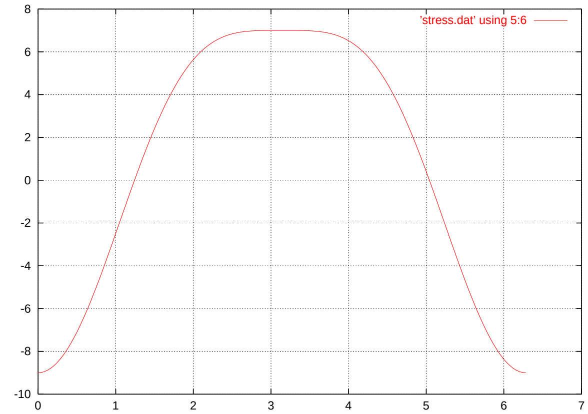

Fig.8.13 T^*_{nn} versus θ with $R_e = 1.0$.

9 $-\frac{\alpha_1}{\rho a^2}$ を変化させた時 $(a = 1.0, Re = 1.0)$

円柱の半径を $a = 1$, レイノルズ数を $Re = 1.0$ とし , $-\frac{\alpha_1}{\rho a^2}$ を 0, 1, 2, 3, 4, 5 と変化させた時の静止円柱表面の応力の分布と, T^*_{nn} と θ の関係のグラフを Fig.9.1 から Fig.9.12 に示し, $-\frac{\alpha_1}{\rho a^2}$ が応力の分布にどのように影響するのかを調べる. なお, 応力の分布図に関しては, 実際の値の 1/100 で表示させている. 一方, T^*_{nn} と θ の関係のグラフについては, 実際の値で表示させている.

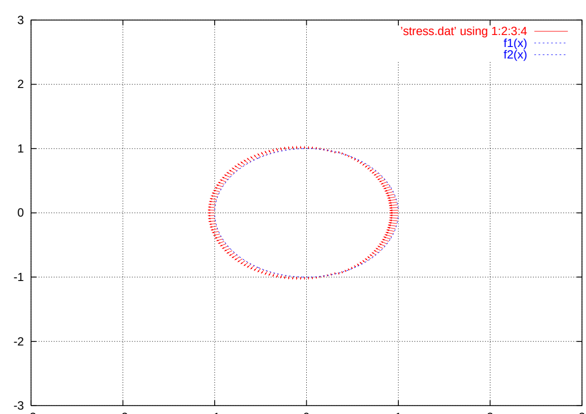

Fig.9.1 Distribution of $T^*_{nn}/100$ with $-\frac{\alpha_1}{\rho a^2} = 0$.

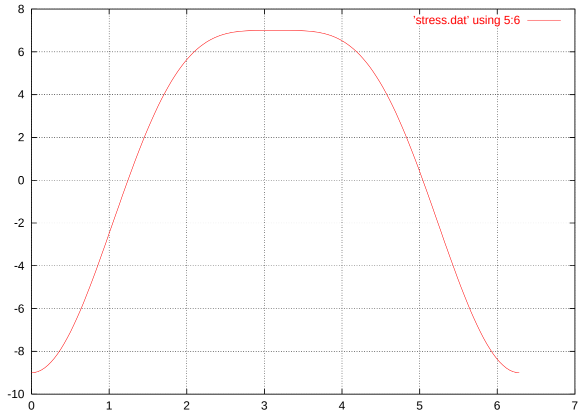

Fig.9.2 T^*_{nn} versus θ with $-\frac{\alpha_1}{\rho a^2} = 0$.

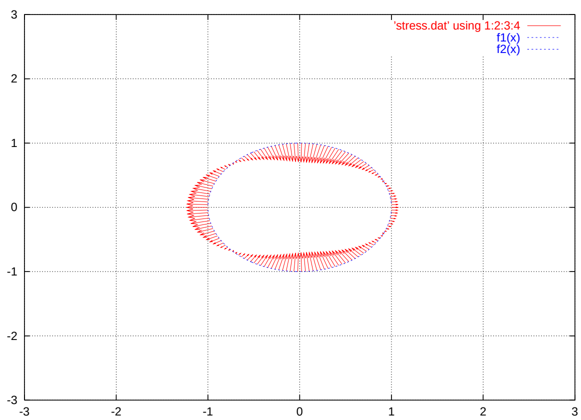

Fig.9.3 Distribution of $T^*_{nn}/100$ with $-\frac{\alpha_1}{\rho a^2} = 1$.

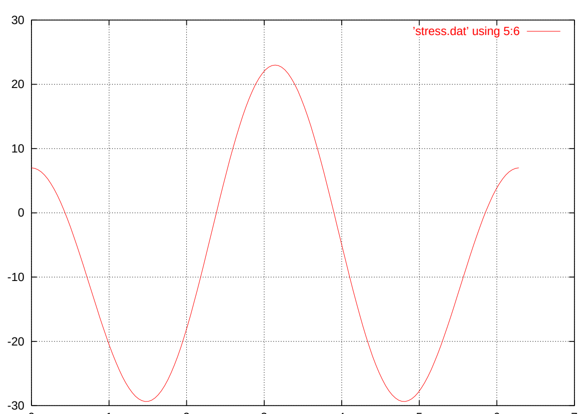

Fig.9.4 T^*_{nn} versus θ with $-\frac{\alpha_1}{\rho a^2} = 1$.

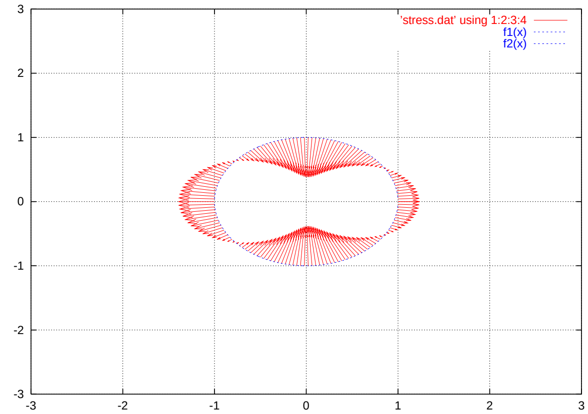

Fig.9.5 Distribution of $T^*_{nn}/100$ with $-\frac{\alpha_1}{\rho a^2} = 2$.

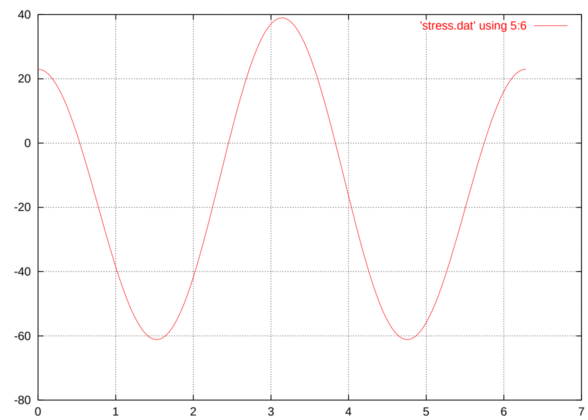

Fig.9.6 T^*_{nn} versus θ with $-\frac{\alpha_1}{\rho a^2} = 2$.

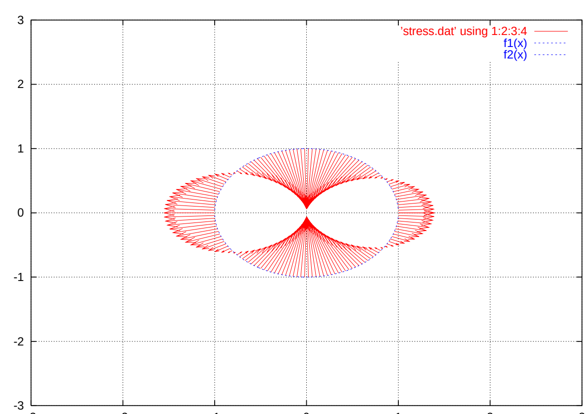

Fig.9.7 Distribution of $T^*_{nn}/100$ with $-\frac{\alpha_1}{\rho a^2} = 3$.

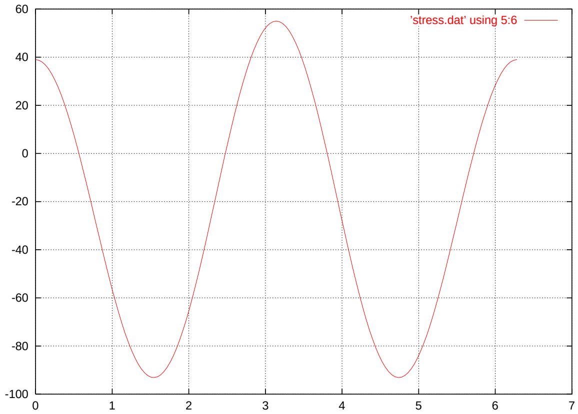

Fig.9.8 T^*_{nn} versus θ with $-\frac{\alpha_1}{\rho a^2} = 3$.

