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Energy equation for irrotational theories of gas-liquid flow:: viscous potential flow (VPF), viscous potential flow with pressure correction (VCVPF), dissipation method (DM)

12.1 Viscous potential flow (VPF)

The effects of viscosity on the irrotational motions of spherical cap bubble, Taylor bubbles in round tubes, Rayleigh-Taylor and Kelvin-Helmholtz instability described in previous chapters were obtained by evaluating the viscous normal stress on potential flow. In gas-liquid flows, the viscous normal stress does not vanish and it can be evaluated on the potential. It can be said that in the case of gas-liquid flow, the appropriate formulation of the irrotational problem is the same as the conventional one for inviscid fluids with the caveat that the viscous normal stress is included in the normal stress balance. This formulation of viscous potential flow is not at all subtle; it is the natural and obvious way to express the equations of balance when the flow is irrotational and the fluid viscous.

We shall use the acronym VPF, viscous potential flow, to stand for the irrotational theory in which the viscous normal stresses are evaluated on the potential.

In gas-liquid flows we may assume that the shear stress in the gas is negligible so that no condition need be enforced on the tangential velocity at the free surface, but the shear stress must be zero. The condition that the shear stress is zero at each point on the free surface is dropped in the irrotational approximations. In general, you get an irrotational shear stress from the irrotational analysis. The discrepancy between this shear stress and the zero shear stress required for exact solutions will generate a vorticity layer. In many cases this layer is thin and its influence on the bulk motion and on the irrotational effects of viscosity is minor. Exact solutions cannot be the same as any irrotational approximation for cases in which the irrotational shear stress does not vanish.

12.2 Dissipation method according to Lamb

Another purely irrotational theory of flow of viscous fluids is associated with the dissipation method. The acronym DM is used for the dissipation method. The DM extracts information from evaluating the equation governing the evolution of energy and dissipation on irrotational flow. VPF does not rely on arguments about energy and dissipation and the results of the two irrotational theories, DM and VPF, are usually different.

The dissipation method was used by Lamb (1932) to compute the decay of irrotational waves due to viscosity. This method can be traced back to Stokes (1851). A pressure correction was not introduced in these studies. Lamb presented some exact solutions from which the irrotational approximation can be evaluated and the nature of the boundary layer rigorously examined. Lamb (1932) studied the viscous decay of free oscillatory waves on deep water § 348 and small oscillations of a mass of liquid about the spherical form § 355, using the dissipation method. Lamb showed that in these cases the rate of dissipation can be calculated with sufficient accuracy by regarding the motion as irrotational.

Lamb's (1932) calculations are based on the assumption that the speed of a progressive wave is the same as in an inviscid fluid. Padrino (Padrino and Joseph 2006, §14.2.4 and 14.3.1) noticed that a more complete calculation of the dissipation method yields a different result; the growth rates are the same as in Lamb's calculation but the speed of the wave depends on the viscosity of the liquid.

12.3 Drag on a spherical gas bubble calculated from the viscous dissipation of an irrotational flow

The computation of the drag D on a sphere in potential flow using the dissipation method seems to have been given first by Bateman in 1932 (see Dryden, Murnaghan, and Bateman 1956) and repeated by Ackeret (1952). They found that $D = 12\pi a\mu U$ where μ is the viscosity, a radius of the sphere and U its velocity. This drag is twice the Stokes drag and is in better agreement with the measured drag for Reynolds numbers in excess of about 8.

The drag on a spherical gas bubble of radius a rising in a viscous liquid at modestly high Reynolds numbers was calculated, using the dissipation method by various investigators beginning with Levich (1949), who obtained the value $12\pi a\mu U$ or equivalently the drag coefficient $48/R$, where $R = 2aU\rho/\mu$ is the Reynolds number, by calculating the dissipation of the irrotational flow around the bubble. Moore (1959) calculated the drag directly by integrating the pressure and viscous normal stress of the potential flow and neglecting the viscous shear stress (which physically should be zero), obtaining the value $8\pi a\mu U$.

