CDC01-REG1221

Worst-case performance analysis with constrained uncertainty

Pete Seiler* and Andrew K. Packard
Department of Mechanical Engineering
University of California at Berkeley
Email: pseiler@newton.me.berkeley.edu
pack@me.berkeley.edu

Abstract

In this paper, we examine the worst-case performance of a linear system with real parametric uncertainty. In particular, we will analyze the worst-case gain from disturbances to errors of a system subjected to 2 real, scalar uncertainties. The 2 scalar uncertainties are typically normalized so that they have absolute value less than or equal to one. In the parameter space, this constrains the uncertainties to lie in the unit cube. The contribution of this paper is that we also assume that the 2 scalar parameters are correlated. This correlation is represented by an additional offset rectangle constraint in the parameter space. The motivation for this problem is to use our knowledge of parameter correlation to remove some of the conservativeness in the standard performance analysis.

1 Introduction

In this paper, we examine the worst-case performance of a linear system with real parametric uncertainty. In particular, we will analyze the worst-case gain from disturbances to errors of a system subjected to 2 real, scalar uncertainties. We will focus on the constant matrix problem. However, this can be extended to analysis of linear, dynamic systems by analyzing the transfer function frozen at a finite, but densely gridded, number of frequencies. In this problem, the 2 scalar uncertainties are typically normalized so that they have absolute value less than or equal to one. In the parameter space, this constrains the uncertainties to lie in the unit cube. The contribution of this paper is that we also assume that the 2 scalar parameters are correlated. This correlation is represented by an additional offset rectangle constraint in the parameter space. The motivation for this problem is to use our knowledge of parameter correlation to remove some of the conservativeness in the standard performance analysis.

If the two scalar uncertainties are only restricted to lie in the unit cube, this analysis can be accomplished, in theory, by bisection involving a set of structured singular value (μ) problems or by directly solving a skewed-μ problem. We refer the unfamiliar reader to [5] for a tutorial exposition of the structured singular value. Since computing μ is NP hard [2], computationally tractable upper/lower bounds are used. Specifically, the μ upper bound scaled for worst-case performance is simply a linear matrix inequality (LMI) obtained by applying the $\mathcal{S}$-procedure to the skewed-μ problem.

When additional constraints are placed on the scalar uncertainties, the analysis becomes more complicated. Previous work on analysis of μ with additional linear constraints in the parameter space is given in [3]. Khatri gives three approaches for computing upper bounds when the parameters are constrained to lie in the intersection of a wedge, i.e. between two hyperplanes, and the unit box. The first is a generalization of a spherical-μ upper bound which approximates the wedge constraints with a highly eccentric ellipsoid. The second involves a change of variables in the parameter space leading to a standard μ problem which upper bounds the constrained problem. The third uses constraints on the signals in the uncertain system which implicitly enforce the constraints in the parameter space. The upper bounds appear to be good for the case where the system matrix is low rank, but the methods apparently give poor bounds for general matrices.

The key idea given by Khatri is that constraints in the parameter space can be converted into constraints in the signal space. It is this idea which is applied in this paper to derive an upper bound on the worst-case performance for the constrained uncertainty problem. Our approach differs in one seemingly minor way from Khatri's work. The additional constraint is given by an offset, rotated rectangle instead of a wedge. The result of this minor change is that the related signal constraints lead to a nice generalization of the standard upper bound for worst case performance. This upper bound always performs at least as good as the standard upper bound on either the unit cube or the offset rectangle.

Before proceeding, we present some notation. $F_u(M, \Delta_u)$ denotes the linear fractional transformation (LFT) of M with the upper loop closed by Δ_u: $F_u(M, \Delta_u) = M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}$, where M has been properly 2×2 block partitioned. $F_l(M, \Delta_u)$ is similarly defined with Δ_u closing the lower loop around M. We will also use the star-product, $\mathcal{S}(J, M)$, which is a generalization of F_u and F_l. $\mathcal{S}(J, M)$ denotes the upper loop of M wrapped with J as in Figure 4. The signal dimensions will be clear from the context.

*2162 Etcheverry Hall, University of California at Berkeley, Berkeley, CA 94704

2 Performance Problem

The uncertain linear system is represented by an LFT involving a known matrix, $M \in \mathbb{C}^{(n+2)\times(n+2)}$, and an uncertainty block, Δ_u (Figure 1). We assume that the uncertainty block, Δ_u, is restricted to lie in the following set:

$$\boldsymbol{\Delta_u} = \left\{ \left[\begin{smallmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{smallmatrix}\right] \ : \ \delta_i \in \mathbb{R} \ , \ |\delta_i| \leq 1 \ , \ \left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right] \in \text{rect}(\theta, d_1, d_2, c) \right\}$$

where $\text{rect}(\theta, d_1, d_2, c)$ specifies the offset, rotated rectangle shown in Figure 2. The shaded area in this figure is intersection of the unit cube and this rectangle, i.e. it is the allowable region for $\left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right]$. Thus the set of allowable models is given by $\{F_u(M, \Delta_u) : \ \Delta_u \in \boldsymbol{\Delta_u}\}$. We will assume that the LFT is well defined for each Δ_u in the allowable set. In other words, every allowable system is 'stable' and it makes sense to look at its performance gain.