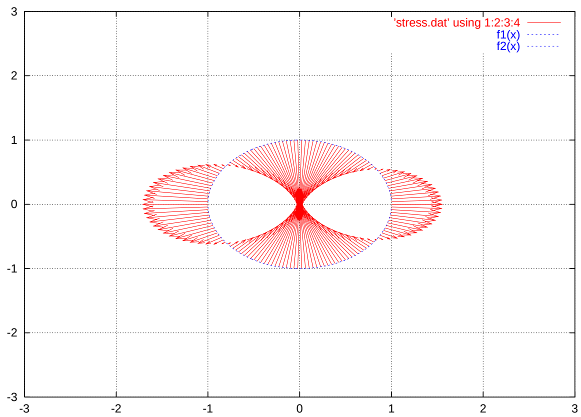

Fig.9.9 Distribution of $T^*_{nn}/100$ with $-\frac{\alpha_1}{\rho a^2} = 4$.

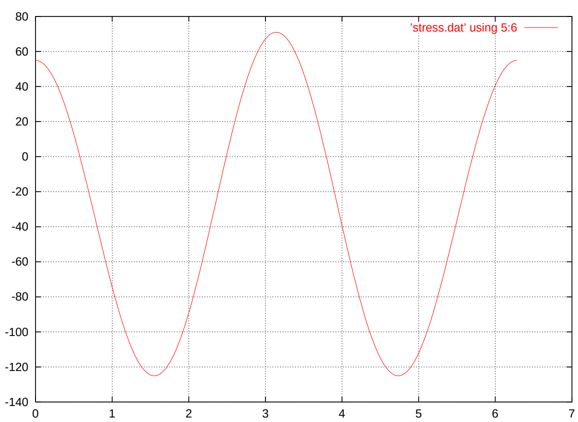

Fig.9.10 T^*_{nn} versus θ with $-\frac{\alpha_1}{\rho a^2} = 4$.

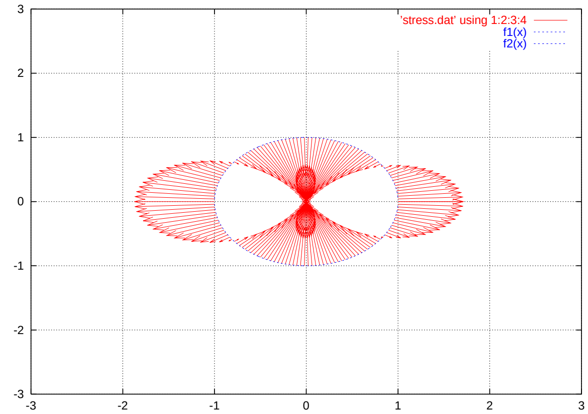

Fig.9.11 Distribution of $T^*_{nn}/100$ with $-\frac{\alpha_1}{\rho a^2} = 5$.

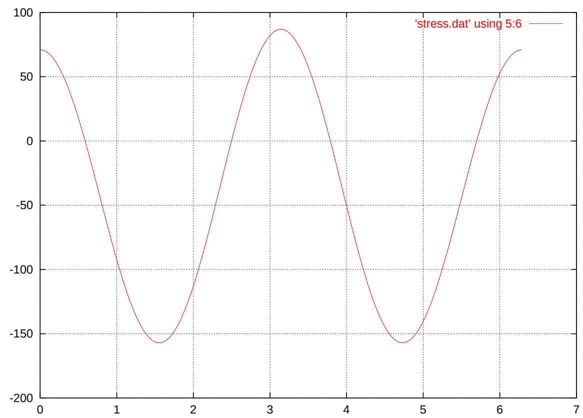

Fig.9.12 T^*_{nn} versus θ with $-\frac{\alpha_1}{\rho a^2} = 5$.

Data used in computations for flows around an elliopsoid

T.Funada & N.Tashiro, June 20, 2004 / ellipsoid-table-jun20c-24.tex / printed July 16, 2004

1 Review of the previous reports

In terms of the elliptic coordinates $(\lambda_1, \lambda_2, \lambda_3)$ satisfying

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1, \tag{1.1}$$

the velocity potential ϕ for which $\boldsymbol{v} = (u, v, w)$ in Cartesian frame (x, y, z) may be expressed as

$$\phi = \frac{abcUx}{2-\alpha_0} \int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}, \tag{1.2}$$

which is for flows around an ellipsoid moving with uniform velocity $-U$ in the x direction, and α_0 is defined as

$$\alpha_0 = abc \int_0^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}. \tag{1.3}$$

The basic part of the elliptic integral is given by

$$\int_\lambda^\infty \frac{\mathrm{d}\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} = \int_s^0 \frac{-\mathrm{d}s/s^2}{\sqrt{(a^2+1/s)(b^2+1/s)(c^2+1/s)}} = \int_0^s \frac{\mathrm{d}s}{abc\sqrt{\varphi(s)}}, \tag{1.4}$$

$$\int_\lambda^\infty \frac{\mathrm{d}\lambda}{(a^2+\lambda)\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} = \int_s^0 \frac{-\mathrm{d}s/s^2}{(a^2+1/s)\sqrt{(a^2+1/s)(b^2+1/s)(c^2+1/s)}} = \int_0^s \frac{s\,\mathrm{d}s}{a^2abc\,(s+1/a^2)\sqrt{\varphi(s)}}, \tag{1.5}$$

where $s = 1/\lambda$ and $\varphi(s)$ is given by

$$\varphi(s) = s\left(s+1/a^2\right)\left(s+1/b^2\right)\left(s+1/c^2\right). \tag{1.6}$$

The selection of two quadratics from $\varphi(s)$ may be taken as in the following table.

type	$\varphi(s)$	Characteristic equation for ξ
(A)	$\varphi(s) = \left(s^2+\frac{1}{a^2}s\right)\left(s^2+\left(\frac{1}{b^2}+\frac{1}{c^2}\right)s+\frac{1}{b^2c^2}\right)$	$\left(-\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\xi^2+\frac{2}{b^2c^2}\xi+\frac{1}{a^2b^2c^2}=0$
(B)	$\varphi(s) = \left(s^2+\frac{1}{b^2}s\right)\left(s^2+\left(\frac{1}{a^2}+\frac{1}{c^2}\right)s+\frac{1}{a^2c^2}\right)$	$\left(\frac{1}{a^2}-\frac{1}{b^2}+\frac{1}{c^2}\right)\xi^2+\frac{2}{c^2a^2}\xi+\frac{1}{a^2b^2c^2}=0$
(C)	$\varphi(s) = \left(s^2+\frac{1}{c^2}s\right)\left(s^2+\left(\frac{1}{a^2}+\frac{1}{b^2}\right)s+\frac{1}{a^2b^2}\right)$	$\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right)\xi^2+\frac{2}{a^2b^2}\xi+\frac{1}{a^2b^2c^2}=0$

We will see later that type (A) is suitable when a is the maximum semi-axis among the semi-axes a, b, c. Then, type (B) is suitable when b is the maximum, and type (C) is suitable when c is the maximum.

