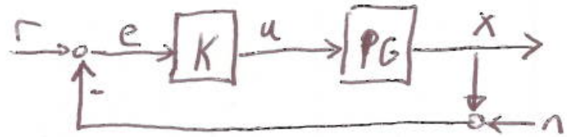
Loopshaping Design Example

$\dot{x} = 10u$

G: $G(s) = \frac{10}{s}$



Performance Specifications
- Closed-loop stability
- Loop crossover frequency near 1 rad/sec

- Tracking Performance:

$$|S(j0.01)| \leq 0.0004$$
$$|S(j0.1)| \leq 0.04$$
$$|S(j0.2)| \leq 0.2$$

S(s) is the transfer function from $r \to e$. These bounds ensure that the error amplitude is small if r is a low frequency sinusoid.

- Noise Rejection:

$$|T(j100)| \leq 0.0004$$
$$|T(j10)| \leq 0.04$$
$$|T(j3)| \leq 0.3$$

T(s) is the transfer function from $-n \to x$. These bounds ensure that high frequency noise has small impact on the output.

We'll use loopshaping to design a controller that meets these specifications. Note that we're starting from frequency domain specifications on S and T. Part of the design work is to actually figure out what specifications on S and T are reasonable. You'll have some practice doing this on the homework.

Step 1 Translate specs on S and T into specs on L the loop transfer function $L = GK$.

a) Bounds on $S(s) = \frac{1}{1+G(s)K(s)} = \frac{1}{1+L(s)}$

If we have an upper bound: $|S(j\omega)| = \left|\frac{1}{1+L(j\omega)}\right| \leq b$

Then we need: (this is equivalent) $|1+L(j\omega)| \geq 1/b$

If $|L(j\omega)| \geq \frac{1}{b}+1$ then $|1+L(j\omega)| \geq |L(j\omega)| - 1 \geq 1/b$

This follows from the triangle inequality: $|a+b| \leq |a|+|b|$

Specifically

$$|L(j\omega)| = |L(j\omega)+1-1| \leq |L(j\omega)+1| + \underbrace{|-1|}_{=1}$$

$$\Rightarrow |L(j\omega)| - 1 \leq |L(j\omega)+1|$$

$$\boxed{|L(j\omega)| \geq 1 + 1/b \Rightarrow |S(j\omega)| \leq b}$$

ω	Upper Bound on $\lvert S(j\omega)\rvert$	Lower Bound on $\lvert L(j\omega)\rvert$
0.01	0.0004	2501
0.1	0.04	26
0.2	0.2	6

← If the bound on $|S(j\omega)|$ is $b \ll 1$ then the bound on $|L(j\omega)|$ is $\approx 1/b$.

Lower Bounds on $L(j\omega)$ in dB
67.96
28.3
15.6

b) Bounds on $T(s) = \dfrac{G(s)K(s)}{1+G(s)K(s)} = \dfrac{L(s)}{1+L(s)}$

If we have an upper bound: $|T(j\omega)| = \left|\dfrac{L(j\omega)}{1+L(j\omega)}\right| \leq b$

Then an equivalent bound on L is:

$$\frac{1}{b} \leq \left|\frac{1+L(j\omega)}{L(j\omega)}\right| = \left|1+\frac{1}{L(j\omega)}\right|$$

If $\left|\frac{1}{L(j\omega)}\right| \geq 1/b + 1$ then $\left|1+\frac{1}{L(j\omega)}\right| \geq \left|\frac{1}{L(j\omega)}\right| - 1 \geq 1/b$

Again use the triangle inequality as on p185.

$$\therefore \quad \left|\frac{1}{L(j\omega)}\right| \geq \frac{b+1}{b} \Rightarrow |T(j\omega)| \leq b$$

$$\Rightarrow \boxed{|L(j\omega)| \leq \frac{b}{b+1} \Rightarrow |T(j\omega)| \leq b}$$

ω	Upper Bound on $\lvert T(j\omega)\rvert$	Upper Bound on $\lvert L(j\omega)\rvert$	Upper Bound on $\lvert L(j\omega)\rvert$ in dB
3	0.3	0.23	−12.7
10	0.04	0.0385	−28.3
100	0.0004	0.0004	−67.96

If the bound on $|T(j\omega)|$ is $b \ll 1$ then the bound on $|L(j\omega)|$ is $\approx b$.