12.4 The idea of a pressure correction

The discrepancy between the two drag values led G. K. Batchelor, as reported by Moore (1963), to suggest the idea discrepancy could be resolved by introducing a viscous correction to the irrotational pressure. We do not know whether we should praise or blame Batchelor for this idea since it comes to us second hand. There is no doubt that the pressure in the liquid depends on the viscosity and it could take on its largest values in boundary layer next to the free surface but the velocity there could also come to depend on the viscosity. Moore (1963) performed a boundary layer analysis and his pressure correction is readily obtained by setting $y = 0$ in his equation (2.37):

$$p_v = (4/R)\,(1-\cos\theta)^2\,(2+\cos\theta)\,/\sin^2\theta \tag{12.4.1}$$

which is singular at the separation point where $\theta = \pi$. The presence of separation is a problem for the application of boundary layers to the calculation of drag on solid bodies. To find the drag coefficient Moore (1963) calculated the momentum defect, and obtained the Levich value $48/R$ plus contributions of order $R^{-3/2}$ or lower.

Kang & Leal (1988a, b) put the shear stress of the potential flow on the bubble surface to zero and calculated a pressure correction. They obtained the drag coefficient given by Levich's dissipation approximation by direct integration of the normal viscous stress and pressure over the bubble surface. They accomplished this by expanding the pressure correction as a spherical harmonic series and noting that only one term of this series contributes to the drag, no appeal to the boundary-layer approximation being necessary. A similar result was obtained by Joseph & Wang (2004a) using their theory described below, for computing a pressure correction.

The accepted idea, starting with Batchelor, is that the pressure correction is a real viscous pressure which varies from the pressure in the irrotational flow outside a narrow vorticity layer near the gas-liquid surface, to the required value at the interface. This boundary layer has not been computed. Maybe the idea is not quite correct. In the case of the decay of free irrotational gravity waves computed by Lamb (1932), a viscous pressure correction does not appear. The pressure correction computed by Wang & Joseph (2006d) as shown in §14.1 gives the same result as the dissipation method and is in excellent agreement with the exact solution but it is not a viscous correction in a boundary layer of the exact solution.

12.5 Energy equation for irrotational flow of a viscous fluid

Consider the equations of motion for an incompressible Newtonian fluid with gravity as a body force per unit mass

$$\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla\Phi + \mu\nabla^2\mathbf{u} \tag{12.5.1}$$

where

$$\Phi = p + \rho g z. \tag{12.5.2}$$

The stress is given by

$$\mathbf{T} = -p\mathbf{1} + \boldsymbol{\tau} \tag{12.5.3}$$

where

$$\boldsymbol{\tau} = 2\mu\mathbf{D}[\mathbf{u}]$$

and

$$\nabla\cdot\boldsymbol{\tau} = \mu\nabla^2\mathbf{u}$$

The mechanical energy equation corresponding to (12.5.1) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho}{2}|\mathbf{u}|^2\ \mathrm{d}\Omega = \int_S \mathbf{n}\cdot\tilde{\mathbf{T}}\cdot\mathbf{u}\mathrm{d}S - 2\mu\int_\Omega \mathbf{D}:\mathbf{D}\ \mathrm{d}\Omega, \tag{12.5.4}$$

where S is the boundary of Ω, with outward normal $\mathbf{n}$. On solid boundaries no-slip is imposed; say $\mathbf{u}=0$ there, and on the free surface

$$z = \eta(x,y,t)$$

and the shear stress

$$\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{e}_\mathrm{s} = 0, \tag{12.5.5}$$

where $\mathbf{e}_s$ is any vector tangent to the free surface S_f. On S_f we have

$$\int_S \mathbf{n}\cdot\tilde{\mathbf{T}}\cdot\mathbf{u}\mathrm{d}S = \int_{S_f}\mathbf{n}\cdot\tilde{\mathbf{T}}\cdot\mathbf{u}\mathrm{d}S$$

where

$$\tilde{\mathbf{T}} = \mathbf{T} - \rho g\eta\mathbf{1}, \tag{12.5.6}$$

$$\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n} = -\gamma\nabla_{\mathrm{II}}\cdot\mathbf{n}. \tag{12.5.7}$$

