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PRELIMINARY INVESTIGATION OF MESH STABILITY FOR LINEAR SYSTEMS

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ABSTRACT

Damping of disturbances as they propagate through a chain of interconnected systems, termed string stability, has been the subject of significant research. In this paper, we investigate mesh stability, which is the two-dimensional extension of string stability. We review the key results used for string stability analysis and then generalize the conditions for MIMO systems. These results are then applied to a simple class of linear systems which form a mesh in two-dimensions. It is shown (as in the one-dimensional case) that communicating the velocity and acceleration of the lead vehicle to all subsystems is sufficient for mesh stability. This result is then verified by simulation.

INTRODUCTION

The focus of this paper is the mesh stability of a class of decentralized systems. A decentralized system is the union of a countable number of interacting dynamical subsystems. Every subsystem in the decentralized system makes individual control decisions and manipulates its inputs so that a collective task can be completed. For example, a formation of unmanned helicopters performing a combat mission is a decentralized dynamical system.

In a large coordinated system such as this, there is the possibility for error "waves" to amplify as they propagate through the mesh (i.e. the system). We can understand this propagation effect by considering the vehicle following case (Hedrick, 1993), (Swaroop, 1994). Each vehicle in the chain is coupled to it's predecessor via the feedback control law. Specifically, a radar on each vehicle measures the position relative to it's predecessor and uses this information for control. Thus, a disturbance acting on the lead vehicle will propagate through the chain and effect the performance of the following vehicles. It is possible that this error wave may amplify as it propagates down the chain if the control law is improperly designed. If the chain is long enough, this error amplification might cause a vehicle collision. Systems in which the error does not amplify are termed *string stable*.

In an infinite chain, the system is unstable if the error amplifies at each point in the chain. However, a system which consists of a finite number of connected subsystems could be stable in the classical (Lyapunov) sense and yet string unstable. Since the string is finite, errors never "blow up" as they propagate and the overall system can still be classically stable. Practically, we must still make the system string stable to prevent actuator saturation and/or error growth which may result in vehicle collisions. This wave damping characteristic is independent of classical system stability and is needed to prevent errors present at one point from disrupting the operation at another point in the chain.

Mesh stability is the two dimensional extension of string stability. In this paper, we will use a grid formation of linear point masses to investigate mesh stability. This formulation is suitable for initial analysis due to its simple structure and will allow us to obtain intuition about the multi-dimensional problem. Specifically, we will determine if control laws using various information structures (e.g. with and without reference vehicle information) are mesh stable. This paper will have the following structure: First, we will review some useful results in the analysis of string stability. Next, we will outline the problem formulation. Then we will compare the mesh stability characteristics of two sliding controllers. In the final section, we will present simulation results which confirm our analysis.

BACKGROUND

Connective stability in one dimension is called string stability and has been studied by (Chu, 1974), (Eyre, 1998), (Hedrick, 1993), (Swaroop, 1994), and (Swaroop 1996). For string stability, we would like the maximum spacing error to decrease as it propagates down the chain. We will use the following norm definitions: $||f(\cdot)||_\infty = \sup_{t\geq 0}|f(t)|$ and $||f(\cdot)||_1 = \int_0^\infty |f(\tau)|d\tau$. If ε_i and ε_{i+1} are the errors at the i^{th} and $i+1^{th}$ point in the chain, then we need $||\varepsilon_{i+1}||_\infty \leq ||\varepsilon_i||_\infty$ for string stability.

From linear system theory (Desoer, 1975), if $y = h * u$, then we have the following relationship:

$$||y(t)||_\infty \leq ||h(t)||_1 \, ||u(t)||_\infty \tag{1}$$

Using a sliding control law, Hedrick and Swaroop (Hedrick, 1993) found an LTI convolution kernel, $h(t)$, which relates the errors in a vehicle following chain by: $\varepsilon_{i+1} = h * \varepsilon_i$. Thus string stability of the chain of vehicles can be determined by analyzing the one-norm of the error propagation impulse response, h(t). Since this norm represents the maximum amplification of any error as it propagates down the chain, it provides a useful metric for string stability. If this norm is less than one, then all input errors will be attenuated (in the ∞-norm sense) as they propagate down the chain. If this norm is greater than one, then the system is string unstable and there exists an input error which will be amplified as it propagates.