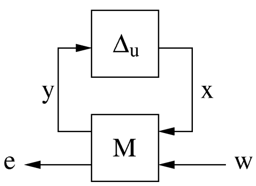

Figure 1: LFT Representation of Model Uncertainty

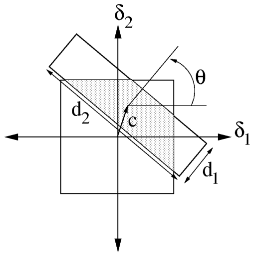

Figure 2: Constraints on the Scalar Uncertainties

Given $\Delta_u \in \boldsymbol{\Delta_u}$, $F_u(M, \Delta_u)$ is an allowable model mapping disturbances, $w \in \mathbb{C}^n$, to errors, $e \in \mathbb{C}^n$. The objective of the worst-case performance analysis is to find the maximum gain from the disturbances to the errors over the set of allowable linear systems. Mathematically, we want to solve:

$$\sup_{\Delta_u \in \boldsymbol{\Delta_u}} \ \bar{\sigma}(F_u(M, \Delta_u)) \tag{1}$$

where we are measuring the gain using the induced $2 \rightarrow 2$ norm, i.e. the maximum singular value. The following theorem, which is a simple version of the Main Loop Theorem [5], answers whether or not this worst case gain is $\geq \gamma$. This theorem will be used in the following section to generate quadratic constraints on the signals in the loop equations.

Theorem 1 *There exists a $\Delta_u \in \boldsymbol{\Delta_u}$ such that the gain from w to e, $\bar{\sigma}(F_u(M, \Delta_u))$, is $\geq \gamma$ if and only if there exists a matrix $\Delta_p \in \mathbb{C}^{n\times n}$, $\bar{\sigma}(\Delta_p) \leq 1/\gamma$ such that the loop in Figure 3 has a nontrivial solution, (x, w, e, y).*

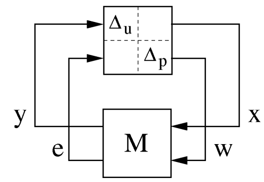

Figure 3: M-Δ Loop for Performance Analysis

3 Signal Constraints

3.1 Reformulation of Rectangle Constraint

In the introduction, we mentioned that specifying the additional constraint in terms of a rotated, offset rectangle instead of a wedge leads to useful signal constraints. The first step of this process is to note that the rectangle can be characterized by an affine mapping from a different unit cube. That is, $\left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right] \in \text{rect}(\theta, d_1, d_2, c)$ if and only if there exists real numbers n_1 and n_2 such that $|n_i| \leq 1$ and:

$$\left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right] \left[\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right] + \left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right] \tag{2}$$

We can now specify the allowable uncertainty set as:

$$\begin{aligned} \boldsymbol{\Delta_u} \ = \ & \left\{ \left[\begin{smallmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{smallmatrix}\right] \ : \ \delta_i \in \mathbb{R} \ , \ |\delta_i| \leq 1 \ , \right. \\ & \left. \exists n_1, n_2 \in \mathbb{R} \ , \ |n_i| \leq 1 \text{ s.t. Equation 2 holds} \right\} \end{aligned}$$

Before proceeding, we rewrite the constraint given by Equation 2 in the following form:

$$\Delta_u = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} = L\Delta_N R + C \tag{3}$$

where $L = \left[\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right]$, $\Delta_N = \left[\begin{smallmatrix} n_1 I_2 & 0_2 \\ 0_2 & n_2 I_2 \end{smallmatrix}\right]$, $R = \left[\begin{smallmatrix} d_1\cos\theta & 0 \\ 0 & d_1\sin\theta \\ -d_2\sin\theta & 0 \\ 0 & d_2\cos\theta \end{smallmatrix}\right]$, and $C = \left[\begin{smallmatrix} c_1 & 0 \\ 0 & c_2 \end{smallmatrix}\right]$.

Equation 3 is just Equation 2 with δ_1 and δ_2 fanned out to make a diagonal matrix. Define $J = \left[\begin{smallmatrix} 0_4 & R \\ L & C \end{smallmatrix}\right]$ so that Equation 3 can be compactly written as: $\Delta_u = F_u(J, \Delta_N)$. The constraint set, $\boldsymbol{\Delta_u}$, says that any allowable Δ_u must look like $\left[\begin{smallmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{smallmatrix}\right]$ where $\left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right]$ lies in the unit cube. It also says that we must be able to represent Δ_u by $F_u(J, \Delta_N)$ where $\left[\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right]$ also lie in the unit cube. Graphically, the original loop in Figure 1 must have the form shown in Figure 4 where x, $y \in \mathbb{C}^2$ and $\tilde{x}$, $\tilde{y} \in \mathbb{C}^4$.

