Cladistic Asset Pricing*

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Abstract

In most contexts in which agents possess long-lived private information about a firm's value, they will also have repeated access to new information over time. We develop a method for analyzing informed trade in such environments. We exploit the stationary structure of the economy to map informed agents' optimization problems in the time domain—maximize discounted expected trading profits—into the frequency domain. We then use control-theoretic arguments to solve for equilibrium trading strategies, pricing, profits and the information content of prices. We derive *explicit* characterizations of equilibrium outcomes—trading strategies, pricing, profits and information transmission—when informed agents see distinct AR(1) innovations to the asset value each period. We characterize analytically how equilibrium outcomes are affected by the amounts of private information or noise trade. Finally, we provide a practical method to approximate equilibrium trading strategies. We prove that the approximation converges to the true equilibrium, and then use it to characterize the equilibrium quantitatively to derive the impact of persistence in firm value and the degree of competition among informed agents on equilibrium outcomes.

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1. INTRODUCTION

This paper develops a method for solving for equilibrium outcomes in stationary strategic settings in which agents are informationally large and understand how their actions affect the information content of prices. We use the method to characterize speculative trade and stock price dynamics when agents acquire private, long-lived information on a recurring basis, and trade strategically.

In most contexts, few speculative traders will have private information about any given stock. The private information they possess will be long-lived and they will have repeated access to new information over time. For example, corporate insiders have information about current and past earnings, and, in the future, they will again be privy to news about earnings. So, too, investors often focus on a small number of stocks on which to do detailed research. Their accumulated expertise leaves them better situated to evaluate information both immediately after they learn it, and when it arrives in the future.

Such speculators will trade strategically, recognizing that their trades will affect prices and, hence, convey information to other speculators; and that prices will contain information about the signals of other speculators that they, themselves, can use. Agents understand that over time, information about past signals will leak out through equilibrium prices, reducing their value somewhat, but not to zero. Hence, speculators can continue to trade profitably on information acquired in the past. To solve for equilibrium outcomes, one must first determine how speculative traders combine current and past signals together with the information in current and past prices, then determine how much they trade at each date, and finally put this all together and solve for the associated equilibrium pricing.

Currently, there is no known method even to solve such a model. We develop precisely such a method. We then use it to answer questions such as: How does competition among such speculators affects trading and equilibrium price dynamics? Does increased competition lead prices to reveal more information? Information about recent innovations? Past innovations? How do the characteristics of private information—its quantity and persistence over time—affect trading strategies, pricing, profit and information transmission?

We analyze an environment in which the asset value evolves according to an AR stochastic process, and investors privately observe innovations to the asset's value. We develop an iterative best-response mapping of speculators to conjectured trading strategies of other agents. We then use a novel contraction mapping argument to prove that there exists a unique linear equilibrium, and to derive the form of equilibrium trading strategies. In particular, we prove that equilibrium trading strategies are given by infinite sums of AR(1) functions of the private information and public-information net order flow processes. We then provide tight analytical characterizations of how strategic trading behavior, market prices, volume and profits are affected by the amounts of private information and noise trade in the economy. Finally, we use the iterative best-response algorithm to characterize numerically how equilibrium outcomes—trading strategies, pricing, volume, information revealed, profits—vary with the primitives that describe the environment.

It is useful to highlight what makes the analysis so difficult when multiple informed agents have continuing access to private information. First, traders should use *all* of their private information—the *entire* history of private signals—to process the information in the history of equilibrium prices. This immediately implies that we must resolve "forecasting the forecasts" issues that arise when strategic heterogeneously-informed agents process the information in prices. Prices reflect the trades of other speculators, and hence their forecasts of other speculators, generating an infinite hierarchy of forecasts. In particular, because private signals are not perfectly correlated, individual trader's forecasts will differ given the public information prices (see Townsend (1985) and Pearlman and Sargent (2002), where these issues do not arise). Once we have resolved these issues, we must determine how intensively agents should trade on each private signal, and on the information contained in each price. Trading more aggressively on information raises current expected period profits, but reveals more information through price, reducing future profits. Finally, we must put this all together and solve for the equilibrium of the game: competitive market makers face a similar inference problem, and set prices to break even in expectation at each date given the equilibrium implications of their pricing for informed trade.

This problem is so formidable that researchers have turned to simpler settings. The seminal paper, Kyle (1985), considers a monopolist speculator who learns the asset's terminal value at the beginning of the trading game. Back and Pederson (1998) extend this analysis with a monopolist speculator to allow for information arrival over time, but, again, strategic interactions between agents remain absent. Subsequently, Back, Cao and Willard (2000), Holden and Subrahmanyam (1992), and Foster and Viswanathan (1994, 1996) extend Kyle's analysis to allow multiple agents to receive symmetrically distributed signals, but these papers still assume that information arrives only at date zero. Bernhardt and Miao (2004) extend these analyses to characterize equilibrium in arbitrary finite horizon environments in which agents can acquire distinct signals at different dates of varying quality and correlations. However, while Bernhardt and Miao prove that trading strategies are linear functions of unrevealed private information, they cannot provide more specific analytical characterizations. For example, they cannot derive how equilibrium outcomes are affected by the amounts of private information or noise trade. Also, their numerical characterizations are largely limited to three-period settings. Finally, their theoretical approach simply does not extend to stationary settings. They conjecture the form of trading strategies and value functions, and solve for the equilibrium recursively beginning at the terminal date. In an infinite horizon setting, there is no last period, information sets "blow up", and an infinite number of parameters characterize equilibrium strategies and value functions.