The characteristic equation for ξ is given symbolically by

$$a_2\xi^2 + 2a_1\xi + a_0 = 0 \quad \rightarrow \quad \xi = -\frac{a_1}{a_2} \pm \sqrt{\left(\frac{a_1}{a_2}\right)^2 - \frac{a_0}{a_2}} = \xi_1,\ \xi_2 \text{ say}. \tag{1.7}$$

Transformation from s $(0 < s < \infty)$ to t $(t_0 < t < -1$ where $t_0 = -\xi_2/\xi_1 = -\nu)$ may be given by

$$s = \frac{\xi_1 t + \xi_2}{t+1}, \quad \mathrm{d}s = \frac{\xi_1 - \xi_2}{(t+1)^2}\mathrm{d}t, \tag{1.8}$$

thus

$$\int_0^s \frac{\mathrm{d}s}{abc\sqrt{\varphi(s)}} = \int_{t_0}^t \frac{(\xi_1-\xi_2)\,\mathrm{d}t}{abc\sqrt{\varphi(\xi_1)\,(t^2\pm\nu^2)\,(t^2\pm\mu^2)}} = \frac{(\xi_1-\xi_2)}{abc\sqrt{|\varphi(\xi_1)|}}\int_{t_0}^t \frac{\mathrm{d}t}{\sqrt{\pm\,(t^2\pm\nu^2)\,(t^2\pm\mu^2)}}, \tag{1.9}$$

$$\int_0^s \frac{1}{abc}\frac{s}{a^2\,(s+1/a^2)}\frac{\mathrm{d}s}{\sqrt{\varphi(s)}} = \frac{(\xi_1-\xi_2)}{a^2abc\sqrt{|\varphi(\xi_1)|}}\int_{t_0}^t \frac{S(t)\mathrm{d}t}{\sqrt{\pm\,(t^2\pm\nu^2)\,(t^2\pm\mu^2)}}, \tag{1.10}$$

with $S(t)$ defined as

$$\frac{s}{(s+1/a^2)} = \left[\frac{\xi_1}{\xi_1+1/a^2} - \frac{(\xi_1-\xi_2)\,/a^2}{(\xi_1+1/a^2)\,[(\xi_1+1/a^2)\,t+(\xi_2+1/a^2)]}\right] = S(t). \tag{1.11}$$

The sign of $\varphi(\xi_1)$, $\pm\nu^2$ and $\pm\mu^2$ are given in the table:

type	$\varphi(\xi_1)$	$\pm\nu^2\to-\nu^2$	$\pm\mu^2\to-\mu^2$
(A)	$\varphi(\xi_1)<0$	$\frac{\xi_2^2+\xi_2/a^2}{\xi_1^2+\xi_1/a^2}=-\left(\frac{\xi_2}{\xi_1}\right)^2$	$\frac{(\xi_2+1/b^2)\,(\xi_2+1/c^2)}{(\xi_1+1/b^2)\,(\xi_1+1/c^2)}=-\left(\frac{\xi_2+1/b^2}{\xi_1+1/b^2}\right)^2$
(B)	$\varphi(\xi_1)<0$	$\frac{\xi_2^2+\xi_2/b^2}{\xi_1^2+\xi_1/b^2}=-\left(\frac{\xi_2}{\xi_1}\right)^2$	$\frac{(\xi_2+1/c^2)\,(\xi_2+1/a^2)}{(\xi_1+1/c^2)\,(\xi_1+1/a^2)}=-\left(\frac{\xi_2+1/c^2}{\xi_1+1/c^2}\right)^2$
(C)	$\varphi(\xi_1)<0$	$\frac{\xi_2^2+\xi_2/c^2}{\xi_1^2+\xi_1/c^2}=-\left(\frac{\xi_2}{\xi_1}\right)^2$	$\frac{(\xi_2+1/a^2)\,(\xi_2+1/b^2)}{(\xi_1+1/a^2)\,(\xi_1+1/b^2)}=-\left(\frac{\xi_2+1/a^2}{\xi_1+1/a^2}\right)^2$

The elliptic integral:

$$\int_0^s \frac{\mathrm{d}s}{\sqrt{\varphi(s)}} = \frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\int_{t_0}^t \frac{\mathrm{d}t}{\sqrt{\pm\,(t^2\pm\mu^2)\,(t^2\pm\nu^2)}} = \int_{t_0}^t \frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\frac{\mathrm{d}t}{\sqrt{(t^2-\mu^2)\,(\nu^2-t^2)}}, \tag{1.12}$$

is transformed into the expression

$$\int_0^s \frac{\mathrm{d}s}{\sqrt{\varphi(s)}} = \int_0^u \frac{c\ \mathrm{d}u}{\sqrt{(1-u^2)\,(1-k^2u^2)}} \qquad (c:\text{constant})\,, \tag{1.13}$$

using the transformation of type (ii):

type	denominator in Eq(1.12)	transformation $(t\to u)$	$k^2\ (0<k^2<1)$
(ii)	$(t^2-\mu^2)\,(\nu^2-t^2)$	$t^2=\nu^2\,(1-k^2u^2)\quad[\mu^2<t^2<\nu^2]$	$(\nu^2-\mu^2)\,/\nu^2\quad[\mu^2<\nu^2]$

The transformation $\lambda\to s\to t\to u\to\theta$ and the inverse $\theta\to u\to t\to s\to\lambda$ are arranged in Table 1.

Table 1 Transformation $\lambda\to s\to t\to u\to\theta$ and the inverse $\theta\to u\to t\to s\to\lambda$.

λ	$s=1/\lambda$	$t=-\frac{s-\xi_2}{s-\xi_1}$	$u=\sqrt{\frac{\nu^2-t^2}{\nu^2-\mu^2}}$	$\theta=\sin^{-1}u$
0	∞	-1	$u_0=\sqrt{\frac{\nu^2-1}{\nu^2-\mu^2}}$	$\theta_0=\sin^{-1}u_0$
$\lambda=1/s$	$s=(\xi_1t+\xi_2)\,/\,(t+1)$	$t=-\sqrt{\nu^2\,(1-k^2u^2)}$	$u=\sin\theta$	θ
∞	0	$t_0=-\xi_2/\xi_1=-\nu$	0	0
$0<\lambda<\infty$	$0<s<\infty$	$-\nu<t<-1<-\mu$	$0<u<u_0<1$	$0<\theta<\theta_0<\pi/2$

1.1 Type (A) with (ii)

The transformation given by $t^2=\nu^2\left(1-k^2u^2\right)$ in $\mu^2<t^2<\nu^2$ leads to $t=-\sqrt{\nu^2\,(1-k^2u^2)}$ in $-\nu<t<-1$ corresponding to the region $0<s<\infty$, for which $0<u<u_0$ (where $u_0\equiv\sqrt{(\nu^2-1)\,/\,(\nu^2-\mu^2)}$), $2t\mathrm{d}t=-\nu^2k^22u\mathrm{d}u$ and the integral in u is

$$\int_0^s \frac{\mathrm{d}s}{\sqrt{\varphi(s)}} = \int_{-\xi_2/\xi_1}^t \frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\frac{\mathrm{d}t}{\sqrt{(t^2-\mu^2)\,(\nu^2-t^2)}} = \int_0^u \frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\frac{\mathrm{d}u/\nu}{\sqrt{(1-u^2)\,(1-k^2u^2)}}. \tag{1.14}$$

Then we have

$$\begin{aligned}\frac{s}{s+1/a^2} &= \frac{\xi_1}{\xi_1+1/a^2} - \frac{(\xi_1-\xi_2)/a^2}{\xi_1+1/a^2}\frac{\left(\xi_1+1/a^2\right)t-\left(\xi_2+1/a^2\right)}{\left(\xi_1+1/a^2\right)^2 t^2-\left(\xi_2+1/a^2\right)^2}\\ &= \frac{\xi_1}{\xi_1+1/a^2} + \frac{(\xi_1-\xi_2)/a^2}{\xi_1+1/a^2}\frac{\left(\xi_1+1/a^2\right)\sqrt{\nu^2\left(1-k^2u^2\right)}+\left(\xi_2+1/a^2\right)}{\left(\xi_1+1/a^2\right)^2\nu^2\left(1-k^2u^2\right)-\left(\xi_2+1/a^2\right)^2}\\ &= \frac{\xi_1}{\xi_1+1/a^2}+A\frac{b_1\sqrt{(1-k^2u^2)}+b_0}{-a_2u^2}\end{aligned} \tag{1.15}$$

with

$$A=\frac{(\xi_1-\xi_2)/a^2}{\xi_1+1/a^2},\quad a_0=\left(\xi_1+1/a^2\right)^2\nu^2-\left(\xi_2+1/a^2\right)^2=0,\quad a_2=\left(\xi_1+1/a^2\right)^2\nu^2k^2,$$
$$b_1=\left(\xi_1+1/a^2\right)\nu,\quad b_0=\left(\xi_2+1/a^2\right)=b_1. \tag{1.16}$$

Therefore the integral is now expressed as

$$\begin{aligned}&\frac{1}{a^2abc}\frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\int_0^u\left[\frac{\xi_1}{\xi_1+1/a^2}+\frac{A}{a_2}\frac{b_0+b_1\sqrt{1-k^2u^2}}{-u^2}\right]\frac{\mathrm{d}u/\nu}{\sqrt{(1-u^2)(1-k^2u^2)}}\\ &=\frac{1}{a^2abc}\frac{\xi_1-\xi_2}{\nu\ \sqrt{|\varphi(\xi_1)|}}\left\{\frac{\xi_1}{\xi_1+1/a^2}F(\theta,k)\ +\frac{Ab_0}{a_2}\left[F(\theta,k)-E(\theta,k)-\frac{1}{u}\sqrt{(1-u^2)(1-k^2u^2)}+\frac{\sqrt{1-u^2}}{u}\right]\right\},\end{aligned} \tag{1.17}$$

by taking $u=\sin\theta$ and $0<k^2<1$.