If $h(t)$ does not change sign, the string stability condition, $||h(t)||_1 \leq 1$, can be equivalently satisfied in the frequency domain if the magnitude of the associated transfer function, $H(j\omega)$, is less than one at all frequencies, i.e. $||H(j\omega)||_\infty \leq 1$. In (Hedrick, 1993), they found that the sliding control law resulted in a string stable system if reference vehicle information was used.

The SISO input/output norm results are easily generalized to the MIMO case. Let $f : \mathbf{R}_+ \to \mathbf{R}^n$ and define $||f(\cdot)||_\infty = \max_i \sup_{t\geq 0}|f_i(t)|$. If $h(t)$ is the convolution kernel for an n-input, n-output MIMO system, and $y = h * u$, then the input-output relationship is given by (Desoer, 1975):

$$||y(t)||_\infty \leq \left(\max_i \sum_{j=1}^{n} ||h_{ij}(t)||_1\right) \cdot ||u(t)||_\infty \tag{2}$$

This can also be related to an equivalent frequency domain condition if none of the entries of the convolution kernel changes sign. Let $H(j\omega)$ be the $n \times n$ transfer function matrix for the LTI system given by $h(t)$. If none of the $h_{ij}(t)$ change sign, then:

$$||y(t)||_\infty \leq \left(\max_i \sum_{j=1}^{n} ||H_{ij}(j\omega)||_\infty\right) \cdot ||u(t)||_\infty \tag{3}$$

As an extension to the string problem mentioned previously, we can consider the leader following problem in two dimensions. This example has practical application to a chain of Mobile Offshore Bases moving at sea (Hedrick, 1998). We consider a generic 3 DOF double integrator model, which could be a nonlinear dynamical system after feedback linearization or a linear system with decoupled states: $\ddot{\eta} = u$. Here $\eta = [X;\ Y;\ \Psi]^T$, where (X,Y) is the position in the plane and Ψ is the heading. If we follow a procedure similar to the vehicle following case (Hedrick, 1993), the transfer function matrix relating errors in the chain is given by: $\varepsilon_{i+1}(s) = H(s)\varepsilon_i(s)$ where $H(s) = diag(H_{11}(s), H_{22}(s), H_{33}(s))$. We can apply the linear MIMO input-output result to determine whether or not this chain is string stable. In this case, the result is trivial since the states have been decoupled and the row sum condition for string stability reduces to $||H_{ii}(s)||_\infty \leq 1,\ i = 1,2,3$. For string stability, the control law must be designed so that each direction (X, Y, Ψ) is independently string stable. As in the vehicle following scenario, the chain satisfies these conditions for string stability if reference vehicle information is communicated to each follower.

PROBLEM FORMULATION

Symbolically, the simple class of meshed systems to be discussed in this paper are as follows:

$$\dot{x}_{i,j} = f_{i,j}(x_{i,j};\ x_{i,j-1};\ x_{i-1,j};\ x_{1,1}) \tag{4}$$

where $x_{i,j} \in \mathbf{R}^n\ \forall i,j \in \mathbf{N}$. Also, $x_{i,j}(t) \equiv x_{i,1}(t)$ if $j \leq 1$ and $x_{i,j}(t) \equiv x_{1,j}(t)$ if $i \leq 1$ (these are mesh boundary conditions). In this scenario, each $x_{i,j}(t)$ could represent the state vector of a single helicopter moving in two dimensions. Each subsystem, $x_{i,j}(t)$, is dynamically linked to its predecessor along the rows and columns of the mesh and possibly to the reference or leader subsystem, $x_{1,1}(t)$. Figure 1 schematically shows the interconnections of the mesh when the subsystems do not use the additional reference vehicle information.

Assume that the helicopters are point masses with motions restricted to a plane, so the state vector is $x_{i,j} = [y_{i,j};\ z_{i,j};\ \dot{y}_{i,j};\ \dot{z}_{i,j}]^T$, where we have used the Y-Z coordinates defined in Figure 1. To simplify the notation, define the position, velocity and acceleration vectors as: $p_{i,j} = [y_{i,j};\ z_{i,j}]^T$; $v_{i,j} = \dot{p}_{i,j}$;