A second approach has been to assume that private information is short-lived: as new private information becomes available, old information is revealed to the market (Admati and Pfleiderer (1988)). Then agents when deciding how much to trade do not have to consider how to trade off current for future profit, dramatically simplifying characterizations.

A third approach has been to build noisy REE models of stock price dynamics with asymmetrically-informed traders (*e.g.* Wang (1994), He and Wang (1995), Malinova and Smith (2003)). These papers characterize how the underlying asset value processes affect volume and price dynamics in environments where there is *no* strategic behavior. That is, informed agents are individually small price takers who ignore the price impact of their trades. Assuming that agents are informationally small *circumvents* individual strategic behavior, as agents do not have to trade off profit-taking against information release. But, allowing for strategic informed trade is important—in practice, informed agents for a given stock are few in number, and these speculators understand that significant trading has price impacts that they should anticipate and internalize.

It is worth noting that all of these models of strategic speculator behavior share the competitive dealership market structure of Kyle (1985). There is, however, no real world analog to this market design. In sharp contrast, Kyle (1989) develops a model in which agents submit demand schedules that detail how much they want to trade at each price, and the equilibrium price clears the market, equating supply to demand. This institutional structure is used to determine opening prices on most exchanges, including the NYSE, NASDAQ, Paris Bourse and Toronto Stock Exchange. Bernhardt and Taub (2004) show that if there are large numbers of uninformed, but strategic liquidity providers in this demand submission market, then equilibrium outcomes correspond to those in a competitive dealership market in which speculators see both their private signals *and* the market-clearing price, and then determine how much to trade. This current paper exploits this insight to provide the first dynamic analysis of a Kyle (1989) demand-submission market design.

Our approach is first to recognize that if the underlying asset values are stationary, then equilibrium strategies, although quite complicated, will be stationary, linear, functions of the history of private information signals and public information prices. We then use the methods of Whiteman (1985) to convert an informed agent's optimization problem in the time domain—maximize expected discounted lifetime trading profits—into an optimization problem in the frequency domain.

We next use variational methods to find the optimal policy *function*. This optimization method exploits certainty equivalence: trading strategies and pricing are linear functions of states, so that we can convert the original problem, which is to optimize using conditional information, to an unconditional optimization problem. Essentially one solves an infinite-horizon model in which the constraints associated with each state are stated implicitly, generating an Euler equation. The Euler equation takes the form of a Wiener-Hopf equation, which we solve to obtain optimal trading strategies.

We use the frequency domain setting to construct a proof that there exists a unique stationary linear equilibrium. We construct an iterative-best response mapping, and show that it converges. More specifically, we first conjecture (falsely) that speculators adopt linear trading strategies on private information that match the valuation process. Given this conjecture we solve for the implied pricing and trading on the information in prices. We then assume that each speculator believes that other speculators will trade according to these implied first round outcomes, and solve for the optimal best response. We then iterate on this best response mapping, using a contraction mapping argument to prove that there is a unique fixed point. Our proof method should generalize to settings such as oligopoly games with random demand and technology shocks where firms are informationally large and take into account how output choices affect the information content of prices.

We then characterize equilibrium outcomes. We prove that in equilibrium, an agent's net total order is equal to his forecast of the error in the market maker's forecast of his trade on private information. That is, from an agent's trade on private information, he subtracts off its projection onto the history of net order flows—he subtracts off the market maker's forecast of his trade on private information. This interpretation of equilibrium trading strategies is very general. We then derive the form of equilibrium trading strategies—we prove that both the private and public information trading strategy components are infinite sums of AR(1) terms, and that there is no simpler representation of strategies. However, the market maker's pricing function remains an AR(1), as he undoes the complicated autoregressive structure of the order flow generated by the trading strategies and converts it back to a process with the same AR(1) structure as the asset value process.

We then derive how the amounts of private information and noise trade affect equilibrium outcomes. We prove that the weights on the private information component of trading strategies are proportional to the square root of the noise trade variance to informed signal innovation variance ratio, $\frac{\sigma_u}{\sigma_e}$. In turn, this implies that the market maker's pricing function is inversely proportional to $\frac{\sigma_u}{\sigma_e}$, and that informed profits are proportional to $\sigma_u \sigma_e$. Thus, we prove that properties of simpler models of informed trade such as Kyle (1985) hold very generally.

Finally, we exploit the iterative-best response algorithm to characterize equilibrium outcomes quantitatively. We derive the quantitative impact of (1) the degree of competition among informed agents, (2) the persistence in asset valuations, and (3) the correlation of private information, on the equilibrium levels of (a) informed trading intensity on current and past information, (b) informed profit, (c) market maker pricing, and (d) the amount of information that is revealed through price about current and past innovations. To ease presentation, our theoretical analysis presumes that the private signals speculators receive are uncorrelated. However, our numerical analysis sheds light on how correlation in information across traders affects equilibrium outcomes.

Section 2 presents the economic environment. Section 3 analyzes trading strategies and pricing. Section 4 develops the contraction mapping argument used to prove existence of the stationary linear equilibrium. Section 5 analyzes properties of equilibrium strategies. Section 6 quantitatively characterizes equilibrium outcomes. We conclude by discussing how our approach can be extended to other settings. Appendix A presents our frequency-domain methods. Appendix B provides proofs of theorems.