1.2 Type (B) or (C) with (ii)

The transformation given by $t^2=\nu^2\left(1-k^2u^2\right)$ in $\mu^2<t^2<\nu^2$ leads to $t=-\sqrt{\nu^2\left(1-k^2u^2\right)}$ in $-\nu<t<-1$ corresponding to the region $0<s<\infty$, for which $0<u<u_0$ (where $u_0\equiv\sqrt{(\nu^2-1)/(\nu^2-\mu^2)}$), $2t\mathrm{d}t=-\nu^2k^2 2u\mathrm{d}u$ and the integral in u is

$$\int_0^s\frac{\mathrm{d}s}{\sqrt{\varphi(s)}}=\int_{-\xi_2/\xi_1}^t\frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\frac{\mathrm{d}t}{\sqrt{(t^2-\mu^2)(\nu^2-t^2)}}=\int_0^u\frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\frac{\mathrm{d}u/\nu}{\sqrt{(1-u^2)(1-k^2u^2)}}. \tag{1.18}$$

Then we have

$$\begin{aligned}\frac{1/a^2}{s+1/a^2} &= \frac{\xi_1}{\xi_1+1/a^2} - \frac{(\xi_1-\xi_2)/a^2}{\xi_1+1/a^2}\frac{\left(\xi_1+1/a^2\right)t-\left(\xi_2+1/a^2\right)}{\left(\xi_1+1/a^2\right)^2 t^2-\left(\xi_2+1/a^2\right)^2}\\ &= \frac{\xi_1}{\xi_1+1/a^2} + \frac{(\xi_1-\xi_2)/a^2}{\xi_1+1/a^2}\frac{\left(\xi_1+1/a^2\right)\sqrt{\nu^2\left(1-k^2u^2\right)}+\left(\xi_2+1/a^2\right)}{\left(\xi_1+1/a^2\right)^2\nu^2\left(1-k^2u^2\right)-\left(\xi_2+1/a^2\right)^2}\\ &= \frac{\xi_1}{\xi_1+1/a^2}+A\frac{b_1\sqrt{(1-k^2u^2)}+b_0}{a_0-a_2u^2}\end{aligned} \tag{1.19}$$

with

$$A=\frac{(\xi_1-\xi_2)/a^2}{\xi_1+1/a^2},\quad a_0=\left(\xi_1+1/a^2\right)^2\nu^2-\left(\xi_2+1/a^2\right)^2,\quad a_2=\left(\xi_1+1/a^2\right)^2\nu^2k^2=a_0,$$
$$b_1=\left(\xi_1+1/a^2\right)\nu,\quad b_0=\left(\xi_2+1/a^2\right)=\left(\xi_1+1/a^2\right)\mu=b_1\frac{\mu}{\nu}=b_1k'. \tag{1.20}$$

Therefore the integral is now expressed as

$$\frac{1}{a^2abc}\frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\int_0^u \left[1-\frac{1/a^2}{\xi_1+1/a^2}+\frac{A}{a_0}\frac{b_0+b_1\sqrt{1-k^2u^2}}{1-u^2}\right]\frac{\mathrm{d}u/\nu}{\sqrt{(1-u^2)\,(1-k^2u^2)}}$$

$$=\frac{1}{a^2abc}\frac{\xi_1-\xi_2}{\nu\;\sqrt{|\varphi(\xi_1)|}}\left\{\left[1-\frac{1/a^2}{\xi_1+1/a^2}\right]F(\theta,k)\;+\frac{Ab_0}{a_0}\left[F(\theta,k)-\frac{1}{k'^2}E(\theta,k)+\frac{1}{k'^2}\tan\theta\sqrt{1-k^2\sin^2\theta}\right]\right.$$
$$\left.-2\frac{Ab_1}{a_0}\left[\sqrt{1-u^2}-1\right]\right\},\quad (1.21)$$

by taking $u=\sin\theta$, $0<k^2<1$ and $k'^2=1-k^2$.

2 Data of a, b, c and Elliptic integrals

The analysis is confirmed by numerical computations shown below.

No.	a	b	c
1	1.0000E+00	2.0000E+00	3.0000E+00
2	3.0000E+00	1.0000E+00	2.0000E+00
3	2.0000E+00	3.0000E+00	1.0000E+00
4	1.0000E+00	3.0000E+00	2.0000E+00
5	3.0000E+00	2.0000E+00	1.0000E+00
6	2.0000E+00	1.0000E+00	3.0000E+00
7	3.0000E+00	2.0000E+00	5.0000E-01
8	5.0000E-01	3.0000E+00	2.0000E+00
9	2.0000E+00	5.0000E-01	3.0000E+00
10	1.0500E+00	1.0100E+00	1.0000E+00
11	1.0000E+00	1.0500E+00	1.0100E+00
12	1.0100E+00	1.0000E+00	1.0500E+00

No.	type	ξ_1	ξ_2	$-\nu^2$	$-\mu^2$	ν	μ
1	A	2.5648E-01	-1.6952E-01	-4.3687E-01	-2.5249E-02	6.6096E-01	1.5890E-01
1	C	-6.5255E-02	-3.7377E-01	-3.2808E+01	-4.4883E-01	5.7279E+00	6.6995E-01
2	A	-6.5255E-02	-3.7377E-01	-3.2808E+01	-4.4883E-01	5.7279E+00	6.6995E-01
2	B	2.5648E-01	-1.6952E-01	-4.3687E-01	-2.5249E-02	6.6096E-01	1.5890E-01
3	B	-6.5255E-02	-3.7377E-01	-3.2808E+01	-4.4883E-01	5.7279E+00	6.6995E-01
3	C	2.5648E-01	-1.6952E-01	-4.3687E-01	-2.5249E-02	6.6096E-01	1.5890E-01
4	A	2.5648E-01	-1.6952E-01	-4.3687E-01	-2.5249E-02	6.6096E-01	1.5890E-01
4	B	-6.5255E-02	-3.7377E-01	-3.2808E+01	-4.4883E-01	5.7279E+00	6.6995E-01
5	A	-6.5255E-02	-3.7377E-01	-3.2808E+01	-4.4883E-01	5.7279E+00	6.6995E-01
5	C	2.5648E-01	-1.6952E-01	-4.3687E-01	-2.5249E-02	6.6096E-01	1.5890E-01
6	B	2.5648E-01	-1.6952E-01	-4.3687E-01	-2.5249E-02	6.6096E-01	1.5890E-01
6	C	-6.5255E-02	-3.7377E-01	-3.2808E+01	-4.4883E-01	5.7279E+00	6.6995E-01
7	A	-6.4044E-02	-4.1918E-01	-4.2840E+01	-8.2768E-01	6.5452E+00	9.0977E-01
7	C	1.8254E-01	-1.6727E-01	-8.3972E-01	-3.6579E-02	9.1636E-01	1.9126E-01
8	A	1.8254E-01	-1.6727E-01	-8.3972E-01	-3.6579E-02	9.1636E-01	1.9126E-01
8	B	-6.4044E-02	-4.1918E-01	-4.2840E+01	-8.2768E-01	6.5452E+00	9.0977E-01
9	B	1.8254E-01	-1.6727E-01	-8.3972E-01	-3.6579E-02	9.1636E-01	1.9126E-01
9	C	-6.4044E-02	-4.1918E-01	-4.2840E+01	-8.2768E-01	6.5452E+00	9.0977E-01
10	A	-8.3724E-01	-9.8951E-01	-1.3968E+00	-4.1513E-03	1.1819E+00	6.4430E-02
10	C	-9.5658E-01	-1.0475E+00	-1.1992E+00	-8.0419E+00	1.0951E+00	2.8358E+00
11	A	-9.5658E-01	-1.0475E+00	-1.1992E+00	-8.0419E+00	1.0951E+00	2.8358E+00
11	B	-8.3724E-01	-9.8951E-01	-1.3968E+00	-4.1513E-03	1.1819E+00	6.4430E-02
12	B	-9.5658E-01	-1.0475E+00	-1.1992E+00	-8.0419E+00	1.0951E+00	2.8358E+00
12	C	-8.3724E-01	-9.8951E-01	-1.3968E+00	-4.1513E-03	1.1819E+00	6.4430E-02