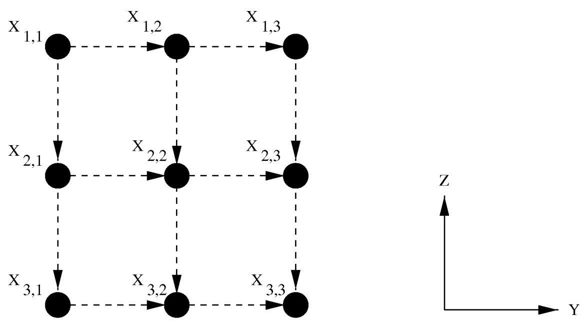

Figure 1. MESH SCHEMATIC

$a_{i,j} = \dot{v}_{i,j}$. We will assume that the helicopter models are simple double integrators: $\ddot{p}_{i,j} = a_{i,j} = u_{i,j}$ where $u_{i,j} \in \mathbf{R}^2$ is the input helicopter acceleration.

We will also find it useful to define the following mesh spacing errors:

$$\varepsilon_{i,j} = \delta_{1,des} - (p_{i,j-1} - p_{i,j}) \quad (5)$$

$$\gamma_{i,j} = \delta_{2,des} - (p_{i-1,j} - p_{i,j}) \quad (6)$$

Figure 2 shows the specific error vectors with $i = j = 2$. For simplicity, only one of the position vectors, $p_{2,1}$, is shown. The $\varepsilon_{i,j}$'s are the errors with respect to the 'left' neighbor along the row and the $\gamma_{i,j}$'s are the errors with respect to the 'above' neighbor along the column. These errors are each vectors in $\mathbf{R}^2$, as depicted in Figure 2. Also, $\delta_{1,des}$ and $\delta_{2,des}$ are the desired spacing vectors. To obtain the spatial arrangement shown in the schematic (Figure 1), we will define $\delta_{1,des} = [-L;\ 0]^T$ and $\delta_{2,des} = [0;\ L]^T$ for all i,j. Notice however, that the formation is not required to have this rigid grid shape. By properly choosing $\delta_{1,des}$ and $\delta_{2,des}$ at each i, j we can obtain a myriad of spatial arrangements while maintaining the subsystem dependencies.

MESH STABILITY ANALYSIS

In this section, we will derive a sliding control law so that each subsystem maintains its relative position in the mesh. Mesh movement can be coordinated by commanding only the leader helicopter to change position. First define the following 2x1 vector sliding function:

$$S_{i,j} = (\dot{\varepsilon}_{i,j} + \dot{\gamma}_{i,j}) + q_1 \cdot (\varepsilon_{i,j} + \gamma_{i,j}) \quad (7)$$

We will find it useful to define a composite error vector, $e_{i,j} = \varepsilon_{i,j} + \gamma_{i,j}$, at each point in the mesh. Using this composite error, the vector sliding function has the standard appearance: $S_{i,j} = \dot{e}_{i,j} + q_1 \cdot e_{i,j}$. For subsystems on the boundaries of the mesh, one of the two error terms ($\varepsilon_{i,j}$ or $\gamma_{i,j}$) will be identically

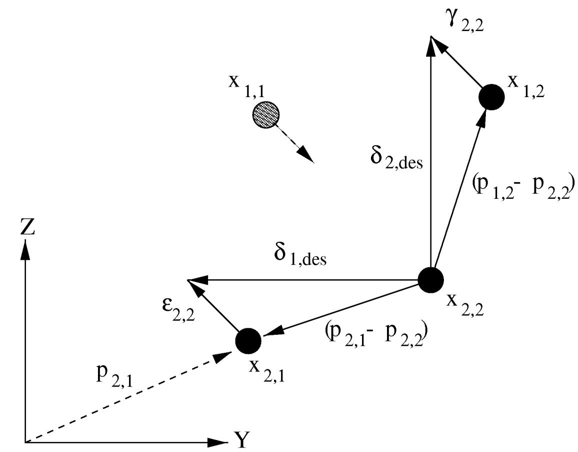

Figure 2. ERROR DEFINITIONS

zero. The feedback controller for a subsystem on the boundary receives only half the amount of information. As a result, the sliding function errors (and hence the control effort) for boundary subsystems are roughly half those of internal subsystems. To normalize the control effort, the composite error for boundary subsystems is defined to be $e_{i,j} = 2\varepsilon_{i,j}$ or $e_{i,j} = 2\gamma_{i,j}$, as appropriate.