2. ASSET VALUATIONS AND TRADER OPTIMIZATION IN A STATIONARY SETTING

We now develop a stationary model of strategic informed trade by agents who have recurring access to distinct long-lived private information about a firm's value. N risk neutral informed traders and exogenous noise traders trade claims to the firm in a market made by risk neutral competitive, uninformed market makers. Agents share a common discount factor $\tilde{\beta} \in [0, 1)$. At each date t there is a constant probability $\pi \geq 0$ that the firm will be liquidated at the end of the period, so that with probability $1 - \pi$ the firm continues on to another period. We assume that $\tilde{\beta}(1 - \pi) < 1$, so that expected discounted informed trading profits are bounded. A firm that is liquidated at date T pays

$$\tilde{v}_T = \tilde{v}_0 + \delta_1 + \delta_2 + \ldots + \delta_T,$$

where $\tilde{v}_0$ is public information, and δ_t is the period-t innovation to earnings. We assume that δ_t evolves stochastically over time according to the sum of N AR(1) processes:

$$\delta_t = \bar{\delta} + \sum_{j=1}^{N} [\tilde{e}_{jt} + \rho \tilde{e}_{jt-1} + \rho^2 \tilde{e}_{jt-2} + \ldots] \equiv \bar{\delta} + \phi(L) \sum_{j=1}^{N} \tilde{e}_{jt},$$

where $\tilde{e}_{jt} \sim N(0, \frac{\tilde{\sigma}^2}{N})$, $j = 1, \ldots, N$, is independently and identically distributed across the N processes and time, and $\rho \in (0, 1]$. The normalization of the mean to zero is without loss of generality. The AR(1) formulation allows for a rich class of earnings processes. In particular, the innovation $e_\tau \equiv \sum_{j=1}^{N} \tilde{e}_{jt}$ can contribute not only to period-τ revenues, but also to future revenues. The greater is ρ, the more persistent is the contribution. For example, $\tilde{e}_\tau$ may reflect earnings due to a new product, and consumers will want to buy the product in future periods, albeit in decreasing numbers.

Information: We introduce heterogeneous privately-informed agents into this economy by assuming that each AR(1) process is observed by a single agent. That is, informed agent i has private information about the $\tilde{e}_i$ process. Thus, there are N informed agents who are symmetrically-situated, but have heterogeneous information: at date t, informed agent i knows the time-t history of those innovations that he has observed, $(\tilde{e}_{it}, \tilde{e}_{it-1}, \tilde{e}_{it-2}, \ldots, \tilde{e}_{i0})$.

In addition, each informed agent knows the time t history of prices, including the period t price at which his order will be executed. Bernhardt and Taub (2004) show that the equilibrium when speculators see both their signals and the current price corresponds to that in Kyle's (1989) demand submission market design when there are many uninformed, but strategic, liquidity providers. In demand submission markets, agents submit demand schedules, which are aggregated to determine the equilibrium, market-clearing price. This market design closely mirrors that used on most exchanges to determine prices at open. Our paper is the first to solve for equilibrium outcomes in a dynamic demand submission market.

Often, researchers assume that agents observe prices with a lag as in Kyle (1985), so that agent i's date t information is $(\mathbf{\tilde{e}_{it}}, \mathbf{\Omega_{t-1}})$: agents do not know the price they will get when submitting orders. This approach may ease analysis, but at a cost—there is no close real world analog to this market design.

Pricing: Let x_{it} be informed trader i's order, $i = 1, \ldots, N$, at date t. In addition to trade from informed agents, there is exogenous liquidity trade of u_t. Liquidity trade is independently and identically normally distributed each period according to $N(0, \sigma_u^2)$. Let $X_t = \sum_{j=1}^{N} x_{jt}$ be the total informed trade at date t. Thus, $X_t + u_t$ is the total net order flow at date t and $\mathbf{\Omega_{t-1}} \equiv (X_{t-1} + u_{t-1}, \ldots X_1 + u_1, \tilde{v}_0)$ is the date-t history of net order flows and the date 0 expected value of the firm. Market makers see this net order flow history. Competition between market makers leads them to set price equal to the expected value of the asset given this date-t public information, so that

$$p_t = E[\sum_{T=t}^{\infty} \pi(1-\pi)^{T-t} \tilde{\beta}^{T-t} \tilde{v}_T \mid X_t + u_t, \mathbf{\Omega_{t-1}}],$$

where $\pi(1-\pi)^{T-t}$ is the probability the firm is liquidated at the end of date $T \geq t$ after date T trading given that it has not been liquidated prior to date t.

We focus on equilibria in which market makers set prices that are linear functions of the order flow history. As a result, knowing the history of prices is equivalent to knowing the history of order flows. Thus, public information evolves according to $\mathbf{\Omega_t} = \{X_t + u_t, \mathbf{\Omega_{t-1}}\}$, where $\mathbf{\Omega_0} = \tilde{v}_0$. It follows that in period t, trader i knows the earnings innovation history that he has observed, $\mathbf{\tilde{e}_{it}} \equiv (\tilde{e}_{it}, \tilde{e}_{it-1}, \ldots \tilde{e}_{i1})$, his past orders, $\mathbf{x_{it}} \equiv (x_{it}, x_{it-1}, \ldots x_{i1})$, and the history of prices, which is informationally equivalent to $\mathbf{\Omega_t}$.