We have now placed the problem in a form where the parameter constraints can be easily written as quadratic constraints on the loop signals. Our goal is to analyze the performance loop in Figure 3 where $\left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right]$ is restricted to the intersection of the rectangle and unit cube. In this form, we can implicitly enforce the box constraint on $\left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right]$ and the norm constraint on Δ_p by quadratic constraints on the signals x, y, w and e. Next we analyze the equivalent loop in Figure 5. This equivalent form follows by writing $\Delta_u = F_u(J, \Delta_N)$ as in Figure 4. Then pull out the uncertainty block, Δ_N, and form a new model $\tilde{M} = \mathcal{S}(J, M)$. In this form, the box constraint on $\left[\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right]$ (which is actually a rotated rectangle constraint on $\left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right]$) and the norm constraint on Δ_p are enforced by quadratic constraints on the signals $\tilde{x}$, $\tilde{y}$, w, and e. Finally, we will enforce an interconnection constraint which ties signals x, y, w, e to signals $\tilde{x}$, $\tilde{y}$, w and e. It is the interconnection constraint that will allow us to generate an LMI upper bound which enforces both of these constraints simultaneously.

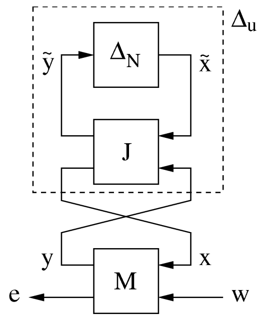

Figure 4: LFT Uncertainty Model, $F_u(M, \Delta_u)$

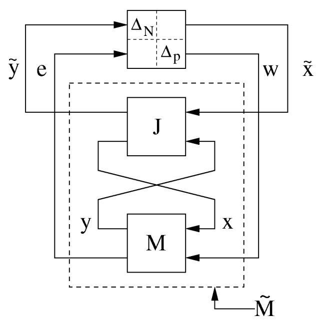

Figure 5: $\tilde{M} - \tilde{\Delta}$ Loop For Performance Analysis

3.2 Signal Constraints: $M - \Delta$ Form

The signal constraints follow from two simple lemmas. The key point is that the Lemmas allow us to write signal constraints which are independent of the uncertainty blocks.

Lemma 1 *Given x, $y \in \mathbb{C}^n$, there exists $\delta \in \mathbb{R}$, $|\delta| \leq 1$ such that $x = \delta y$ if and only if $yy^* - xx^* \geq 0_n$ and $yx^* - xy^* = 0_n$.*

Lemma 2 *Given w, $e \in \mathbb{C}^n$, there exists $\Delta \in \mathbb{C}^{n \times n}$, $\bar{\sigma}(\Delta) \leq 1/\gamma$ such that $w = \Delta e$ if and only if $e^*e - \gamma^2 w^* w \geq 0$.*

We will consider the performance loop given in Figure 3. The box constraint on $\left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right]$ implies that $x_i = \delta_i y_i$ where $\delta_i \in \mathbb{R}$, $|\delta_i| \leq 1$ for $i = 1, 2$. Apply Lemma 1 to get the equivalent constraints which are independent of δ_i:

$$y_i y_i^* - x_i x_i^* \geq 0 \text{ and } y_i x_i^* - x_i y_i^* = 0 \text{ for } i = 1, 2 \tag{4}$$

We can eliminate y_i from these constraints by using the constraint induced by the matrix, M. Block partition the rows of M compatibly with the output of M, $[y_1 \; y_2 \; e^T]^T$:

$$M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} \tag{5}$$

where M_1, $M_2 \in \mathbb{C}^{1 \times (n+2)}$ and $M_3 \in \mathbb{C}^{n \times (n+2)}$. Then:

$$y_i = M_i \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \text{ for } i = 1, 2 \tag{6}$$

Finally, we can rewrite the constraints given in Equations 4 and 6 to arrive at quadratic constraints on $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ which are equivalent to the box constraint on $\left[\begin{smallmatrix} \delta_1 \\ \delta_2 \end{smallmatrix}\right]$:

$$M_i \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]^* M_i^* - E_{i,1}^T \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]^* E_{i,1} \geq 0 \tag{7}$$

$$M_i \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]^* E_{i,1} - E_{i,1}^T \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]^* M_i^* = 0 \tag{8}$$

for $i = 1, 2$. The matrix $E_{i,j}$ is an $(n+2) \times j$ matrix which is block row partitioned compatibly with M, i.e. it has three block rows. It is defined by placing I_j in the i^{th} block. Thus, $E_{i,1}$ is just the i^{th} standard basis vector.