Figure 2 shows a typical mesh configuration when the leader, $x_{1,1}$, accelerates at -45^o angle into the mesh. $x_{1,2}$ and $x_{2,1}$ react to this movement before $x_{2,2}$, which results in the error vectors $\varepsilon_{2,2}$ and $\gamma_{2,2}$. The $\varepsilon_{2,2}$ and $\gamma_{2,2}$ vectors are equal and point at $+45^o$. The equality of $\varepsilon_{2,2}$ and $\gamma_{2,2}$ is due to the equal dynamics of each subsystem and is independent of the leader movement direction. Also, these vectors tend to point in the opposite direction of the initial leader movement. By the same general argument we can expect that at each point in the mesh that $\gamma_{i,j} = \varepsilon_{i,j}$, so $e_{i,j} = 2\gamma_{i,j} = 2\varepsilon_{i,j}$ is a good measure of mesh deformation.

As in the standard sliding control formulation, we would like the control input to force $S_{i,j}$ to converge to 0. If $S_{i,j} = 0$, then by the arguments above, we may assume that $\varepsilon_{i,j} = \gamma_{i,j} = 0$ and the mesh is in the desired formation. The following control input will force $S_{i,j}$ to a boundary layer of zero:

$$\bar{u}_{i,j} = \frac{1}{2} \cdot [-K \cdot S_{i,j} - q_1 \cdot \dot{e}_{i,j} + (a_{i,j-1} + a_{i-1,j})] \quad (8)$$

where K and q_1 are greater than zero for classic stability. The overbar is used to indicate that this is the desired helicopter acceleration along the Y and Z directions. We will assume that the actual vehicle acceleration is delayed due to processing and actuator dynamics modeled by the first order system:

$$(\tau \cdot I_{2x2})\dot{u}_{i,j} + u_{i,j} = \bar{u}_{i,j} \quad (9)$$

If we add Equations 5 and 6 and differentiate twice, we obtain the relation:

$$u_{i,j} = 0.5 \cdot (\ddot{e}_{i,j} + a_{i,j-1} + a_{i-1,j}) \quad (10)$$

After substituting Equations 8 and 10 into Equation 9 and rearranging terms, we obtain a differential equation which relates $\varepsilon_{i,j}$ to $\varepsilon_{i-1,j}$ and $\varepsilon_{i,j-1}$. After taking Laplace transforms, we get:

$$E_{i,j}(s) = \begin{bmatrix} H(s) & 0 \\ 0 & H(s) \end{bmatrix} \left[\frac{E_{i-1,j}(s) + E_{i,j-1}(s)}{2} \right] \quad (11)$$

with:

$$H(s) = \frac{s^2 + (q_1 + K)s + q_1 K}{\tau s^3 + s^2 + (q_1 + K)s + q_1 K} \quad (12)$$

$E_{i,j}(s)$ is the Laplace transform (element-wise) of $e_{i,j}(t)$, hence it is a 2x1 column vector. The same is true of $E_{i-1,j}(s)$ and $E_{i,j-1}(s)$, resulting in a 2x2 propagation matrix.

This formulation is a direct extension of the work by Hedrick and Swaroop in (Hedrick, 1993). It is not surprising that this H(s) has the exact form of the transfer function relation in their vehicle following chain relation (when no leader information is used), $\varepsilon_{i+1} = H(s)\varepsilon_i$. Hedrick and Swaroop showed that this transfer function has magnitude greater than one at sufficiently small frequencies. Applying the MIMO norm condition (Equation 2) is again trivial since the states are decoupled. Since $||h(t)||_1 \geq ||H(j\omega)||_\infty > 1$ for small ω, we expect that $||e_{2,2}||_\infty$ will be greater than the average of $||e_{1,2}||_\infty$ and $||e_{2,1}||_\infty$. In general, maximum errors will, on the average, grow as they propagate through the mesh. We will see in the Results section that symmetry of the problem will make the error amplification monotonic as a function of $i + j$.

As an alternative, we can try the following sliding function:

$$S_{i,j} = \dot{e}_{i,j} + q_1 \cdot e_{i,j} + q_2 \cdot (v_{i,j} - v_{1,1}) \quad (13)$$

Notice that this function differs from Equation 7 by the addition of the last term. This term ties each subsystem to the leader, which acts as a reference for the mesh. In the one-dimensional case, the addition of this reference term was found to be sufficient for string stability (Hedrick, 1993).