Step 2 Use the various design "stages" to get a controller that satisfies the requirements on $|L(j\omega)|$.

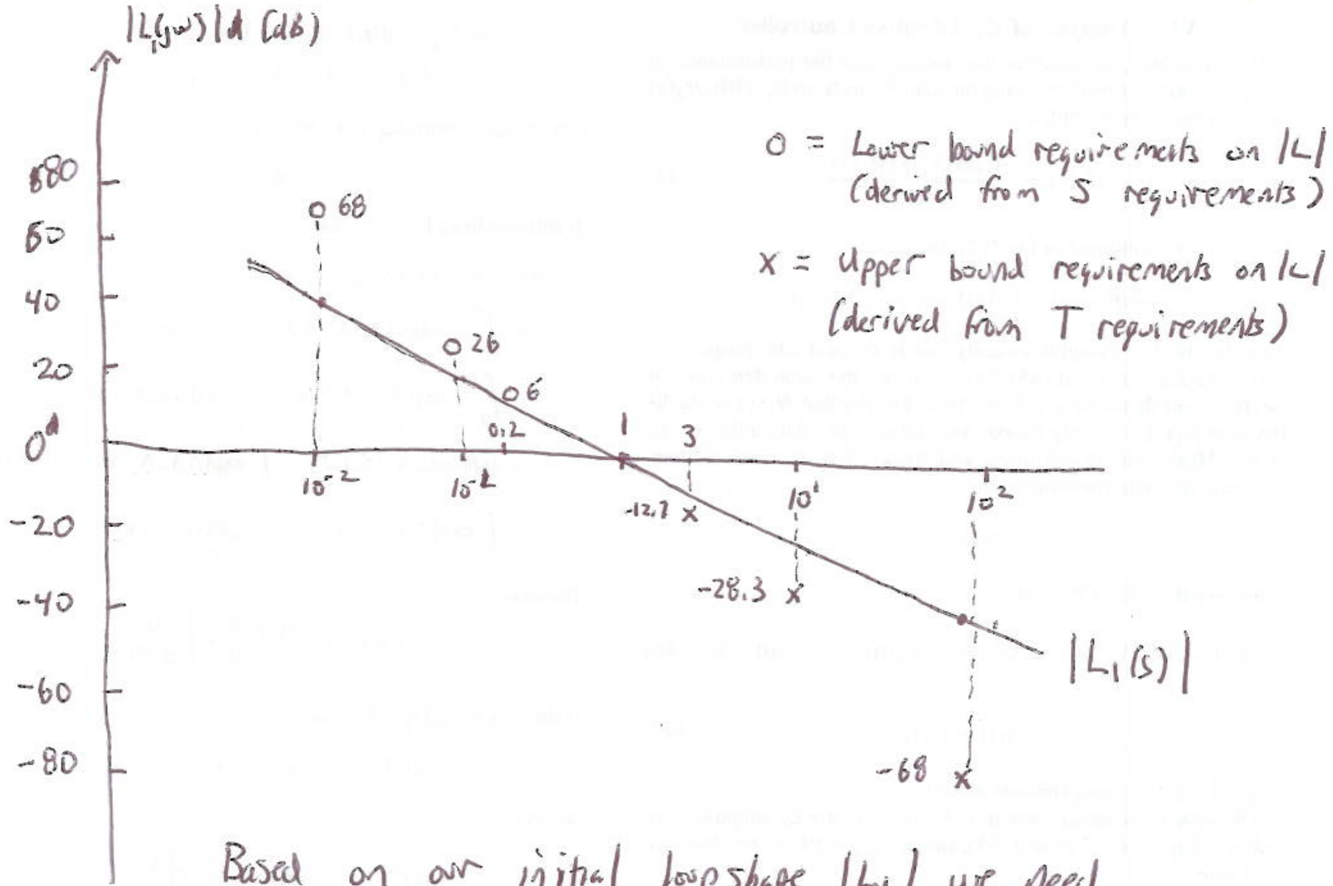
a) $G(s) = 10/s$

First use a proportional gain $K_1(s) = \beta$ so that the loop $G(s)K_1(s)$ has the desired cross-over frequency of $\omega_c = 1$ rad/sec.

$$1 = |G(j\omega_c) K_1(j\omega_c)| = \frac{10}{\omega_c} \cdot \beta \quad \rightarrow \quad \beta = \frac{\omega_c}{10} = 1/10$$

$\therefore$ Choose $K_1(s) = 1/10$

Initial Loop shape $\Rightarrow L_1(s) = G(s)K_1(s) = 1/s$



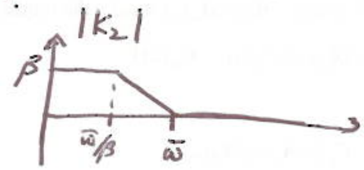
Based on our initial loopshape $|L_1|$ we need to increase the low-frequency gain and reduce the high frequency gain.

Important Rule of Thumb

b) We can use either a low frequency boost or integral boost stage to increase the low frequency gain.

We'll use a low-frequency boost:

$$K_2(s) = \frac{s+\bar{\omega}}{s+\bar{\omega}/\beta}$$



- At $\omega = 0.2$ rad/sec our loopshape has gain $|L_1(j0.2)| = \frac{1}{|j0.2|} = 5$ and the lower bound requirement is $|L(j0.2)| \geq 6$.

Thus we need the boost to at least slightly increase the gain at $\omega = 0.2$. Thus we should pick $\bar{\omega} > 0.2$.

- At $\omega = 10^{-2}$, our loopshape has gain $|L_1(0.01)| = \frac{1}{0.01} = 100$.

The lower bound requirement is $|L(j0.01)| \geq 2501$.

Thus $K_2(s)$ needs to increase the low frequency gain by $\frac{2501}{100} \approx 25$