Next we replace the performance constraint $w = \Delta_p e$, $\bar{\sigma}(\Delta_p) \leq 1/\gamma$ by a quadratic constraint on $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$. Apply Lemma 2 to get an equivalent constraint which is independent of Δ_p: $e^*e - \gamma^2 w^* w \geq 0$. Then eliminate e by using the constraint from M, $e = M_3 \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$. The constraint on Δ_p can be written as the quadratic signal constraint:

$$\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]^* (M_3^* M_3 - \gamma^2 E_{3,n} E_{3,n}^T) \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \geq 0 \tag{9}$$

where $E_{3,n}$ as defined above has I_n in the third block row.

3.3 Signal Constraints: $\tilde{M} - \tilde{\Delta}$ Form

In this section, we consider the performance loop given in Figure 5. We will again make use of Lemmas 1 and 2. Partition $\tilde{y}$ compatibly with Δ_N: $\tilde{y} = \left[\begin{smallmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{smallmatrix}\right]$ where $\tilde{y}_i \in \mathbb{C}^2$ for $i = 1, 2$. Block partition the rows of $\tilde{M}$ compatibly with the output of $\tilde{M}$, $[\tilde{y}_1 \; \tilde{y}_2 \; e^T]^T$:

$$\tilde{M} = \begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \\ \tilde{M}_3 \end{bmatrix} \tag{10}$$

where $\tilde{M}_1$, $\tilde{M}_2 \in \mathbb{C}^{2 \times (n+4)}$ and $\tilde{M}_3 \in \mathbb{C}^{n \times (n+4)}$. Using the same steps as in Section 3.2, we get the following constraints on $\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]$ which are equivalent to the box constraints on $\left[\begin{smallmatrix} n_1 \\ n_2 \end{smallmatrix}\right]$:

$$\tilde{M}_i \left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]^* \tilde{M}_i^* - \tilde{E}_{i,2}^T \left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]^* \tilde{E}_{i,2} \geq 0_2 \tag{11}$$

$$\tilde{M}_i \left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]^* \tilde{E}_{i,2} - \tilde{E}_{i,2}^T \left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]^* \tilde{M}_i^* = 0_2 \tag{12}$$

for $i = 1, 2$. The matrix $\tilde{E}_{i,j}$ is an $(n+4) \times j$ matrix which is block row partitioned compatibly with $\tilde{M}$, i.e. it has three block rows. It is defined by placing I_j in the i^{th} block.

The performance constraint $w = \Delta_p e$, $\bar{\sigma}(\Delta_p) \leq 1/\gamma$ is again replaced by the following quadratic constraint on $\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]$ by applying Lemma 2.

$$\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]^* (\tilde{M}_3^* \tilde{M}_3 - \gamma^2 \tilde{E}_{3,n} \tilde{E}_{3,n}^T) \left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \geq 0 \tag{13}$$

3.4 Interconnection Constraint

Figure 4 shows that the relation between $\tilde{y}$, x and $\tilde{x}$, y is given by $\left[\begin{smallmatrix} \tilde{y} \\ x \end{smallmatrix}\right] = J \left[\begin{smallmatrix} \tilde{x} \\ y \end{smallmatrix}\right]$. Recall that $J = \left[\begin{smallmatrix} 0_4 & R \\ L & C \end{smallmatrix}\right]$ so that $x = L\tilde{x} + Cy$.

Define a 2×2 block partition of M compatible with the input signals, $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$, and output signals, $\left[\begin{smallmatrix} y \\ e \end{smallmatrix}\right]$:

$$M = \left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right] \tag{14}$$

where $M_{11} \in \mathbb{C}^{2\times 2}$, $M_{12} \in \mathbb{C}^{2\times n}$, $M_{21} \in \mathbb{C}^{n\times 2}$, and $M_{22} \in \mathbb{C}^{n\times n}$. Using this notation, $x = L\tilde{x} + C\left[\begin{smallmatrix} M_{11} & M_{12} \end{smallmatrix}\right]\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$. We will assume that $F_u(M, C)$ is 'stable', i.e. $I - CM_{11}$ is nonsingular[1]. Then we can find x in terms of $\tilde{x}$ and w:

$$x = (I - CM_{11})^{-1}[L\tilde{x} + CM_{12}w]$$

Finally, we can write the constraint which relates $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]$:

$$\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} (I-CM_{11})^{-1}L & (I-CM_{11})^{-1}CM_{12} \\ 0_{n\times 4} & I_n \end{smallmatrix}\right]\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \equiv T\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \tag{15}$$

where T is the given transformation matrix from $\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$.

3.5 Set of Allowable Loop Signals

Let us review the purpose of the preceding sections. As stated in Section 2, our goal is to find the maximum gain from disturbances to errors over the allowable set of linear systems. It is easier to answer the related question of whether or not this worst case gain is $\geq \gamma$. By Theorem 1, this is equivalent to the loop shown in Figure 3 having nontrivial signals for some $\Delta_u \in \boldsymbol{\Delta_u}$ and $\Delta_p \in \mathbb{C}^{n\times n}$, $\bar{\sigma}(\Delta_p) \leq 1/\gamma$. Our goal in this section is to write the set of allowable signals, $\mathcal{S}_{allow}$, in this loop. If this set only contains the trivial solution, then we can conclude that the worst case gain is $< \gamma$.