Trader Optimization: Consider trader i's perspective at some date t given that the firm has not yet been liquidated. If the firm is liquidated at some future date T, then trader i's net signed position at liquidation would be $\sum_{s \leq T} x_{is}$. If the realized value of the firm was $\tilde{v}_T$, then the date t value of such a position would be $\tilde{\beta}^{T-t} \tilde{v}_T \sum_{s \leq T} x_{is}$. The date t cost of this position would be $\sum_{s \leq T} \tilde{\beta}^{t-s} p_s x_{is}$, because the signed order by trader i at date s, namely x_{is}, was executed at price p_s, and from a date t perspective is discounted by $\tilde{\beta}^{s-t}$. Thus, if the firm is liquidated at date T, informed agent i's trading profits just equal the difference between the value of his position and its cost,

$$\sum_{s \leq T} (\tilde{\beta}^{T-t} \tilde{v}_T - \tilde{\beta}^{s-t} p_s) x_{is}.$$

Integrating over future possible liquidation dates, we see that at date t, informed agent i trades to maximize expected discounted lifetime trading profits:

$$\max_{\{x_{iT}\}_{T \geq t}} E_t \left[\sum_{T=t}^{\infty} \pi(1-\pi)^{T-t} \left[\sum_{\tau \leq T} (\tilde{\beta}^{T-t} \tilde{v}_T - \tilde{\beta}^{\tau-t} p_\tau) x_{i\tau} \right] \mid \mathbf{\tilde{e}_{it}}, \mathbf{\Omega_t}, \mathbf{x_{it-1}} \right]. \qquad (1)$$

Note that we can decompose informed agent i's expected discounted lifetime trading profits into a component due to past trading that he no longer controls, and a component reflecting current and future trading that he seeks to maximize. As a result, agent i's optimization problem is identical in structure each period.

3. ANALYSIS

Our equilibrium analysis mirrors that of Back *et al.* (2000) in that we solve a control problem and restrict attention to equilibrium path outcomes. We also focus on stationary linear equilibria. As we later discuss, our equilibrium corresponds to the unique linear Markov equilibrium of the limiting finite horizon economy. When agents adopt stationary strategies, the private- and public-information histories $\tilde{\mathbf{e}}_{\mathbf{it}}$ and $\mathbf{\Omega_t}$ fully determine $\mathbf{x_{it}}$. Hence, conditioning on $\mathbf{x_{it-1}}$ is redundant.

We next rewrite agent i's optimization problem in an analytically more way in which the dating on prices, orders and expected contributions to the value process all correspond.

LEMMA 3.1: *Agent i's objective can be written as:*

$$\max_{\{x_{i\tau}\}_{\tau\geq t}} E_t\left[\sum_{\tau=t}^{\infty}[\tilde{\beta}(1-\pi)]^{\tau-t}\left(\frac{\pi}{1-\rho\tilde{\beta}(1-\pi)}\tilde{v}_\tau - p_\tau\right)x_{i\tau} \mid \tilde{\mathbf{e}}_{\mathbf{it}}, \mathbf{\Omega_t}\right]. \tag{2}$$

PROOF: The proof to this lemma and all succeeding results are presented in Appendix B. ■

Here, $\frac{\pi}{1-\rho\tilde{\beta}(1-\pi)}\tilde{v}_\tau$ is essentially a one-period security that pays off in period τ, corresponding to the firm's expected liquidation value given date τ information, integrating over possible liquidation dates.

Letting $\beta = \tilde{\beta}(1-\pi)$, and $m = \frac{\pi}{1-\rho\tilde{\beta}(1-\pi)}$, we compactly rewrite informed agent i's objective as

$$\max_{\{x_{i\tau}\}_{\tau\geq t}} E_t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left(m\tilde{v}_\tau - p_\tau\right)x_{i\tau} \mid \tilde{\mathbf{e}}_{\mathbf{it}}, \mathbf{\Omega_t}\right].$$

Define the adjusted innovations

$$e_{i\tau} \equiv m\tilde{e}_{i\tau}.$$

The sequel will treat these as the innovations, so that informed agent i's restated objective becomes

$$\max_{\{x_{it}\}} E_t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left(v_\tau - p_\tau\right)x_{i\tau} \mid \mathbf{e_{it}}, \mathbf{\Omega_t}\right],$$

where $v \equiv m\tilde{v}$.

Linear Trading Strategies and Pricing: We next develop the consequences of linear trading strategies and pricing for the objectives of the informed agents and the market maker. We can write the market maker's linear pricing function as

$$p_t = \sum_{s=0}^{\infty}\left[\lambda^s(X_{t-s} + u_{t-s})\right] \equiv \lambda(L)(X_t + u_t),$$

where we use the lag operator notation to simplify presentation. Thus, the market maker constructs price by linearly weighting current and past net order flow, where λ^s is the weight on net order flow from period $t-s$. We further conjecture that the informed traders adopt period trading strategies that are linear functions of both the history of their private signals and the history of public information net order flows. Specifically, we conjecture that trader i's order in period t is given by:

$$x_{it} = \sum_{s=0}^{\infty}\Big[b_{ie}^{s} e_{it-s} + b_{i\Omega}^{s}(X_{t-s} + u_{t-s})\Big] \equiv b_{ie}(L)e_{it} + b_{i\Omega}(L)(X_t + u_t). \tag{3}$$

We call b_{ie} the *private-information trading intensity filter* and call $b_{i\Omega}$ the *public-information trading intensity filter*. In equilibrium, the sum of order flows from each informed agent induced by these strategies together with liquidity trade must generate the total net order flow, so that

$$X_t + u_t = \sum_{i=1}^{N} x_{it} + u_t = \sum_{i=1}^{N}\big[b_{ie}(L)e_{it} + b_{i\Omega}(L)\big(\sum_{j=1}^{N} x_{jt} + u_t\big)\big] + u_t.$$

Solving for the net period order flow yields

$$\sum_{i=1}^{N} x_{it} + u_t = \sum_{i=1}^{N}\left[\frac{b_{ie}(L)}{\left[1 - \sum_{j=1}^{N} b_{j\Omega}(L)\right]} e_{it}\right] + \frac{1}{\left[1 - \sum_{j=1}^{N} b_{j\Omega}(L)\right]} u_t.$$