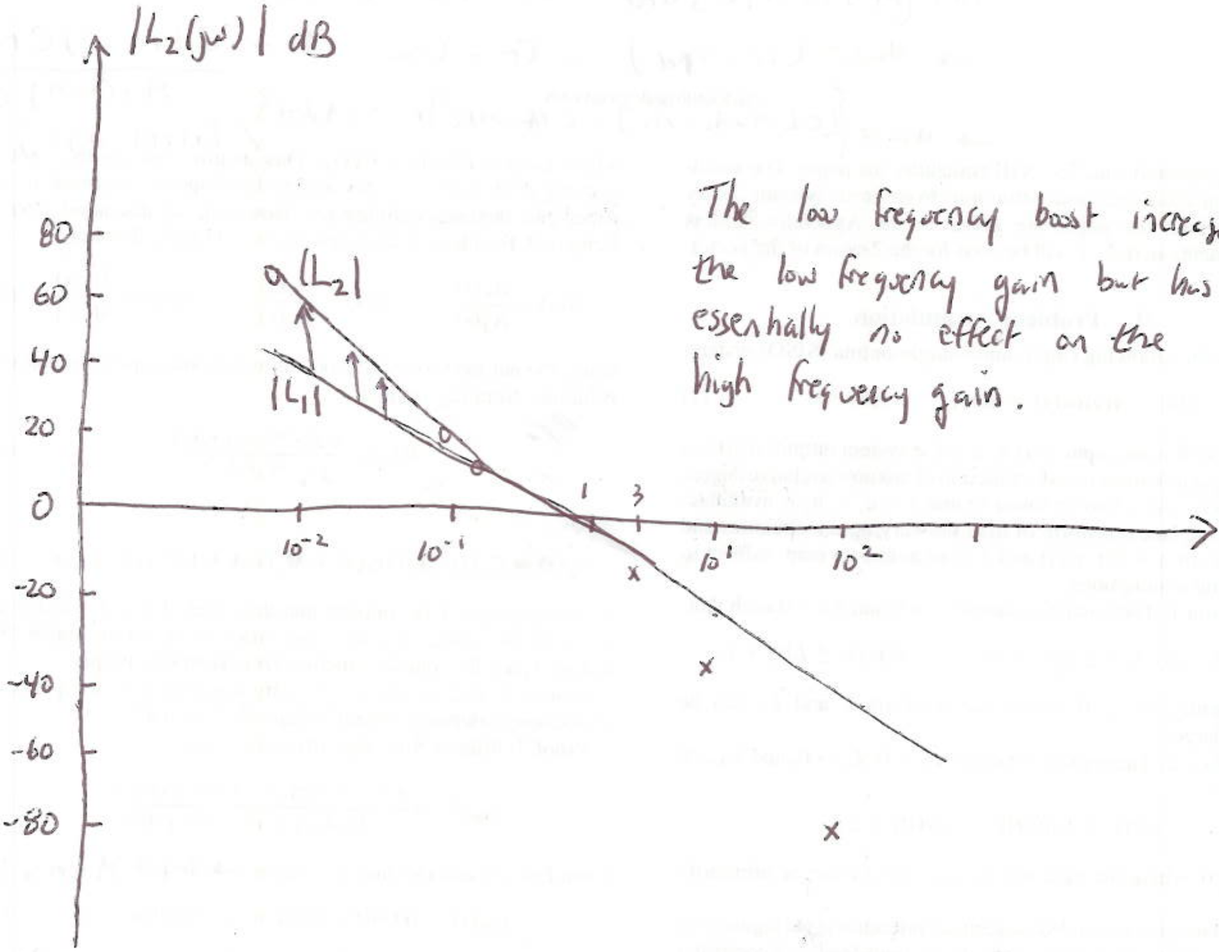
[illegible]

Our straight-line Bode plots are only rough approximations. I could give you precise formulas for picking $\bar{\omega}$ and β given the specifications above but I think it is easier to iterate with the values. After a few tries I found the following values are very close to satisfying the requirements

$$\bar{\omega} = 0.3 \quad \text{and} \quad \beta = 50$$

Our controller is now $K_1(s)K_2(s) = \frac{1}{10}\,\frac{s+0.3}{s+0.3/50}$

and the second loopshape is $L_2(s) = G(s)[K_1(s)K_2(s)]$



The low frequency boost increases the low frequency gain but has essentially no effect on the high frequency gain.

c) Next we can use a high frequency roll-off to decrease the loop gain at high ~~frequency~~ frequencies so that we are under the upper bound specifications.

At $\omega = 3$ rad/sec, our second loop-shape has $|L_2(j3)| = 0.32$ and the upper bound requirement is $|L_2(j3)| \leq 0.23$