Given signals w and x, we can determine signals y and e in Figure 3. Suppose signals w and x satisfy the performance loop equations for some allowable Δ_u, Δ_p. In Section 3.2, we showed that the box constraint on Δ_u and the norm constraint on Δ_p imply that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ satisfies the quadratic constraints given by Equations 7 - 9. In Section 3.1, we showed that the rectangle constraint on Δ_u could be reformulated by writing Δ_u as $F_u(J, \Delta_N)$ and imposing a box constraint on Δ_N. This introduced signals, $\tilde{x}$ and $\tilde{y}$, which are internal to Δ_u. Then in Section 3.3, we showed that the box constraint on Δ_N and the norm constraint on Δ_p imply that $\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]$ satisfies the quadratic constraints given by Equations 11 - 13. Finally in Section 3.4 we showed that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]$ are related by Equation 15. This constraint allows us to rewrite the constraints on $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ with constraints on $\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]$. Specifically, Equations 7-9 become:

$$M_i T\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]^* T^* M_i^* - E_{i,1}^T T\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]^* T^* E_{i,1} \geq 0 \tag{16}$$

$$M_i T\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]^* T^* E_{i,1} - E_{i,1}^T T\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]^* T^* M_i^* = 0 \tag{17}$$

$$\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]^* T^*(M_3^* M_3 - \gamma^2 E_{3,n} E_{3,n}^T) T\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \geq 0 \tag{18}$$

Thus the set of allowable signals in the loop can be specified in terms of $\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]$:

$$\mathcal{S}_{allow} = \{\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \ : \ \left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \text{ satisfy Equations 11-13 and 16-18}\} \tag{19}$$

[1]This assumption is essentially without loss of generality. If $I - CM_{11}$ is singular, we can find a rectangle center matrix arbitrarily close to C such that the matrix becomes nonsingular. The constrained area is then changed by an arbitrarily small amount.

In fact, constraints 13 and 18 are redundant. After some algebra, $\tilde{M} = \mathcal{S}(J, M)$ can be written as:

$$\tilde{M} = \begin{bmatrix} R & 0_{4\times n} \\ 0_{n\times 2} & I_n \end{bmatrix}\left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right] T \tag{20}$$

where we have employed the 2×2 block partition of M given in Equation 14. Using the block row decompositions of M and $\tilde{M}$ given in Equations 5 and 10, respectively, this relation gives: $\tilde{M}_3 = M_3 T$. Thus constraints 13 and 18 are indeed redundant and we can write $\mathcal{S}_{allow}$ as:

$$\mathcal{S}_{allow} = \{\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \ : \ \left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \text{ satisfy Equations 11-13, and 16-17 }\} \tag{21}$$

If $\mathcal{S}_{allow} = \left\{\begin{bmatrix} 0_{4\times 1} \\ 0_{n\times 1} \end{bmatrix}\right\}$ then the gain is $< \gamma$. In the next section we will derive a sufficient condition for $\mathcal{S} = \left\{\begin{bmatrix} 0_{4\times 1} \\ 0_{n\times 1} \end{bmatrix}\right\}$.

4 Worst-Case Gain Upper Bound

As stated in the introduction, finding the exact value of the worst case performance is computationally hard. Instead, we will derive an upper bound on the worst case performance which is computed by solving a linear matrix inequality (LMI). This derivation uses the $\mathcal{S}$-procedure [1] to generate a sufficient condition for $\mathcal{S}_{allow} = \left\{\begin{bmatrix} 0_{4\times 1} \\ 0_{n\times 1} \end{bmatrix}\right\}$. We will require two lemmas to prove the sufficiency of the condition we derive:

Lemma 3 *Given any $\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]$ satisfying Equation 11 and any $\tilde{D}_i \in \mathbb{C}^{2\times 2}$ such that $\tilde{D}_i^* = \tilde{D}_i > 0$, then:*

$$\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]^* \left(\tilde{M}_i^* \tilde{D}_i \tilde{M}_i - \tilde{E}_{i,2} \tilde{D}_i \tilde{E}_{i,2}^T\right)\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] \geq 0$$

Lemma 4 *Given any $\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]$ satisfying Equation 12 and any $\tilde{G}_i \in \mathbb{C}^{2\times 2}$ such that $\tilde{G}_i^* = \tilde{G}_i$, then:*

$$\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right]^* \left(\tilde{E}_{i,2} \tilde{G}_i \tilde{M}_i - \tilde{M}_i^* \tilde{G}_i \tilde{E}_{i,2}^T\right)\left[\begin{smallmatrix} \tilde{x} \\ w \end{smallmatrix}\right] = 0$$

Application of the $\mathcal{S}$-procedure leads to the following sufficient condition.