No.	$\xi_1 - \xi_2$	ξ_2/ξ_1	$\varphi(\xi_1)$	k	k'	type
1	4.2600E-01	-6.6096E-01	5.9996E-02	9.7067E-01	2.4041E-01	A
1	3.0851E-01	5.7279E+00	-5.1675E-04	9.9314E-01	1.1696E-01	C (ii)
2	3.0851E-01	5.7279E+00	-5.1675E-04	9.9314E-01	1.1696E-01	A (ii)
2	4.2600E-01	-6.6096E-01	5.9996E-02	9.7067E-01	2.4041E-01	B
3	3.0851E-01	5.7279E+00	-5.1675E-04	9.9314E-01	1.1696E-01	B (ii)
3	4.2600E-01	-6.6096E-01	5.9996E-02	9.7067E-01	2.4041E-01	C
4	4.2600E-01	-6.6096E-01	5.9996E-02	9.7067E-01	2.4041E-01	A
4	3.0851E-01	5.7279E+00	-5.1675E-04	9.9314E-01	1.1696E-01	B (ii)
5	3.0851E-01	5.7279E+00	-5.1675E-04	9.9314E-01	1.1696E-01	A (ii)
5	4.2600E-01	-6.6096E-01	5.9996E-02	9.7067E-01	2.4041E-01	C
6	4.2600E-01	-6.6096E-01	5.9996E-02	9.7067E-01	2.4041E-01	B
6	3.0851E-01	5.7279E+00	-5.1675E-04	9.9314E-01	1.1696E-01	C (ii)
7	3.5513E-01	6.5452E+00	-2.2063E-03	9.9029E-01	1.3900E-01	A (ii)
7	3.4981E-01	-9.1636E-01	9.6975E-02	9.7798E-01	2.0871E-01	C
8	3.4981E-01	-9.1636E-01	9.6975E-02	9.7798E-01	2.0871E-01	A
8	3.5513E-01	6.5452E+00	-2.2063E-03	9.9029E-01	1.3900E-01	B (ii)
9	3.4981E-01	-9.1636E-01	9.6975E-02	9.7798E-01	2.0871E-01	B
9	3.5513E-01	6.5452E+00	-2.2063E-03	9.9029E-01	1.3900E-01	C (ii)
10	1.5227E-01	1.1819E+00	-1.3605E-03	9.9851E-01	5.4515E-02	A (ii)
10	9.0967E-02	1.0951E+00	4.8807E-05	2.3887E+00	2.5896E+00	C
11	9.0967E-02	1.0951E+00	4.8807E-05	2.3887E+00	2.5896E+00	A
11	1.5227E-01	1.1819E+00	-1.3605E-03	9.9851E-01	5.4515E-02	B (ii)
12	9.0967E-02	1.0951E+00	4.8807E-05	2.3887E+00	2.5896E+00	B
12	1.5227E-01	1.1819E+00	-1.3605E-03	9.9851E-01	5.4515E-02	C (ii)

3 Flows around an elipsoid at rest

For flows around an elipsoid at rest, the velocity potential ϕ is given by

$$\phi = Ux + xf_0. \tag{3.1}$$

For Type (A) with (ii), f_0 is expressed as

$$\begin{aligned} f_0 &= \frac{abcU}{2-\alpha_0}\frac{1}{a^2abc}\frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\int_0^u \left[\frac{\xi_1}{\xi_1+1/a^2}+\frac{A}{a_2}\frac{b_0+b_1\sqrt{1-k^2u^2}}{-u^2}\right]\frac{\mathrm{d}u/\nu}{\sqrt{(1-u^2)\,(1-k^2u^2)}} \\ &= \frac{U\,(\xi_1-\xi_2)}{a^2\,(2-\alpha_0)\,\nu\sqrt{|\varphi(\xi_1)|}}\left\{\frac{\xi_1}{\xi_1+1/a^2}F(\theta,k) + \frac{Ab_0}{a_2}\left[F(\theta,k)-E(\theta,k)+\frac{\sqrt{1-u^2}}{u}\left(1-\sqrt{(1-k^2u^2)}\right)\right]\right\}, \end{aligned} \tag{3.2}$$

with α_0

$$\begin{aligned} \alpha_0 &= \frac{abc}{a^2abc}\frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\int_0^{u_0} \left[\frac{\xi_1}{\xi_1+1/a^2}+\frac{A}{a_2}\frac{b_0+b_1\sqrt{1-k^2u^2}}{-u^2}\right]\frac{\mathrm{d}u/\nu}{\sqrt{(1-u^2)\,(1-k^2u^2)}} \\ &= \frac{(\xi_1-\xi_2)}{a^2\nu\sqrt{|\varphi(\xi_1)|}}\left\{\frac{\xi_1}{\xi_1+1/a^2}F(\theta,k) + \frac{Ab_0}{a_2}\left[F(\theta,k)-E(\theta,k)+\frac{\sqrt{1-u^2}}{u}\left(1-\sqrt{(1-k^2u^2)}\right)\right]\right\}_0^{u_0=\sin\theta_0}, \end{aligned} \tag{3.3}$$

by taking $u = \sin\theta$ and $0 < k^2 < 1$. It is noted for this case that A, a_0, a_2, b_0, b_1 are given in (1.16).

For Type (B) or (C) with (ii), f_0 is expressed as

$$\begin{aligned} f_0 &= \frac{abcU}{2-\alpha_0}\frac{1}{a^2abc}\frac{\xi_1-\xi_2}{\sqrt{|\varphi(\xi_1)|}}\int_0^u \left[\frac{\xi_1}{\xi_1+1/a^2}+\frac{A}{a_0}\frac{b_0+b_1\sqrt{1-k^2u^2}}{1-u^2}\right]\frac{\mathrm{d}u/\nu}{\sqrt{(1-u^2)\,(1-k^2u^2)}} \\ &= \frac{U\,(\xi_1-\xi_2)}{a^2\,(2-\alpha_0)\,\nu\sqrt{|\varphi(\xi_1)|}}\left\{\frac{\xi_1}{\xi_1+1/a^2}F(\theta,k) + \frac{Ab_0}{a_0}\left[F(\theta,k)-\frac{1}{k'^2}E(\theta,k)+\frac{1}{k'^2}\tan\theta\sqrt{1-k^2\sin^2\theta}\right]\right. \\ &\qquad \left. -2\frac{Ab_1}{a_0}\left[\sqrt{1-u^2}-1\right]\right\}, \end{aligned} \tag{3.4}$$

with α_0

$$\alpha_0 = \frac{abc}{a^2 abc} \frac{\xi_1 - \xi_2}{\sqrt{|\varphi(\xi_1)|}} \int_0^{u_0} \left[\frac{\xi_1}{\xi_1 + 1/a^2} + \frac{A}{a_0} \frac{b_0 + b_1\sqrt{1-k^2u^2}}{1-u^2} \right] \frac{\mathrm{d}u/\nu}{\sqrt{(1-u^2)(1-k^2u^2)}}$$
$$= \frac{(\xi_1 - \xi_2)}{a^2\nu\sqrt{|\varphi(\xi_1)|}} \left\{ \frac{\xi_1}{\xi_1 + 1/a^2} F(\theta, k) + \frac{Ab_0}{a_0} \left[F(\theta, k) - \frac{1}{k'^2} E(\theta, k) + \frac{1}{k'^2} \tan\theta \sqrt{1 - k^2 \sin^2\theta} \right] \right.$$
$$\left. -2\frac{Ab_1}{a_0} \left[\sqrt{1-u^2} - 1 \right] \right\}_0^{u_0 = \sin\theta_0}, \tag{3.5}$$

by taking $u = \sin\theta$, $0 < k^2 < 1$ and $k'^2 = 1 - k^2$. It is noted for this case that A, a_0, a_2, b_0, b_1 are given in (1.20).

4 Drawing

4.1 Plane fixed at z =constant

Consider a region:

$$-L_x \leq x \leq L_x, \quad -L_y \leq y \leq L_y, \quad -L_z \leq z \leq L_z \tag{4.1}$$

which includes an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{4.2}$$

or

$$-a \leq x \leq a, \quad -b \leq y \leq b, \quad -c \leq z \leq c. \tag{4.3}$$

A point in the region is taken by steps:

$$\Delta x = \frac{L_x}{N_x}, \quad \Delta y = \frac{L_y}{N_y}, \quad \Delta z = \frac{L_z}{N_z}. \tag{4.4}$$

When $z = 0$, we may take

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \to \quad y = \pm b\sqrt{1 - \frac{x^2}{a^2}} = \pm y_0, \tag{4.5}$$

or

$$x = a\cos\theta, \quad y = b\sin\theta, \tag{4.6}$$

in $0 \leq \theta \leq \pi$.