Trader i controls how he trades on his private signals, b_{ie}, and the public information order flow, $b_{i\Omega}$, to maximize expected lifetime trading profits. Rather than solve this problem directly, we reformulate trader i's optimization problem so that he consciously controls the market quantities influenced by his trade. Define the adjusted public-information filter

$$\gamma_i(L) \equiv \frac{b_{i\Omega}(L)}{1 - \sum_{j=1}^{N} b_{j\Omega}(L)}.$$

In equilibrium, agent i controls his public-information filter $b_{i\Omega}(L)$, taking as given the strategies $\tilde{b}_{j\setminus i\Omega}(L)$ of other traders. To emphasize that trader i controls γ_i in the same sense as he controls $b_{i\Omega}$, write γ_i as

$$\gamma_i(L) \equiv \frac{b_{i\Omega}(L)}{1 - \sum_{j=1}^{N} \tilde{b}_{j\setminus i\Omega}(L) - b_{i\Omega}(L)}.$$

Trader i's objective, equation (2), can be stated in terms of the γ_i and $\{b_{ie}\}$. Expressing the objective in matrix form and suppressing the lag operator notation yields

$$\max_{b_{ie},\gamma_i} E\left[\sum_{t=0}^{\infty}\beta_i^t \left(\begin{pmatrix} \phi - \lambda(b_{1e} + \sum_{j=1}^{N}\gamma_j b_{1e}) & \dots & \phi - \lambda(b_{Ne} + \sum_{j=1}^{N}\gamma_j b_{Ne}) & -\lambda(\sum_{j=1}^{N}\gamma_j + 1)\end{pmatrix}\begin{pmatrix} e_{1,t} \\ \vdots \\ e_{N,t} \\ u_t\end{pmatrix}\right)\right.$$
$$\left.\times\left(\begin{pmatrix}\gamma_i b_{1e} & \dots & (b_{ie} + \gamma_i b_{ie}) & \dots & \gamma_i b_{Ne} & \gamma_i\end{pmatrix}\begin{pmatrix} e_{1,t} \\ \vdots \\ e_{N,t} \\ u_t\end{pmatrix}\right)\right]. \tag{4}$$

The product in the first line of trader's i's objective is the difference between the firm's value, represented by $\phi(L)\sum_{k=1}^{N} e_{kt}$, and the market-maker's forecast of the firm's value, $\lambda(L)(1+\sum_{j=1}^{N}\gamma_j(L))(\sum_{k=1}^{N} b_{je}(L)e_{kt}+u_t)$: it is the market maker's forecast error of the firm's value. The second line is agent i's trade, which, when we multiply out and re-arrange, we see is the sum of his filtering of the trades of other agents and liquidity trade, $\gamma_i(L)(\sum_j b_{je}(L)e_{j,t}+u_t)$, plus his trade on his own private information, $b_{ie}(L)e_{i,t}$.

Note that we have reposed the speculator's conditional optimization problem (2) as an unconditional optimization problem. We can do this because equlibrium trading strategies and pricing are both linear. That is, the same linear trading rule (i.e., the coefficients of the linear function) maximizes expected profit given *any* equilibrium path of private information and price realizations. Integrating over all such histories yields the unconditional expected profit using this "conditional" trading rule. But the same trading strategy must also maximize unconditional expected profits, i.e., objective (4). To see this, note that a speculator maximizing objective (4) *could* employ the conditional trading rule that maximizes (2), so that the trading rule maximizing (4) must do at least as well as the conditional trading rule. Conversely, in the conditional problem, the trader could choose the unconditional trading rule. But, if he ever finds that unconditional rule suboptimal, then integrating over all realized histories yields an unconditional expected profit that exceeds the unconditional profit attained using the unconditional trading rule. But this contradicts the optimality of the unconditional trading rule. Hence, the conditional and unconditional trading rules must correspond.

Having set the speculator's optimization problem up as a stationary, unconditional optimization problem, we can now exploit certainty equivalance and transform the problem to the frequency domain and solve the transformed problem. Because these techniques are unfamiliar to many economists, Appendix A motivates this transformation and details the algorithm's mechanics. Re-stating informed trader i's objective (4) in frequency domain form yields

$$\max_{b_{ie}(\cdot),\gamma_i(\cdot)} \frac{1}{2\pi i}\oint \mathrm{tr}\left\{ \begin{pmatrix} \phi-\lambda(1+\sum_{j=1}^{N}\gamma_j)b_{1e} \\ \vdots \\ \phi-\lambda(1+\sum_{j=1}^{N}\gamma_j)b_{ie} \\ \vdots \\ \phi-\lambda(1+\sum_{j=1}^{N}\gamma_j)b_{Ne} \\ -\lambda(1+\sum_{j=1}^{N}\gamma_j) \end{pmatrix} \begin{pmatrix} \gamma_i^* b_{1e}^* & \dots & (1+\gamma_i^*)b_{ie}^* & \dots & \gamma_i^* b_{Ne}^* & \gamma_i^* \end{pmatrix} \times \begin{pmatrix} \sigma_{1e}^2 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & \sigma_{Ne}^2 & 0 \\ 0 & 0 & \dots & 0 & \sigma_u^2 \end{pmatrix} \right\}\frac{dz}{z}, \tag{5}$$

where the integration is counterclockwise around the unit circle. The objective takes a vector form because there are $N+1$ fundamental processes: the N innovation processes $\{e_{jt}\}$, and the noise trade process u_t. Since these processes are mutually independent, the covariance matrix is diagonal. The column vector in i's objective again corresponds to the market maker's forecast error of the firm's value. The row vector $(\gamma_i^* b_{1e}^* \dots (1+\gamma_i^*)b_{ie}^* \dots \gamma_i^* b_{Ne}^* \gamma_i^*)$ expresses his order,

$$\gamma_i(L)\Big(b_{1e}(L)e_{1t}+\ldots+b_{ie}(L)e_{it}+\ldots+b_{Ne}(L)e_{Nt}+u_t\Big)+b_{ie}(L)e_{it}.$$

The first term in trader i's order is his trade on the public information net order flow. We will show that it is equal to (minus) the projection of b_{ie} onto the public information history of net order flows.