Theorem 2 *If there exists $\tilde{D}_i^* = \tilde{D}_i \in \mathbb{C}^{2\times 2}$, $\tilde{D}_i > 0$, $\tilde{G}_i^* = \tilde{G}_i \in \mathbb{C}^{2\times 2}$, $d_i \in \mathbb{R}$, $d_i > 0$, $g_i \in \mathbb{R}$ for $i = 1, 2$ and $\tilde{d}_3 \in \mathbb{R}$, $\tilde{d}_3 > 0$ such that:*

$$\begin{aligned}
&\left(\tilde{M}_1^* \tilde{D}_1 \tilde{M}_1 - \tilde{E}_{1,2} \tilde{D}_1 \tilde{E}_{1,2}^T\right) + \left(\tilde{M}_2^* \tilde{D}_2 \tilde{M}_2 - \tilde{E}_{2,2} \tilde{D}_2 \tilde{E}_{2,2}^T\right) \\
&+ \tilde{d}_3\left(\tilde{M}_3^* \tilde{M}_3 - \gamma^2 \tilde{E}_{3,n} \tilde{E}_{3,n}^T\right) + j\left(\tilde{E}_{1,2} \tilde{G}_1 \tilde{M}_1 - \tilde{M}_1^* \tilde{G}_1 \tilde{E}_{1,2}^T\right) \\
&+ j\left(\tilde{E}_{2,2} \tilde{G}_2 \tilde{M}_2 - \tilde{M}_2^* \tilde{G}_2 \tilde{E}_{2,2}^T\right) + d_1 T^*\left(M_1^* M_1 - E_{1,1} E_{1,1}^T\right) T \\
&+ d_2 T^*\left(M_2^* M_2 - E_{2,1} E_{2,1}^T\right) T + j g_1 T^*\left(E_{1,1} M_1 - M_1^* E_{1,1}^T\right) T \\
&+ j g_2 T^*\left(E_{2,1} M_2 - M_2^* E_{2,1}^T\right) T < 0
\end{aligned}$$

then $\mathcal{S}_{allow} = \left\{\begin{bmatrix} 0_{4\times 1} \\ 0_{n\times 1} \end{bmatrix}\right\}$.

Proof. Suppose there exists $\left[\begin{smallmatrix}\tilde{x}\\ w\end{smallmatrix}\right] \in \mathcal{S}_{allow}$ such that $\left[\begin{smallmatrix}\tilde{x}\\ w\end{smallmatrix}\right] \neq \left[\begin{smallmatrix}0_{4\times1}\\ 0_{n\times1}\end{smallmatrix}\right]$. If we hit the terms of the matrix inequality on the left and right by $\left[\begin{smallmatrix}\tilde{x}\\ w\end{smallmatrix}\right]^*$ and $\left[\begin{smallmatrix}\tilde{x}\\ w\end{smallmatrix}\right]$, respectively, we immediately obtain a contradiction. Specifically, the first two terms will be ≥ 0 by Equation 11 and Lemma 3. The next term will be ≥ 0 by Equation 13. Terms 4 and 5 will be $= 0$ by Equation 12 and Lemma 4. The remaining terms will be ≥ 0 by Equations 16 and 17. Thus the left hand side must be ≥ 0. However, the matrix inequality and $\left[\begin{smallmatrix}\tilde{x}\\ w\end{smallmatrix}\right] \neq \left[\begin{smallmatrix}0_{4\times1}\\ 0_{n\times1}\end{smallmatrix}\right]$ imply that it must be strictly negative. Hence $\mathcal{S}_{allow} = \left\{\left[\begin{smallmatrix}0_{4\times1}\\ 0_{n\times1}\end{smallmatrix}\right]\right\}$. □

The matrix inequality condition can be scaled by any positive number without affecting the result. If we scale the condition by $2/\tilde{d}_3$ and combine terms properly, we can rewrite the sufficient LMI as:

$$\begin{aligned}&\tilde{M}^*\begin{bmatrix}\tilde{D} & \\ & I_n\end{bmatrix}\tilde{M} - \begin{bmatrix}\tilde{D} & \\ & \gamma^2 I_n\end{bmatrix} + j\left(\begin{bmatrix}\tilde{G} & \\ & 0_n\end{bmatrix}\tilde{M} - \tilde{M}^*\begin{bmatrix}\tilde{G} & \\ & 0_n\end{bmatrix}\right)\\ &+T^*\left\{M^*\begin{bmatrix}D & \\ & I_n\end{bmatrix}M - \begin{bmatrix}D & \\ & \gamma^2 I_n\end{bmatrix}\right.\\ &\qquad\left. + j\left(\begin{bmatrix}G & \\ & 0_n\end{bmatrix}M - M^*\begin{bmatrix}G & \\ & 0_n\end{bmatrix}\right)\right\}T < 0\end{aligned} \quad (22)$$

where $\tilde{D} = blkdiag(\tilde{D}_1, \tilde{D}_2)$, $\tilde{G} = blkdiag(\tilde{G}_1, \tilde{G}_2)$, $D = diag(d_1, d_2)$, and $G = diag(g_1, g_2)$. In this form, we can solve for an upper bound on the worst case performance by minimizing γ^2 subject to this LMI constraint.