If we repeat the analysis, we again obtain the error propagation relation given in Equation 11, but with the following transfer function:

$$H(s) = \frac{s^2 + (q_1 + K)s + q_1 K}{(1 + q_2)\tau s^3 + (1 + q_2)s^2 + (q_1 + K + q_2 K)s + q_1 K} \quad (14)$$

Again, this H(s) has the same form as found by Hedrick and Swaroop when leader information is used. They showed that $h(t) = L^{-1}\{H(s)\}$ does not change sign for small values of τ, thus $||h(t)||_1 = ||H(j\omega)||_\infty$. Since $||H(j\omega)||_\infty = 1$, maximum errors cannot amplify as they propagate through the mesh. The addition of leader information in the control law has resulted in mesh stability. However, mesh stability is not obtained for free since leader information must be communicated to each vehicle in the mesh.

RESULTS

In this section we will verify our results by simulating a 4x4 mesh with the two control laws derived above. Each subsystem is a double integrator point mass with processing and actuator delays modeled by first order systems. The benefit of the decentralized control structure is that we can move the entire mesh from one point to another by commanding only the leader to change position. All other subsystems in the mesh will try to maintain their relative position in the mesh. The leader will execute a trapezoidal acceleration profile, shown in Figure 3, in the y-direction. The acceleration profile in the z-direction will be the negative of the profile in the figure. These acceleration traces cause the mesh leader to move from the origin and come to rest at the point (147,-147).

Tables 1 and 2 show $||e_{i,j}||_\infty$ for the controllers with and without leader information, respectively. Table 1 shows the error amplification which we expect. Down any column, row, or combination thereof, the maximum error amplifies in the mesh. However, we can see in Table 2 that the inclusion of reference vehicle information damps out these propagating error waves.

Another notable fact is that the tables are completely symmetrical. As previously discussed, this is due to the equality of the subsystem dynamics. Therefore, "level sets" of performance occur, i.e. the metric at each i, j position is only a function of $i + j$. We clearly expect the performance of $e_{1,2}$ and $e_{2,1}$ to be equal due to symmetry. Inductively, the same holds for all subsystems along the boundary; the response of $e_{1,i}$ is the same as $e_{i,1}$. In particular, $E_{3,1}(s) = E_{1,3}(s) = diag(H(s), H(s)) \cdot E_{1,2}$. We also have that $E_{2,2}(s) = diag(H(s), H(s)) \cdot [E_{1,2}(s) + E_{2,1}(s)]/2$. Since $E_{1,2}(s) = E_{2,1}(s)$, we get that $E_{2,2}(s) = E_{3,1}(s) = E_{1,3}(s)$, which is one of the level sets. All other level sets can be justified in the same fashion.

It is interesting to see how this error damping characteristic translates to the time responses. First notice that the symmetry of Tables 1 and 2 also translates into the time domain. Thus we can examine all possible error responses in the mesh by examining only one response per level set, e.g. the responses along the first column and along the bottom row. In Figures 4 and 5, we plot the first element of the $\gamma_{i,j}$ vectors for several of these responses. Figure 4 confirms that errors amplify in the mesh when leader information is not communicated to each member of the mesh.