When $z = z_c$ ($0 < z_c < c$ or $-c < z_c < 0$), we may take

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z_c^2}{c^2} = R_c^2 \quad \to \quad \frac{x^2}{(aR_c)^2} + \frac{y^2}{(bR_c)^2} = 1 \quad \to \quad y = \pm bR_c\sqrt{1 - \frac{x^2}{(aR_c)^2}}, \tag{4.7}$$

or

$$x = aR_c\cos\theta, \quad y = bR_c\sin\theta. \tag{4.8}$$

When $c \leq z \leq L_z$ ($-L_z \leq z \leq -c$), there is no ellipsoid in the plane z =constant.

4.2 Plane fixed at x =constant

Consider a region:

$$-L_x \leq x \leq L_x, \quad -L_y \leq y \leq L_y, \quad -L_z \leq z \leq L_z \tag{4.9}$$

which includes an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{4.10}$$

or

$$-a \leq x \leq a, \quad -b \leq y \leq b, \quad -c \leq z \leq c. \tag{4.11}$$

A point in the region is taken by steps:

$$\Delta x = \frac{L_x}{N_x}, \quad \Delta y = \frac{L_y}{N_y}, \quad \Delta z = \frac{L_z}{N_z}. \tag{4.12}$$

When $x = 0$, we may take

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \rightarrow \quad y = \pm b\sqrt{1 - \frac{z^2}{c^2}} = \pm y_0, \tag{4.13}$$

or

$$y = b\cos\eta, \quad z = c\sin\eta, \tag{4.14}$$

in $0 \leq \eta \leq 2\pi$.

When $x = x_a$ $(0 < x_a < a$ or $-a < x_a < 0)$, we may take

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x_a^2}{a^2} = R_a^2 \quad \rightarrow \quad \frac{y^2}{(bR_a)^2} + \frac{z^2}{(cR_a)^2} = 1 \quad \rightarrow \quad y = \pm bR_a\sqrt{1 - \frac{z^2}{(cR_a)^2}}, \tag{4.15}$$

or

$$y = bR_a\cos\eta, \quad z = cR_a\sin\eta. \tag{4.16}$$

When $a \leq x \leq L_x$ $(-L_x \leq x \leq -a)$, there is no ellipsoid in the plane x =constant.

A Expressions for semi-axes of ellipsoid

The maximum of the semi-axes of ellipsoid a, b, c may give some geometrical relations as shown below.

A.1 Type (A) ($a > b > c > 0$ or $a > c \geq b > 0$)

$$\xi_{1,2} = \frac{-1/\left(b^2c^2\right) \pm \sqrt{1/\left(b^2c^2\right)\left(1/c^2 - 1/a^2\right)\left(1/b^2 - 1/a^2\right)}}{1/b^2 + 1/c^2 - 1/a^2}, \tag{A.1}$$

$$\xi_{1,2} + 1/a^2 = \frac{-\left(1/c^2 - 1/a^2\right)\left(1/b^2 - 1/a^2\right) \pm \sqrt{1/\left(b^2c^2\right)\left(1/c^2 - 1/a^2\right)\left(1/b^2 - 1/a^2\right)}}{1/b^2 + 1/c^2 - 1/a^2}, \tag{A.2}$$

$$\xi_{1,2} + 1/b^2 = \frac{1/b^2\left(1/b^2 - 1/a^2\right) \pm \sqrt{1/\left(b^2c^2\right)\left(1/c^2 - 1/a^2\right)\left(1/b^2 - 1/a^2\right)}}{1/b^2 + 1/c^2 - 1/a^2}, \tag{A.3}$$

$$\xi_{1,2} + 1/c^2 = \frac{1/c^2\left(1/c^2 - 1/a^2\right) \pm \sqrt{1/\left(b^2c^2\right)\left(1/c^2 - 1/a^2\right)\left(1/b^2 - 1/a^2\right)}}{1/b^2 + 1/c^2 - 1/a^2}, \tag{A.4}$$

$$-\nu^2=\frac{\xi_2}{\xi_1}\frac{\left(\xi_2+1/a^2\right)}{\left(\xi_1+1/a^2\right)}=\frac{\xi_2}{\xi_1}\times\frac{-\sqrt{\left(1/c^2-1/a^2\right)\left(1/b^2-1/a^2\right)}-\sqrt{1/\left(b^2c^2\right)}}{-\sqrt{\left(1/c^2-1/a^2\right)\left(1/b^2-1/a^2\right)}+\sqrt{1/\left(b^2c^2\right)}}=-\left(\frac{\xi_2}{\xi_1}\right)^2, \tag{A.5}$$

$$-\mu^2=\frac{\left(\xi_2+1/b^2\right)}{\left(\xi_1+1/b^2\right)}\frac{\left(\xi_2+1/c^2\right)}{\left(\xi_1+1/c^2\right)}=-\left(\frac{\sqrt{1/b^2\left(1/b^2-1/a^2\right)}-\sqrt{1/c^2\left(1/c^2-1/a^2\right)}}{\sqrt{1/b^2\left(1/b^2-1/a^2\right)}+\sqrt{1/c^2\left(1/c^2-1/a^2\right)}}\right)^2, \tag{A.6}$$

$$A=\frac{\left(\xi_1-\xi_2\right)/a^2}{\xi_1+1/a^2},\quad a_0=\left(\xi_1+1/a^2\right)^2\nu^2-\left(\xi_2+1/a^2\right)^2=0,\quad a_2=\left(\xi_1+1/a^2\right)^2\nu^2k^2,$$
$$b_1=\left(\xi_1+1/a^2\right)\nu,\quad b_0=\left(\xi_2+1/a^2\right)=b_1. \tag{A.7}$$

Since $\xi_1<0$ only, $\varphi(\xi_1)=\xi_1(\xi_1+1/a^2)(\xi_1+1/b^2)(\xi_1+1/c^2)<0$. When $c=b$, we have $\xi_2+1/c^2=\xi_2+1/b^2=0$ and $\mu=0$.

A.2 Type (B) ($b>c>a>0$ or $b>a\geq c>0$)

$$\xi_{1,2}=\frac{-1/\left(a^2c^2\right)\pm\sqrt{1/\left(a^2c^2\right)\left(1/c^2-1/b^2\right)\left(1/a^2-1/b^2\right)}}{1/a^2+1/c^2-1/b^2}, \tag{A.8}$$

$$\xi_{1,2}+1/b^2=\frac{-\left(1/c^2-1/b^2\right)\left(1/a^2-1/b^2\right)\pm\sqrt{1/\left(a^2c^2\right)\left(1/c^2-1/b^2\right)\left(1/a^2-1/b^2\right)}}{1/a^2+1/c^2-1/b^2}, \tag{A.9}$$

$$\xi_{1,2}+1/a^2=\frac{1/a^2\left(1/a^2-1/b^2\right)\pm\sqrt{1/\left(a^2c^2\right)\left(1/c^2-1/b^2\right)\left(1/a^2-1/b^2\right)}}{1/a^2+1/c^2-1/b^2}, \tag{A.10}$$

$$\xi_{1,2}+1/c^2=\frac{1/c^2\left(1/c^2-1/b^2\right)\pm\sqrt{1/\left(a^2c^2\right)\left(1/c^2-1/b^2\right)\left(1/a^2-1/b^2\right)}}{1/a^2+1/c^2-1/b^2}, \tag{A.11}$$

$$-\nu^2=\frac{\xi_2}{\xi_1}\frac{\left(\xi_2+1/b^2\right)}{\left(\xi_1+1/b^2\right)}=\frac{\xi_2}{\xi_1}\times\frac{-\sqrt{\left(1/c^2-1/b^2\right)\left(1/a^2-1/b^2\right)}-\sqrt{1/\left(a^2c^2\right)}}{-\sqrt{\left(1/c^2-1/b^2\right)\left(1/a^2-1/b^2\right)}+\sqrt{1/\left(a^2c^2\right)}}=-\left(\frac{\xi_2}{\xi_1}\right)^2, \tag{A.12}$$

$$-\mu^2=\frac{\left(\xi_2+1/a^2\right)}{\left(\xi_1+1/a^2\right)}\frac{\left(\xi_2+1/c^2\right)}{\left(\xi_1+1/c^2\right)}=-\left(\frac{\sqrt{1/a^2\left(1/a^2-1/b^2\right)}-\sqrt{1/c^2\left(1/c^2-1/b^2\right)}}{\sqrt{1/a^2\left(1/a^2-1/b^2\right)}+\sqrt{1/c^2\left(1/c^2-1/b^2\right)}}\right)^2, \tag{A.13}$$

$$A=\frac{\left(\xi_1-\xi_2\right)/a^2}{\xi_1+1/a^2},\quad a_0=\left(\xi_1+1/a^2\right)^2\nu^2-\left(\xi_2+1/a^2\right)^2,\quad a_2=\left(\xi_1+1/a^2\right)^2\nu^2k^2=a_0,$$
$$b_1=\left(\xi_1+1/a^2\right)\nu,\quad b_0=\left(\xi_2+1/a^2\right)=\left(\xi_1+1/a^2\right)\mu=b_1\frac{\mu}{\nu}=b_1k'. \tag{A.14}$$

When $a=c$, we have $\xi_2+1/a^2=\xi_2+1/c^2=0$ and $\mu=0$.