Hence, on S_f

$$\mathbf{n}\cdot\tilde{\mathbf{T}}\cdot\mathbf{n} = -\rho g\eta - \gamma\nabla_{\mathrm{II}}\cdot\mathbf{n}. \tag{12.5.8}$$

and, since the shear stress vanishes on S_f

$$\begin{aligned}\mathbf{n}\cdot\tilde{\mathbf{T}}\cdot\mathbf{u} &= \mathbf{n}\cdot\tilde{\mathbf{T}}\cdot(u_\mathrm{n}\mathbf{n}+u_\mathrm{s}\mathbf{e}_\mathrm{s})\\ &= -(\rho g\eta + \gamma\nabla_{\mathrm{II}}\cdot\mathbf{n})u_\mathrm{n}.\end{aligned} \tag{12.5.9}$$

Hence, (12.5.4) may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho}{2}|\mathbf{u}|^2\ \mathrm{d}\Omega = -\int_{S_\mathrm{f}}(\rho g\eta + \gamma\nabla_{\mathrm{II}}\cdot\mathbf{n})u_\mathrm{n}\mathrm{d}S_\mathrm{f} - 2\mu\int_\Omega \mathbf{D}:\mathbf{D}\ \mathrm{d}\Omega, \tag{12.5.10}$$

Equation (12.5.10) holds for viscous fluids satisfying the Navier–Stokes equations (12.5.1) subject to the vanishing shear stress condition (12.5.5).

We turn now to potential flow $\mathbf{u}=\nabla\phi$, $\nabla^2\phi=0$. In this case, $\nabla^2\mathbf{u}=0$ but the dissipation does not vanish. How can this be? In Chapter 4 we showed that the irrotational viscous stress is self-equilibrated and does give rise to a force density term $\nabla^2\nabla\phi=0$; however, the power of self-equilibrated irrotational viscous stresses

$$\int_{S_\mathrm{f}}\mathbf{n}\cdot 2\mu\nabla\otimes\nabla\phi\cdot\mathbf{u}\mathrm{d}S_\mathrm{f}$$

does not vanish, and it gives rise to an irrotational viscous dissipation

$$\begin{aligned}2\mu\int_\Omega \mathbf{D}[\mathbf{u}]:\mathbf{D}[\mathbf{u}]\mathrm{d}\Omega &= 2\mu\int_\Omega \frac{\partial^2\phi}{\partial x_\mathrm{i}\partial x_\mathrm{j}}\frac{\partial^2\phi}{\partial x_i\partial x_j}\mathrm{d}\Omega = 2\mu\int_S n_\mathrm{j}\frac{\partial^2\phi}{\partial x_\mathrm{j}\partial x_\mathrm{i}}u_\mathrm{i}\mathrm{d}S\\ &= 2\mu\int_{S_\mathrm{f}}\mathbf{n}\cdot\mathbf{D}[\nabla\phi]\cdot\mathbf{u}\mathrm{d}S_\mathrm{f} = 2\mu\int_{S_\mathrm{f}}\mathbf{n}\cdot\mathbf{D}\cdot(u_\mathrm{n}\mathbf{n}+u_\mathrm{s}\mathbf{e}_\mathrm{s})\,\mathrm{d}S_\mathrm{f}\\ &= \int_{S_\mathrm{f}}\left(2\mu\frac{\partial^2\phi}{\partial n^2}u_\mathrm{n} + \tau_\mathrm{s}u_\mathrm{s}\right)\mathrm{d}S_\mathrm{f}\end{aligned} \tag{12.5.11}$$

where τ_{s} is an irrotational shear stress

$$\begin{aligned}\tau_s &= 2\mu\mathbf{n}\cdot\mathbf{D}[\nabla\phi]\cdot\mathbf{e}_{\mathrm{s}},\\ u_{\mathrm{s}} &= \mathbf{u}\cdot\mathbf{e}_{\mathrm{s}}.\end{aligned} \tag{12.5.12}$$

Turning next to the inertial terms, we have

$$(\mathbf{u}\cdot\nabla)\,\mathbf{u} = \nabla\frac{|\mathbf{u}|^2}{2}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho}{2}|\mathbf{u}|^2\mathrm{d}\Omega &= \int_\Omega \rho\left(\mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\frac{|\mathbf{u}|^2}{2}\right)\mathrm{d}\Omega\\ &= \int_\Omega \rho\left[\frac{\partial\phi}{\partial x_{\mathrm{i}}}\frac{\partial}{\partial x_{\mathrm{i}}}\frac{\partial\phi}{\partial t}+\nabla\cdot\left(\mathbf{u}\frac{|\mathbf{u}|^2}{2}\right)\right]\mathrm{d}\Omega\\ &= \int_{S_{\mathrm{f}}}\rho u_{\mathrm{n}}\left(\frac{\partial\phi}{\partial t}+\frac{|\nabla\phi|^2}{2}\right)\mathrm{d}S_{\mathrm{f}}\end{aligned} \tag{12.5.13}$$