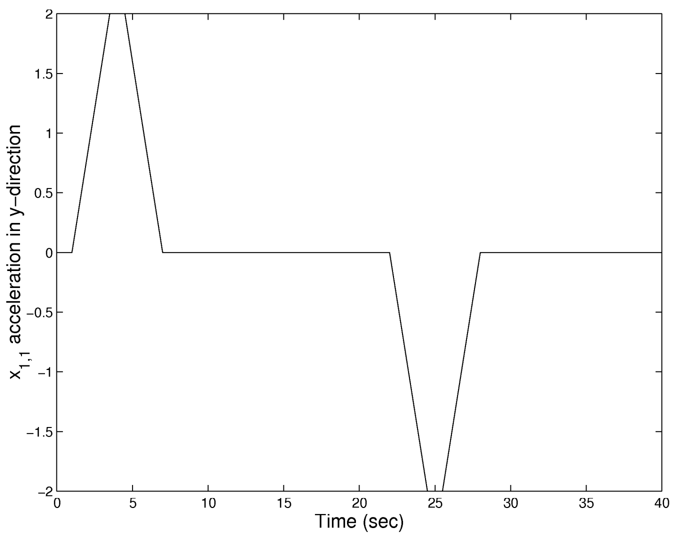

Figure 3. $x_{1,1}$ ACCELERATION PROFILE IN Y DIRECTION

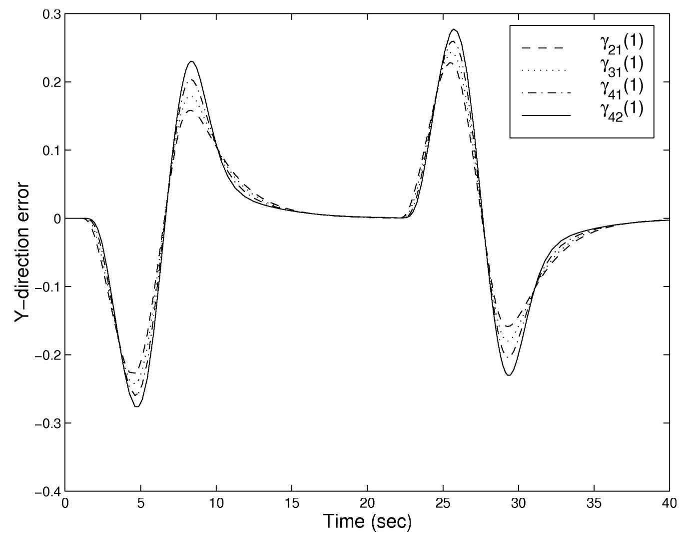

Figure 4. $\gamma_{i,j}(1)$ RESPONSES USING MESH UNSTABLE CONTROLLER

Table 1. $||e_{i,j}||_\infty$ WITHOUT LEADER INFORMATION

$\|\|e_{i,j}\|\|_\infty$	j=1	2	3	4
i= 1	n/a	0.4567	0.4869	0.5193
2	0.4567	0.4869	0.5193	0.5550
3	0.4869	0.5193	0.5550	0.5934
4	0.5193	0.5550	0.5934	0.6363

Table 2. $||e_{i,j}||_\infty$ WITH LEADER INFORMATION

$\|\|e_{i,j}\|\|_\infty$	j=1	2	3	4
i= 1	n/a	0.5822	0.5269	0.4768
2	0.5822	0.5269	0.4768	0.4350
3	0.5269	0.4768	0.4350	0.3977
4	0.4768	0.4350	0.3977	0.3588

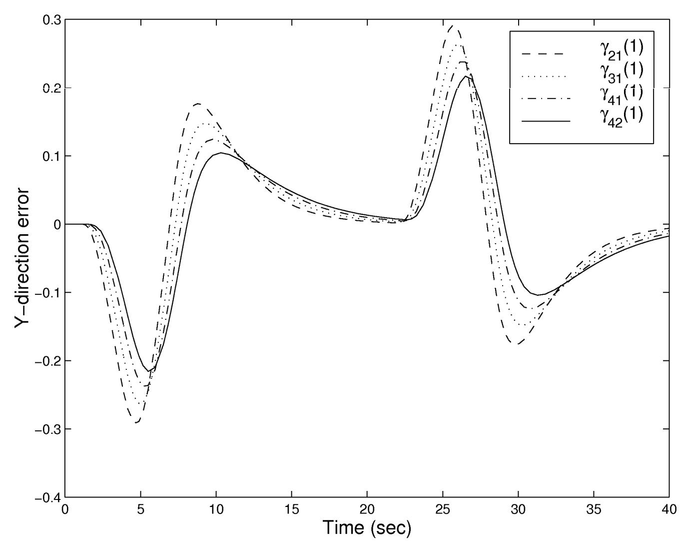

Figure 5. $\gamma_{i,j}(1)$ RESPONSES USING MESH STABLE CONTROLLER

Figure 5 shows that the errors are damped when this information is communicated. The responses for all other errors (the second element of $\gamma_{i,j}$ and the two elements of $\varepsilon_{i,j}$) are similar.

CONCLUSIONS

In this paper we investigated the mesh stability of a simple class of interconnected linear systems with a decentralized control structure. The benefit of this structure is that mesh movement could be coordinated by simply communicating a new desired position to the lead vehicle. We showed that key string stability results, which is the one dimensional analog, could be easily extended to this problem. Thus mesh stability can be ensured by using reference vehicle information in each decentralized controller. The cost of this controller information structure is that lead vehicle velocity and acceleration information must be communicated to each vehicle in the mesh. These results can easily be extended to the 3 dimensional case using a similar analysis.

ACKNOWLEDGMENT

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