Trader i's total order flow is then equal to the difference between his trade on private information, and the projection of this trade on the net order flow—it is equal to his forecast of the error in the market maker's forecast of his trade on private information. Finally, the variance-covariance matrix captures the fact that we have already passed the expectation operator through, so that the e_{jt} do not appear in the row and column vectors as processes.

<u>The market-maker's problem</u>. The market-maker acts competitively, setting the price that yields him zero expected profits given the public information history of net order flows. This implies that the price is equal to the expected value of the firm given the history of net order flows. Since period net order flow is normally distributed, conditional forecasts are linear. Consequently, the market maker's pricing function solves the following linear-least-squares prediction problem:

$$\min_{\lambda} E\left[\sum_{t=0}^{\infty} \beta^t (v_t - \lambda(L)(\sum_{i=1}^{N} x_{it} + u_t))^2\right].$$

That is, the price is equal to the projection of the firm's value onto the net order flow history. We next substitute for the linear trading strategies and express the prediction problem in frequency-domain form:

$$\min_{\lambda(\cdot)} \frac{1}{2\pi i} \oint \mathrm{tr}\left\{ \begin{pmatrix} \phi - \lambda b_{1e}(1+\sum_{j=1}^N \gamma_j) \\ \vdots \\ \phi - \lambda b_{ie}(1+\sum_{j=1}^N \gamma_j) \\ \vdots \\ \phi - \lambda b_{Ne}(1+\sum_{j=1}^N \gamma_j) \\ -\lambda(1+\sum_{j=1}^N \gamma_j) \end{pmatrix} \right.$$
$$\times \left(\phi^* - \lambda^* b_{1e}^*(1+\textstyle\sum_{j=1}^N \gamma_j^*) \ldots \phi^* - \lambda^* b_{ie}^*(1+\sum_{j=1}^N \gamma_j^*) \ldots \phi^* - \lambda^* b_{Ne}^*(1+\sum_{j=1}^N \gamma_j^*) - \lambda^*(1+\sum_{j=1}^N \gamma_j^*) \right)$$
$$\left. \times \begin{pmatrix} \sigma_{1e}^2 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & \sigma_{Ne}^2 & 0 \\ 0 & 0 & \ldots & 0 & \sigma_u^2 \end{pmatrix} \right\} \frac{dz}{z}.$$

We next define our stationary equilibrium. In an equilibrium, (i) all agents optimize and (ii) aggregated information is consistent with that optimization:

DEFINITION 1: *A stationary linear equilibrium is a collection* $\{b_{ie}\}, \{\gamma_i\}, \lambda$ *such that:*

(i) The trading intensity filters $\{b_{ie}\}$ *and public-information filters* $\{\gamma_i\}$ *solve the optimization problem of each trader* i, $i = 1, \ldots, N$.

(ii) The price filter λ *solves the competitive market maker's prediction problem.*

(iii) Public information is equivalent to the history of net order flows arising from the optimization by the informed traders and the competitive market maker.

To construct the equilibrium we develop the first-order conditions to the optimization problems for the informed traders and the market maker.

<u>First-order conditions</u>. The solution process follows the steps outlined in Appendix A. Each trader takes as given the trading intensity filters b_{je}, γ_j of the other traders. Trader i solves his frequency-domain objective by constructing a variation $b_{ie} + \alpha\zeta$ and $\gamma_i + \alpha\zeta$. Taking the variational derivatives for b_{ie} and

γ_i, the first-order conditions for trader i are the following Wiener-Hopf equations:

$$(b_{ie}) \qquad [(\phi - \lambda b_{ie}(1 + \sum_{j=1}^{N} \gamma_j))(1 + \gamma_i^*) - \lambda^* b_{ie}(1 + \sum_{j=1}^{N} \gamma_j^*)(1 + \gamma_i)]\sigma_{ie}^2 = \sum_{-\infty}^{-1}.$$

(γ_i)

$$[(\phi - \lambda b_{1e}(1 + \sum_{j=1}^{N} \gamma_j))b_{1e}^* - \lambda^* b_{1e}^* b_{1e} \gamma_i]\sigma_{1e}^2 + \ldots + [(\phi - \lambda b_{ie}(1 + \sum_{j=1}^{N} \gamma_j))b_{ie}^* - \lambda^* b_{ie}^* b_{ie}(1 + \gamma_i)]\sigma_{ie}^2 + \ldots$$
$$+[(\phi - \lambda b_{Ne}(1 + \sum_{j=1}^{N} \gamma_j))b_{Ne}^* - \lambda^* b_{Ne}^* b_{Ne} \gamma_i]\sigma_{Ne}^2 + [-\lambda(1 + \sum_{j=1}^{N} \gamma_j) - \lambda^* \gamma_i]\sigma_u^2 = \sum_{-\infty}^{-1}.$$

We follow standard convention and use $\sum_{-\infty}^{-1}$ as shorthand for an arbitrary function that has only negative powers of z, and hence cannot be part of the solution to an agent's optimization problem; and we abbreviate notation, writing f instead of $f(z)$, and f^* instead of $f(\beta z^{-1})$.