Collecting the results (12.5.9), (12.5.11), (12.5.12) and (12.5.13) we need to evaluate the energy equation (12.5.4) for (12.5.1) when $\mathbf{u}=\nabla\phi$; we find that

$$\int_{S_{\mathrm{f}}} u_{\mathrm{n}}\left[\rho\left(\frac{\partial\phi}{\partial t}+\frac{|\nabla\phi|^2}{2}+g\eta\right)+2\mu\frac{\partial^2\phi}{\partial n^2}+\gamma\nabla_{\mathrm{II}}\cdot\mathbf{n}\right]\mathrm{d}S_{\mathrm{f}} = -\int_{S_{\mathrm{f}}}\tau_{\mathrm{s}}u_{\mathrm{s}}\mathrm{d}S_{\mathrm{f}}. \tag{12.5.14}$$

Equation (12.5.14) was derived by Joseph; it is the energy equation for the irrotational flow of a viscous fluid. As in the case of the radial motion of a spherical gas bubble (6.0.8), the normal stress balance converts into the energy equation for viscous irrotational flow.

12.6 Viscous correction of viscous potential flow

The viscous pressure p_{v} for VCVPF can be defined by the equation

$$\int_{S_{\mathrm{f}}}(-p_{\mathrm{v}})u_{\mathrm{n}}\mathrm{d}S_{\mathrm{f}} = \int_{S_{\mathrm{f}}}\tau_{\mathrm{s}}u_{\mathrm{s}}\mathrm{d}S_{\mathrm{f}}. \tag{12.6.1}$$

This equation can be said to arise from the condition that the shear stress should vanish but it does not vanish in the irrotational approximation. Batchelor's idea, as reported by Moore (1963), is that the additional drag $4\pi a\mu U$ on a spherical gas-bubble needed to obtain the Levich drag $12\pi a\mu U$ from the value $8\pi a\mu U$, computed using the viscous normal stress, arises from a real viscous pressure in a thin boundary layer at the surface of the bubble.

Our interpretation of Batchelor's idea is to replace the unwanted shear stress term in the energy equation with an additional contribution to the normal stress in the form of a viscous pressure satisfying (12.6.1). An identical interpretation of Batchelor's idea, with a different implementation, has been proposed by Kang and Leal (1988b) and is discussed in §13.1.1.

Suppose now that there is such a pressure correction and Bernoulli equation

$$p_{\mathrm{i}}+\rho\left(\frac{\partial\phi}{\partial t}+\frac{|\nabla\phi|^2}{2}+g\eta\right)=C$$

holds. Since

$$C\int_S \mathbf{u}\cdot\mathbf{n}\mathrm{d}S = C\int_\Omega \nabla\cdot\mathbf{u}\mathrm{d}\Omega = 0$$

we obtain

$$\int_{S_{\mathrm{f}}} u_{\mathrm{n}}\left[-p_{\mathrm{i}}-p_{\mathrm{v}}+2\mu\frac{\partial^2\phi}{\partial n^2}+\gamma\nabla_{\mathrm{II}}\cdot\mathbf{n}\right]\mathrm{d}S_{\mathrm{f}}=0 \tag{12.6.2}$$

The normal stress balance for VCVPF is

$$-p_{\mathrm{i}}-p_{\mathrm{v}}+2\mu\frac{\partial^2\phi}{\partial n^2}+\gamma\nabla_{\mathrm{II}}\cdot\mathbf{n}=0 \tag{12.6.3}$$

Equation (12.6.3) is a working equation in the VCVPF theory only if p_v can be calculated. In the problems discussed in this book, p_v and a velocity correction $\mathbf{u}_v$ are coupled in a linear equation

$$\rho \frac{\partial \mathbf{u}_v}{\partial t} = -\nabla p_v + \mu \nabla^2 \mathbf{u}_v, \qquad \nabla \cdot \mathbf{u}_v = 0, \tag{12.6.4}$$

and since $\mathbf{u}_v$ is harmonic $\nabla^2 p_v = 0$ and p_v can be represented by a series of harmonics functions. In most of the problems in this book, orthogonality conditions show that all the terms in the series but one are zero and this one term is completely determined by (12.6.1). In other problems, such as the flow over bodies studied in Chapter 13, (12.6.1) determines the coefficient of the term in the harmonic series that enters into the direct calculation of the drag; the other coefficients remain undetermined.