Our high-frequency roll-off has the form

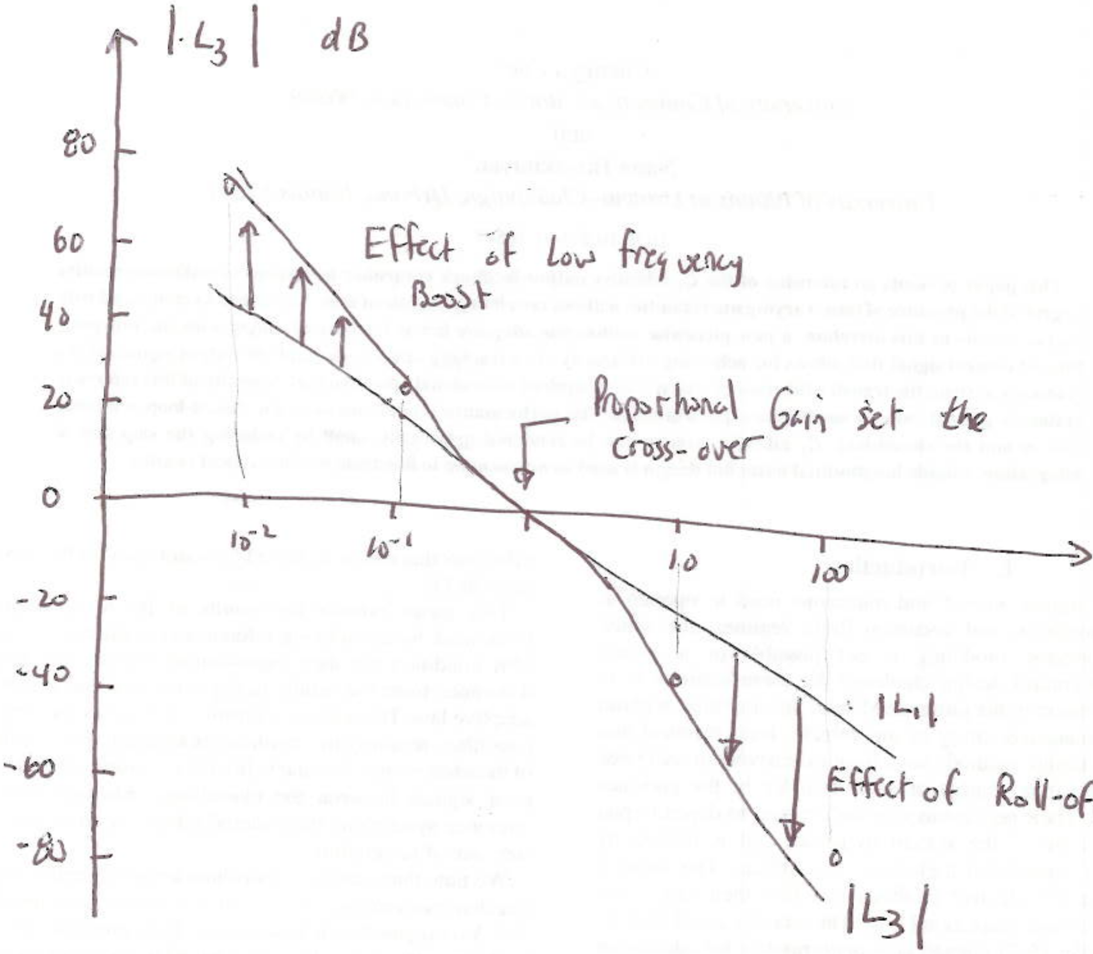
$K_3(s) = \dfrac{\bar{\omega}}{s+\bar{\omega}}$

Since we need to decrease the gain at $\omega = 3$ rad/sec we must choose $\bar{\omega} \leq 3$ rad/sec. After some trial and error, $\bar{\omega} = 2.75$ rad/sec ~~rea~~ satisfies all design requirements.

Our controller is now $K(s) = K_1(s)\, K_2(s)\, K_3(s)$

$$\Rightarrow K(s) = \frac{1}{10}\,\frac{s+0.3}{s+0.3/50}\,\frac{2.75}{s+2.75}$$

The third loop shape is $L_3(s) = G(s)\left[K_1(s)\, K_2(s)\, K_3(s)\right]$



Note

1) L_3 will have a cross-over slightly different than $1^{rad}/sec$ due to the effect of K_2 and K_3. We can use one more proportional gain stage if we want ω_c exactly $= 1^{rad}/sec$

2) $K(s)$ is a 2^{nd} order transfer function with one zero. You can convert this back to a time-domain ODE. It won't be as easy to interpret this ODE as our previous PI + PD control laws.

3) This controller stabilizes $G(s)$ [check this]. What if it didn't? Next we'll introduce tools to understand the effect of loopshaping on stability.