Because the covariance matrix is diagonal, these equations simplify to a one-dimensional form. The market-maker's first-order condition is

$$(\lambda) \qquad \sum_{k=1}^{N}(\phi - \lambda b_{ke}(1 + \sum_{j=1}^{N} \gamma_j))b_{ke}^*(1 + \sum_{j=1}^{N} \gamma_j^*)\sigma_{je}^2 - \lambda(1 + \sum_{j=1}^{N} \gamma_j)(1 + \sum_{j=1}^{N} \gamma_j^*)\sigma_u^2 = \sum_{-\infty}^{-1}.$$

We first develop the solution for b_{ie}. Re-arranging the first-order condition for b_{ie} yields

$$b_{ie}[\lambda(1 + \sum_{j=1}^{N} \gamma_j)(1 + \gamma_i^*) + \lambda^*(1 + \sum_{j=1}^{N} \gamma_j^*)(1 + \gamma_i)]\sigma_{ie}^2 = \phi(1 + \gamma_i^*)\sigma_{ie}^2 + \sum_{-\infty}^{-1}.$$

The coefficient of σ_{ie}^2 on the left-hand side is the sum of complex conjugates. Because of this symmetry, by the theorem of Rozanov (1967), it can be factored into the product of an analytic, invertible function $g_i(z)$ and its conjugate,

$$g_i g_i^* \equiv \lambda(1 + \sum_{j=1}^{N} \gamma_j)(1 + \gamma_i^*) + \lambda^*(1 + \sum_{j=1}^{N} \gamma_j^*)(1 + \gamma_i). \tag{6}$$

Then the first-order condition becomes

$$b_{ie}(g_i g_i^*) = \phi(1 + \gamma_i^*) + \sum_{-\infty}^{-1},$$

with solution

$$b_{ie} = g_i^{-1}[g_i^{*\,-1}\phi(1 + \gamma_i^*)]_+, \tag{7}$$

where $[\cdot]_+$ is the *annihilator operator* that sets the coefficients of negative powers of z in the Laurent expansion to zero, while preserving all coefficients on the non-negative powers of z to obtain a feasible solution to an agent's optimization problem.

Turning to the first order condition for γ_i, rearrange it to obtain

$$[\lambda \sum_{j=1}^{N} \gamma_j + \lambda^* \gamma_i][\sum_{j=1}^{N} b_{je} b_{je}^* \sigma_{je}^2 + \sigma_u^2] = \sum_{j=1}^{N}(\phi - \lambda b_{je})b_{je}^* \sigma_{je}^2 - \lambda^* b_{ie} b_{ie}^* \sigma_{ie}^2 - \lambda \sigma_u^2 + \sum_{-\infty}^{-1}. \tag{8}$$

To solve for γ_i two factorization steps are required. In the first step, we define the process $J(L)w_t$:

$$J(L)w_t = \sum_{j=1}^{N} b_{je}(L)e_{jt} + u_t.$$

Rozanov's factorization theorem again implies that we can choose $J(z)$ and w_t such that

$$JJ^* = \sum_{j=1}^{N} b_{je}b_{je}^*\sigma_{je}^2 + \sigma_u^2, \tag{9}$$

where $J(z)$ is both analytic and invertible, and σ_w^2 is normalized to 1. As a result, the inverse, $J(z)^{-1}$, is analytic. Net order flow is $(1 + \sum_{k=1}^{N} \gamma_k(L))(\sum_{j=1}^{N} b_{je}(L)e_{jt} + u_t)$, so that $J(L)w_t$ is essentially the direct informationally-based order flow, or the order flow gross of the trade based on the informed agents' filtering of net order flow. The function $J(z)$ thus represents this direct order flow process.

We now use the solution for λ to simplify the solution for γ_i.

LEMMA 3.1: $\gamma_i = -J^{-1}[J^{*\,-1} b_{ie} b_{ie}^* \sigma_{ie}^2]_+$.

Lemma 3.1 reveals that γ_i is, indeed, the projection of trader i's private information trading intensity process onto the public information net order flow process. The negative sign in the formula for γ_i emphasizes that this projected information is subtracted from i's gross order flow—trader i trades less aggressively to the degree that his private information can be inferred from the net order flow history.

The following result facilitates a further simplification.

LEMMA 3.2: $1 + \sum_{j=1}^{N} \gamma_j$ *is invertible.*

Exploiting this result, the solution for λ simplifies to

$$\lambda = (1 + \sum_{k=1}^{N} \gamma_k)^{-1} J^{-1} [J^{*\,-1} \phi(\sum_{k=1}^{N} b_{ke}^* \sigma_{ke}^2)]_+. \tag{10}$$

Note that λ is invertible: This is because $(1 + \sum_{k=1}^{N} \gamma_k)^{-1}$ and J^{-1} are invertible by construction, and by an application of Proposition A.1, $[J^{*\,-1} \phi(\sum_{k=1}^{N} b_{ke}^* \sigma_{ke}^2)]_+$ is proportional to ϕ, which is invertible.