A.3 Type (C) ($c>a>b>0$ or $c>b\geq a>0$)

$$\xi_{1,2}=\frac{-1/\left(a^2b^2\right)\pm\sqrt{1/\left(a^2b^2\right)\left(1/a^2-1/c^2\right)\left(1/b^2-1/c^2\right)}}{1/a^2+1/b^2-1/c^2}, \tag{A.15}$$

$$\xi_{1,2}+1/c^2=\frac{-\left(1/a^2-1/c^2\right)\left(1/b^2-1/c^2\right)\pm\sqrt{1/\left(a^2b^2\right)\left(1/a^2-1/c^2\right)\left(1/b^2-1/c^2\right)}}{1/a^2+1/b^2-1/c^2}, \tag{A.16}$$

$$\xi_{1,2}+1/a^2=\frac{1/a^2\left(1/a^2-1/c^2\right)\pm\sqrt{1/\left(a^2b^2\right)\left(1/a^2-1/c^2\right)\left(1/b^2-1/c^2\right)}}{1/a^2+1/b^2-1/c^2}, \tag{A.17}$$

$$\xi_{1,2}+1/b^2=\frac{1/b^2\left(1/b^2-1/c^2\right)\pm\sqrt{1/\left(a^2b^2\right)\left(1/a^2-1/c^2\right)\left(1/b^2-1/c^2\right)}}{1/a^2+1/b^2-1/c^2}, \tag{A.18}$$

$$-\nu^2 = \frac{\xi_2}{\xi_1}\frac{\left(\xi_2 + 1/c^2\right)}{\left(\xi_1 + 1/c^2\right)} = \frac{\xi_2}{\xi_1} \times \frac{-\sqrt{(1/a^2 - 1/c^2)\,(1/b^2 - 1/c^2)} - \sqrt{1/\left(a^2b^2\right)}}{-\sqrt{(1/a^2 - 1/c^2)\,(1/b^2 - 1/c^2)} + \sqrt{1/\left(a^2b^2\right)}} = -\left(\frac{\xi_2}{\xi_1}\right)^2, \tag{A.19}$$

$$-\mu^2 = \frac{\left(\xi_2 + 1/a^2\right)}{\left(\xi_1 + 1/a^2\right)}\frac{\left(\xi_2 + 1/b^2\right)}{\left(\xi_1 + 1/b^2\right)} = -\left(\frac{\sqrt{1/a^2\,(1/a^2 - 1/c^2)} - \sqrt{1/b^2\,(1/b^2 - 1/c^2)}}{\sqrt{1/a^2\,(1/a^2 - 1/c^2)} + \sqrt{1/b^2\,(1/b^2 - 1/c^2)}}\right)^2, \tag{A.20}$$

$$A = \frac{\left(\xi_1 - \xi_2\right)/a^2}{\xi_1 + 1/a^2}, \quad a_0 = \left(\xi_1 + 1/a^2\right)^2 \nu^2 - \left(\xi_2 + 1/a^2\right)^2, \quad a_2 = \left(\xi_1 + 1/a^2\right)^2 \nu^2 k^2 = a_0,$$
$$b_1 = \left(\xi_1 + 1/a^2\right)\nu, \quad b_0 = \left(\xi_2 + 1/a^2\right) = \left(\xi_1 + 1/a^2\right)\mu = b_1\frac{\mu}{\nu} = b_1 k'. \tag{A.21}$$

When $b = a$, we have $\xi_2 + 1/b^2 = \xi_2 + 1/a^2 = 0$ and $\mu = 0$.

B Transformation

The elliptic integral:

$$\int \frac{\mathrm{d}s}{\sqrt{\varphi(s)}} = \frac{\xi_1 - \xi_2}{\sqrt{|\varphi(\xi_1)|}} \int \frac{\mathrm{d}t}{\sqrt{\pm\left(t^2 \pm \mu^2\right)\left(t^2 \pm \nu^2\right)}}, \tag{B.1}$$

is transformed into the expression of u

$$\int \frac{\mathrm{d}s}{\sqrt{\varphi(s)}} = \int \frac{c\,\mathrm{d}u}{\sqrt{\left(1 - u^2\right)\left(1 - k^2u^2\right)}} \qquad \left[c : \text{constant},\ 0 < k^2 < 1\right], \tag{B.2}$$

based on the following table.

type	denominator in Eq(B.1)	transformation ($t \to u$)	k^2
(i)	$\left(t^2 - \mu^2\right)\left(t^2 - \nu^2\right)$	$t^2 = \mu^2 u^2$ $\left[t^2 < \mu^2\right]$; $t^2 = \nu^2/u^2$ $\left[t^2 > \nu^2\right]$	μ^2/ν^2 $\left[\mu^2 < \nu^2\right]$
(ii)	$\left(t^2 - \mu^2\right)\left(\nu^2 - t^2\right)$	$t^2 = \nu^2\left(1 - k^2u^2\right)$ $\left[\mu^2 < t^2 < \nu^2\right]$	$\left(\nu^2 - \mu^2\right)/\nu^2$ $\left[\mu^2 < \nu^2\right]$
(iii)	$\left(t^2 + \mu^2\right)\left(t^2 - \nu^2\right)$	$t^2 = \nu^2/\left(1 - u^2\right)$ $\left[t^2 > \nu^2\right]$	$\mu^2/\left(\mu^2 + \nu^2\right)$
(iv)	$\left(t^2 + \mu^2\right)\left(\nu^2 - t^2\right)$	$t^2 = \nu^2\left(1 - u^2\right)$ $\left[t^2 < \mu^2\right]$	$\nu^2/\left(\mu^2 + \nu^2\right)$
(v)	$\left(t^2 + \mu^2\right)\left(t^2 + \nu^2\right)$	$t^2 = \mu^2u^2/\left(1 - u^2\right)$	$\left(\nu^2 - \mu^2\right)/\nu^2$ $\left[\mu^2 < \nu^2\right]$

The case for which $-\left(t^2 + \mu^2\right)\left(t^2 + \nu^2\right)$ is taken away since the term in the square root is negative.

Our problem may be solved using type (ii).

A note on Elliptic integral

T.Funada, October 3, 2004 / cal-oct3.tex / printed October 3, 2004

1 Derivation of formulas

Funada computed on Sunday: for a triaxial ellipsoid with $a > b > c$, the velocity potentials are given by

$$\phi_x = \frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \left(a^2+\lambda\right)^{-3/2}\left(b^2+\lambda\right)^{-1/2}\left(c^2+\lambda\right)^{-1/2}\mathrm{d}\lambda, \tag{1.1}$$

$$\phi_y = \frac{abcVy}{2-\beta_0}\int_\lambda^\infty \left(a^2+\lambda\right)^{-1/2}\left(b^2+\lambda\right)^{-3/2}\left(c^2+\lambda\right)^{-1/2}\mathrm{d}\lambda, \tag{1.2}$$

$$\phi_z = \frac{abcWz}{2-\gamma_0}\int_\lambda^\infty \left(a^2+\lambda\right)^{-1/2}\left(b^2+\lambda\right)^{-1/2}\left(c^2+\lambda\right)^{-3/2}\mathrm{d}\lambda, \tag{1.3}$$

where

$$\alpha_0 = abc\int_0^\infty \left(a^2+\lambda\right)^{-3/2}\left(b^2+\lambda\right)^{-1/2}\left(c^2+\lambda\right)^{-1/2}\mathrm{d}\lambda, \tag{1.4}$$

$$\beta_0 = abc\int_0^\infty \left(a^2+\lambda\right)^{-1/2}\left(b^2+\lambda\right)^{-3/2}\left(c^2+\lambda\right)^{-1/2}\mathrm{d}\lambda, \tag{1.5}$$

$$\gamma_0 = abc\int_0^\infty \left(a^2+\lambda\right)^{-1/2}\left(b^2+\lambda\right)^{-1/2}\left(c^2+\lambda\right)^{-3/2}\mathrm{d}\lambda. \tag{1.6}$$