Having obtained p_v it is necessary to consider the problem (12.6.4) for $\mathbf{u}_v$. This problem is overdetermined; (12.6.4) is a system of four equations for three unknowns. We must also satisfy the kinematic equation $\mathbf{u}_v = \partial \eta_v / \partial t$ and the normal stress boundary condition. Wang and Joseph (2006d) showed that a purely irrotational $\mathbf{u}_v$ and higher order corrections satisfaying (12.6.1) can be computed in the case in which the normal stress bpundary condition is not corrected. Their solution procedure leads to a series of purely irrotational velocities in powers of ν, or reciprocal powers of the Reynolds number (see §14.12). This same kind of special irrotational solution in reciprocal powers of the Reynolds number was constructed by Funada et al (2005) for capillary instability of a viscous fluid and by Wang et al (2005b) for a capillary instability of a viscoelastic fluid.

There is no reason to think that the higher order irrotational solutions generated by p_v have a hydrodynamical significance. In general, the solution of potential flow problems with pressure corrections can be expected to induce vorticity. This is the case for the nonlinear model of capillary collapse and rupture studied in §17.4.2. The flow from irrotational stagnation points of depletion to irrotational stagnation points of accumulation cannot occur without generating the vorticity given by (17.4.24) at leading order.

In the direct applications of the dissipation method, a pressure correction is not introduced; the power of the irrotational shear stress on the right side of (12.5.14) is computed directly. The energy equation for the system of VCVPF equations with p_v is the same as (12.5.14) when the power of p_v is replaced by the power of the shear stress using (12.6.1).

In the case of rising bubbles we may compute the same value for the drag in a direct computation in which the irrotational shear stress is neglected and p_v is included as we get from the dissipation method using (12.5.14) with $\tau_s \neq 0$ and $p_v = 0$.

The dissipation method requires that we work with the power generated by the shear $u_s \tau_s$ and not with a direct computation using p_v. The direct computation of drag with $p_v = 0$ and $\tau_s \neq 0$ given by potential flow leads to zero drag (13.1.8). The pressure correction allows us to do direct calculations without using the energy equations. The two methods lead to the same results by different routes. This feature of energy analysis is related to the fact that irrotational stresses are self-equilibrated (4.3.1) but do work.

The nature of the construction of VCVPF equations, which have the same energy equations as those used in the dissipation method, guarantees that the same results will be obtained by these two methods.

The VCVPF theory is perhaps artificial in some cases, like the decay of gravity waves in which no viscous pressure appears in the exact solution (see §14.1.3). However, in all cases, VCVPF leads to the same results as the dissipation method.

We have now arrived at the following classification of irrotational solutions:

IPF - inviscid potential flow

VPF - viscous potential flow

VCVPF - viscous correction of viscous potential flow

DM - dissipation method

Recognizing now that DM and VCVPF lead to the same results by different routes, even if p_v is not a true hydrodynamic pressure, we can introduce yet another acronym

VCVPF/DM

and

ES - exact solution.

VCVPF/DM and VPF are different irrotational solutions; they give rise to different results in nearly every instance except in the case of the limit of very high Reynolds numbers where they tend to the inviscid limit IPF. At finite Reynolds number VPF and VCVPF/DM can give significantly better results, closer to the exact solutions than IPF.

In the sequel, VCVPF means VCVPF/DM. We think that it is possible that either VPF or VCVPF give (in different situations which we presently cannot specify *a priori* with precision) the best possible purely irrotational approximations to the exact solution. The problem of the best purely irrotational approximation of real flows is related to the problem of the optimal specification of the irrotational flow in the Helmholtz decomposition studied in Chapter 4.