The first line is simply the μ upper bound scaled for performance if we only know that Δ_u is restricted to the rotated, offset rectangle. The terms in the braces on the second and third lines form the μ upper bound scaled for performance if we only know that Δ_u is restricted to the unit cube. Thus the upper bound for $\Delta_u \in \boldsymbol{\Delta_u}$ is a combination of these two standard upper bounds. It is also interesting that this upper bound obtained by minimizing γ^2 subject to Equation 22 is always less than or equal to the upper bound obtained for either Δ_u restricted to the rectangle or box independently. As just mentioned, if we set $D = G = 0_2$, we recover the upper bound for Δ_u only restricted to the rectangle. The matrices G and D are just extra degrees of freedom which allow the constrained upper bound to do at least as good. To show that the upper bound obtained using Equation 22 is always less than or equal to the upper bound when Δ_u is only restricted to the unit box is more subtle.

5 Numerical Results

In this section, we present some numerical results for the constrained area upper bound (Equation 22). As a simple test example, we consider $M = M_r + jM_i$ where $M_r, M_i \in \mathbb{R}^{4\times4}$:

$$M_r = \begin{bmatrix}-0.077 & -0.408 & -0.223 & -0.386\\ 0.085 & -0.453 & -0.473 & 0.040\\ 0.137 & -0.009 & 0.554 & -0.092\\ -0.254 & 0.169 & -0.235 & -0.253\end{bmatrix}$$

$$M_i = \begin{bmatrix}0.057 & -0.337 & 0.150 & -0.429\\ 0.443 & -0.160 & 0.176 & 0.873\\ -0.291 & 0.803 & 0.476 & -0.278\\ -0.359 & 0.531 & -0.316 & -0.012\end{bmatrix}$$

For this example, MATLAB's LMILab was used to solve the LMI upper bounds for three cases: Δ_u restricted to the unit cube, Δ_u restricted to rect(θ, d_1, d_2, c)= rect$(20deg, 1.5, 0.2, \left[\begin{smallmatrix}0.2\\ 0.5\end{smallmatrix}\right])$ and for Δ_u restricted to the intersection of the cube and the rectangle. For this simple problem, we can find a good estimate of the three gains by gridding up the parameter space. For example, the unit cube is gridded into a 30 by 30 matrix. $F_u(M, \Delta_u)$ is formed at each point and we compute the gain by a maximum singular value computation on the resulting 2×2 matrix. Figure 6 shows this performance curve over the unit cube. The peak gain of 2.592 achieved at $\left[\begin{smallmatrix}\delta_1\\ \delta_2\end{smallmatrix}\right] = \left[\begin{smallmatrix}-1\\ -1\end{smallmatrix}\right]$. This actually gives a lower bound on the worst-case performance. However, we will assume that the gridding of the parameter space is sufficiently fine so that this closely approximates the true worst-case performance. Using the same technique on the rotated rectangle specified above gives a worst-case performance of 1.451. Figure 7 shows the performance on the constrained area. The worst-case performance is 1.200 and is achieved at $\left[\begin{smallmatrix}\delta_1\\ \delta_2\end{smallmatrix}\right] = \left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]$. This example is chosen so that worst-case performance evaluated on either the unit cube or the rectangle independently gives a conservative result.

Table 1 summarizes these lower bounds and the upper bounds for each of the three cases. The constrained upper bound is indeed less conservative than the cube and rectangle upper bounds. We should mention the complexity of the resulting LMIs. The LMI for the constrained upper bound contains 21 variables, while the rectangle upper bound has 17 variables and the unit cube upper bound has only 5 variables. Correspondingly, it took 0.18, 0.49, and 1.52 seconds to compute the cube, rectangle, and constrained area upper bounds, respectively. All three lower bounds were computed in 2.42 seconds.

Cube UB	Rectangle UB	Constrained UB
2.627	1.502	1.329
Cube LB	Rectangle LB	Constrained LB
2.592	1.451	1.200