Putting η, θ and k to

$$\eta = \sqrt{\frac{a^2-c^2}{a^2+\lambda}} = \sin\theta, \quad k = \sqrt{\frac{a^2-b^2}{a^2-c^2}}, \tag{1.7}$$

we may have the elliptic integral in (1.1)

$$\begin{aligned}
&\int_\lambda^\infty \left(a^2+\lambda\right)^{-3/2}\left(b^2+\lambda\right)^{-1/2}\left(c^2+\lambda\right)^{-1/2}\mathrm{d}\lambda \\
&= \int_\eta^0 \left(a^2+\lambda\right)^{-3/2}\left(b^2+\lambda\right)^{-1/2}\left(c^2+\lambda\right)^{-1/2}(-2)\frac{\left(a^2+\lambda\right)^{3/2}}{\sqrt{a^2-c^2}}\mathrm{d}\eta \\
&= \int_0^\eta \left(b^2-a^2+\frac{a^2-c^2}{\eta^2}\right)^{-1/2}\left(c^2-a^2+\frac{a^2-c^2}{\eta^2}\right)^{-1/2}\frac{2}{\sqrt{a^2-c^2}}\mathrm{d}\eta \\
&= \frac{2}{\sqrt{a^2-c^2}\,(a^2-c^2)}\int_0^\eta \eta^2\left(1-k^2\eta^2\right)^{-1/2}\left(1-\eta^2\right)^{-1/2}\mathrm{d}\eta \\
&= \frac{2}{\sqrt{a^2-c^2}\,(a^2-c^2)}\int_0^\eta \frac{1}{k^2}\left[\left(1-k^2\eta^2\right)^{-1/2}-\left(1-k^2\eta^2\right)^{1/2}\right]\left(1-\eta^2\right)^{-1/2}\mathrm{d}\eta \\
&= \frac{2}{\sqrt{a^2-c^2}\,(a^2-b^2)}\int_0^\theta \left[\left(1-k^2\sin^2\theta\right)^{-1/2}-\left(1-k^2\sin^2\theta\right)^{1/2}\right]\mathrm{d}\theta \\
&= \frac{2}{\sqrt{a^2-c^2}\,(a^2-b^2)}\left[F(\theta,k)-E(\theta,k)\right],
\end{aligned} \tag{1.8}$$

in (1.2)

$$
\begin{aligned}
&\int_\lambda^\infty \left(a^2+\lambda\right)^{-1/2}\left(b^2+\lambda\right)^{-3/2}\left(c^2+\lambda\right)^{-1/2}\mathrm{d}\lambda \\
&=\int_\eta^0 \left(a^2+\lambda\right)^{-1/2}\left(b^2+\lambda\right)^{-3/2}\left(c^2+\lambda\right)^{-1/2}(-2)\frac{\left(a^2+\lambda\right)^{3/2}}{\sqrt{a^2-c^2}}\mathrm{d}\eta \\
&=\int_0^\eta \frac{a^2-c^2}{\eta^2}\left(b^2-a^2+\frac{a^2-c^2}{\eta^2}\right)^{-3/2}\left(c^2-a^2+\frac{a^2-c^2}{\eta^2}\right)^{-1/2}\frac{2}{\sqrt{a^2-c^2}}\mathrm{d}\eta \\
&=\frac{2}{\sqrt{a^2-c^2}\left(a^2-c^2\right)}\int_0^\eta \eta^2\left(1-k^2\eta^2\right)^{-3/2}\left(1-\eta^2\right)^{-1/2}\mathrm{d}\eta \\
&=\frac{2}{\sqrt{a^2-c^2}\left(a^2-c^2\right)}\int_0^\theta \sin^2\theta\left(1-k^2\sin^2\theta\right)^{-3/2}\mathrm{d}\theta \\
&=\frac{2}{\sqrt{a^2-c^2}\left(a^2-b^2\right)}\frac{1}{1-k^2}\left[E(\theta,k)-\left(1-k^2\right)F(\theta,k)-\frac{k^2\sin(2\theta)}{2\sqrt{1-k^2\sin^2\theta}}\right],
\end{aligned}
\tag{1.9}
$$

and in (1.3)

$$
\begin{aligned}
&\int_\lambda^\infty \left(a^2+\lambda\right)^{-1/2}\left(b^2+\lambda\right)^{-1/2}\left(c^2+\lambda\right)^{-3/2}\mathrm{d}\lambda \\
&=\int_\eta^0 \left(a^2+\lambda\right)^{-1/2}\left(b^2+\lambda\right)^{-1/2}\left(c^2+\lambda\right)^{-3/2}(-2)\frac{\left(a^2+\lambda\right)^{3/2}}{\sqrt{a^2-c^2}}\mathrm{d}\eta \\
&=\int_0^\eta \frac{a^2-c^2}{\eta^2}\left(b^2-a^2+\frac{a^2-c^2}{\eta^2}\right)^{-1/2}\left(c^2-a^2+\frac{a^2-c^2}{\eta^2}\right)^{-3/2}\frac{2}{\sqrt{a^2-c^2}}\mathrm{d}\eta \\
&=\frac{2}{\sqrt{a^2-c^2}\left(a^2-c^2\right)}\int_0^\eta \eta^2\left(1-k^2\eta^2\right)^{-1/2}\left(1-\eta^2\right)^{-3/2}\mathrm{d}\eta \\
&=\frac{2}{\sqrt{a^2-c^2}\left(a^2-c^2\right)}\int_0^\theta \sin^2\theta\left(1-k^2\sin^2\theta\right)^{-1/2}\left(1-\sin^2\theta\right)^{-1}\mathrm{d}\theta \\
&=\frac{2}{\sqrt{a^2-c^2}\left(a^2-c^2\right)}\int_0^\theta \tan^2\theta\left(1-k^2\sin^2\theta\right)^{-1/2}\mathrm{d}\theta \\
&=\frac{2}{\sqrt{a^2-c^2}\left(a^2-c^2\right)}\frac{1}{1-k^2}\left[\tan\theta\left(1-k^2\sin^2\theta\right)^{1/2}-E(\theta,k)\right].
\end{aligned}
\tag{1.10}
$$

2 Check of the formula

On Saturday (October 2), Sonoda computed the following:

$$
\Gamma=U\left\{\frac{abc}{2-\alpha_0}\frac{2\left[F(\varphi,k)-E(\varphi,k)\right]}{\left(a^2-b^2\right)\sqrt{a^2-c^2}}+1\right\} \tag{2.1}
$$

$$
\frac{\partial\Gamma}{\partial\lambda}=U\frac{2abc}{2-\alpha_0}\frac{1}{\left(a^2-b^2\right)\sqrt{a^2-c^2}}\left[\frac{\partial F}{\partial\lambda}-\frac{\partial E}{\partial\lambda}\right] \tag{2.2}
$$

$$
\begin{aligned}
\left[\frac{\partial F}{\partial\lambda}-\frac{\partial E}{\partial\lambda}\right]&=\frac{\partial F}{\partial\varphi}\frac{\partial\varphi}{\partial\lambda}-\frac{\partial E}{\partial\varphi}\frac{\partial\varphi}{\partial\lambda} \\
=-\frac{1}{2}&\frac{\sqrt{a^2-c^2}\left(a^2-b^2\right)}{\left(\lambda+a^2\right)\sqrt{\left(\lambda+a^2\right)\left(\lambda+b^2\right)\left(\lambda+c^2\right)}}
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
\frac{\partial\varphi}{\partial\lambda}&=-\frac{1}{2}\frac{\sqrt{a^2-c^2}}{\left(\lambda+a^2\right)\sqrt{\lambda+c^2}} \\
\frac{\partial F}{\partial\varphi}&=\frac{1}{\sqrt{1-k^2\sin^2\varphi}}=\sqrt{\frac{\lambda+a^2}{\lambda+b^2}} \\
\frac{\partial E}{\partial\varphi}&=\sqrt{1-k^2\sin^2\varphi}=\sqrt{\frac{\lambda+b^2}{\lambda+a^2}} \\
\varphi&=\arcsin\sqrt{\frac{a^2-c^2}{\lambda+a^2}} \\
k&=\sqrt{\frac{a^2-b^2}{a^2-c^2}}
\end{aligned}
\tag{2.4}
$$

$$
\frac{\partial\Gamma}{\partial\lambda}=\frac{-abcU}{\left(2-\alpha_0\right)\sqrt{\left(\lambda+a^2\right)^3\left(\lambda+b^2\right)\left(\lambda+c^2\right)}} \tag{2.5}
$$