Table 1: Worst-Case Gains

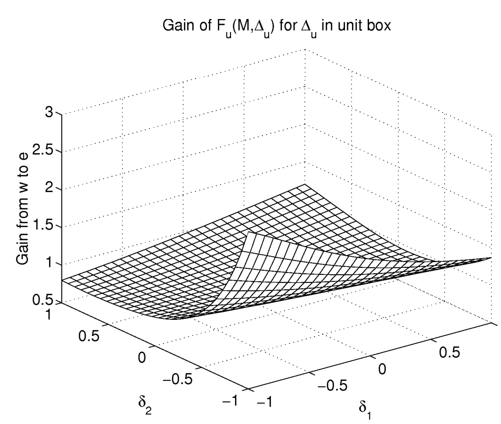

Figure 6: System Performance over Unit Cube

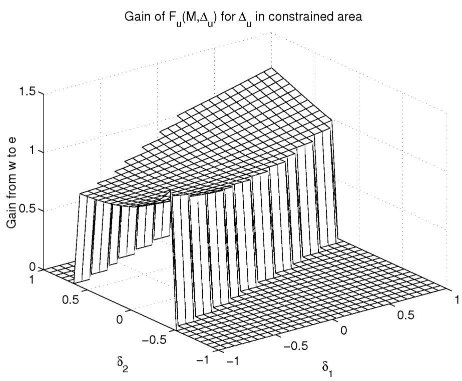

Figure 7: System Performance over Constrained Area

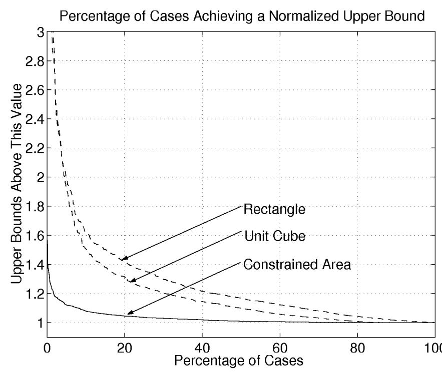

Figure 8: Quality of Upper Bounds for 873 Random Matrices

Next we generated 1000 random 4×4 complex random matrices. If $F_u(M, \Delta_u)$ could not be formed on the unit cube or the rectangle, no data was recorded. It is possible for the system to be unstable on these areas independently, but stable on the constrained area. This would again reduce the conservativeness of the worst-case performance assessment. However, we chose to ignore such matrices for this analysis. For the remaining 873 matrices, we computed the three upper and lower bounds described above using the same rectangle. Since our objective is to find the worst-case performance on the constrained area, we treat the constrained lower bound obtained via gridding as the true worst-case value. The gap between the constrained upper bound and the constrained lower bound is due to the $\mathcal{S}$-procedure (which is only a sufficient condition) and the gap between the constrained lower bound and the true value of the constrained worst-case performance. The rectangle and unit cube upper bounds will have a gap which is due to these two gaps as well as the inaccuracy due to computing the bounds on a larger area than desired. Thus these two bounds will always be $\geq$ the constrained upper bound.

The upper bounds are normalized by dividing by the constrained lower bound. Thus the normalized upper bounds will always be ≥ 1, but we would like them to be as close to one as possible. Figure 8 shows the percentage of matrices which have an upper bound above some threshold. For example this plot shows that only 8.1% of the matrices have a normalized constrained upper bound greater than 1.1 while 49.8% of the unit cube and 65.6% of the rectangle normalized upper bounds are greater than 1.1. The worst value for the normalized constrained upper bound is 1.56. Therefore, the constrained upper bound never differs from the lower bound by more than 56% for these 873 random matrices. The other upper bounds can be arbitrarily far off. In fact, they could be unstable as discussed above. For the 873 matrices, the worst value of the rectangle and cube upper bounds are roughly 19 and 49 times greater, respectively, than the constrained lower bound. Finally, the mean computation times for the matrices are .102, .601, and 1.557 seconds for the cube, rectangle, and constrained areas, respectively. The mean time to compute all three lower bounds is 2.375 seconds.

6 Conclusions

We examined the worst-case performance of a linear system subjected to 2 real, scalar uncertainties which were constrained to the intersection of the unit cube and a rotated offset rectangle. We derived an LMI upper bound which is a mixture of the standard LMI upper bounds obtained when the the uncertainties are restricted to either the unit cube or rectangle independently. This allowed us to analyze the the worst case performance of a system with correlated parameters. This approach was detailed for the very simple case of 2 correlated real scalar uncertainties and one performance block, Δ_p. It is straightforward to extend these results to the case where Δ_p is any mixed block structure. Apparently this method can be generalized to include higher dimensional blocks of real scalars which are restricted to a parallelepiped.

References

[1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, 1994.

[2] R. Braatz, P. Young, J. Doyle, and M. Morari. Computational complexity of μ calculation. *IEEE Transactions on Automatic Control*, 39(5):1000–1002, May 1994.

[3] S. Khatri. *Probabilistic Robustness Analysis and Extensions to the Structured Singular Value*. PhD thesis, California Institute of Technology, October 1998.

[4] A. Packard, G. Balas, R. Liu, and J. Shin. Results on worst-case performance assessment. In *Proceedings of the American Control Conference*, pages 2425–2427, Chicago, Illinois, 2000.

[5] A. Packard and J. Doyle. The complex structured singular value. *Automatica*, 29(1):71–109, 1993.

[6] P. Parrilo. *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, May 2000.