In the market-maker's objective, λ appears only in the product $\lambda(1 + \sum_{j=1}^{N} \gamma_j)$, which, in turn, operates on the net order flow process. Because of this it is convenient to define $\mu \equiv \lambda(1 + \sum_{j=1}^{N} \gamma_j)$ to be the market maker's filter of the direct order flow process, $J(L)w_t$, where

$$\mu \equiv \lambda(1 + \sum_{j=1}^{N} \gamma_j) = J^{-1}[J^{*\,-1} \phi(\sum_{k=1}^{N} b_{ke}^* \sigma_{ke}^2)]_+. \tag{11}$$

One can think of the price process as being captured by $\mu(L)J(L)w_t$, where w_t is the fundamental innovation process for net order flow constructed by the factorization J in (9). Hence, μJ yields the structure of the price process. Recall that the value process is such that $\phi(L)e_{it}$ is AR(1). From Proposition A.1 in appendix A, the annihilate of the product of non-analytic functions $f^* = f(\beta z^{-1})$ with $\phi(z)$ is equal to a constant times $\phi(z)$. Applying Proposition A.1, the price process is given by

$$\mu(L)J(L)w_t = C\phi(L)w_t, \tag{12}$$

where C is the appropriate constant. Equation (12) reveals that the market-maker's price filter undoes the complicated autoregressive structure of the order flow and converts it back to a process with the same AR(1) structure as the value process. That is, λ inverts informed agents' trades on net order flow $1 + \sum \gamma_j$ in order to filter the direct, informationally-based portion of the order flow process, $J(L)w_t$. Of course, the fundamental innovation of the price process w_t differs from the innovation underlying the value process because it is influenced directly by the noise trade process u_t and indirectly by traders' attempts to trade strategically through the structure of the b_{ie} functions.

4. EQUILIBRIUM

In this section we establish the existence of equilibrium. The formulas for $\{b_{ie}\}$, $\{\gamma_i\}$, and λ are the outcomes of the optimization problems that we have presented. The solutions of the optimization problems express the best responses of each trader to the actions of the other traders. The equations are nonlinear so that a closed-form solution cannot be found directly. This leads us to consider a sequence of iterative best responses by an informed agent, taking the k^{th} iteration round as describing the behavior of the other agents, and then solving for the informed agent's best response to obtain the $(k+1)^{th}$ iteration. We show that this best-response approximation converges. This implies that an equilibrium exists and is given by the limit of the best-response approximation.

We begin the iteration by assuming that trading intensities on private information match the valuation process, $b^1_{ie}(L)e_{it} = \frac{1}{1-\rho L} e_{jt}$, where the superscript indexes the iteration. We then solve for pricing, $\lambda^1(L)(X_t + u_t) = \frac{\ell^1_1}{1-\rho L}(X_t + u_t)$, and trading intensities on net order flow, $\gamma^1_j(L)(X_t + u_t) = \frac{g_1}{1-h_1 L}(X_t + u_t)$. We then iterate. We establish inductively that on the k^{th} interation, trading intensity on private information is given by a k-order sum of AR(1) terms,

$$b^k_{ie} = c^k_0 \Big(\frac{c^k_1}{1-\rho z} + \sum_{\ell=2}^{k} \frac{c^k_\ell}{1-a^k_\ell z} \Big), \tag{13}$$

where $\sum_{\ell=1}^{k} c^k_\ell = 1$ and $a^k_1 = \rho > a^k_2 > a^k_3 > ... > a^k_k > 0$. Thus, weights on AR(1) terms in successive iterations are interspersed and the first autoregressive coefficient is ρ, the autoregressive parameter of the valuation process. Pricing continues to evolve according to an AR(1),

$$\lambda^k(L)(X_t + u_t) = \frac{\ell^k_1}{1-\rho L}(X_t + u_t),$$

as the market maker unravels the strategic trading to match the valuation process. Finally, trading on the public information net order flows is the k-order sum of AR(1) terms,

$$\gamma^k(L)(X_t + u_t) = \sum_{i=1}^{k} \frac{g^k_i}{1-h^k_i L}(X_t + u_t).$$

The limit of this mapping generates a fixed point in which the sum in (13) is infinite; this expresses the infinite iteration of the best responses of each trader to the strategies of the other traders. Our main propositions establish that this solution is consistent with the optimization of the informed traders and the market maker and the imposition of the definition of informational equilibrium.

Iteration $k+1$ first uses the internal structure of the private information trading intensity filters $\{b^k_{ie}\}$ and maps them into a list of iterated functions $(\{\gamma^k_i\}, \{g^k_i\}, J^k, \mu^k)$. Specifically, given b^k_{ie}, we compute J^k from the definition of J in equation (9). Then J^k and b^k_{ie} determine γ^k_i using Lemma 3.1. Finally, b^k_{ie} and J^k are used to calculate μ^k using equation (11). We then use $(\{\gamma^k_i\}, \{g^k_i\}, J^k, \mu^k)$ to calculate the next value of the trading intensity filter b^{k+1}_{ie} using equation (7). We express these steps notationally by the mapping T, with $T : \{b^k_{ie}\} \mapsto \{b^{k+1}_{ie}\}$.

Definition 2: *Let ϕ be* AR*(1) with $\phi = (1-\rho z)^{-1}$. Define $\mathcal{H}(\beta)$ to be the space of analytic functions on $\Delta(\beta) \equiv \{z : |z| < \beta^{1/2}\}$. Define C to be the space of all functions on a domain in the complex plane. Define $T : \mathcal{H}(\beta) \to C(\Delta(\beta))$ to be the mapping generated by the equations defining the functions $\{b_{ie}\}$, $i = 1, \ldots, N$, and the corresponding $\{\gamma_i\}$, $\{g_i\}$, J and μ functions.*

Definition 3: *Define $\tilde{T} : \ell_\infty \times \ell_\infty \to \ell_\infty \times \ell_\infty$ to be the mapping T in terms of the coefficients c^k_ℓ and a^k_ℓ of the analytic functions in b_{ie}.*

<u>An iterative formula for a^k_i</u>. Start with $k = 1$. Set $a^1_1 = \rho$ with $a^1_j = 0$ for $j > 1$, and set $c^1_0 = c^1_1 = 1$ with $c^1_j = 0$ for $j > 1$. Calculating $\tilde{T}[(1, 1, 0, \ldots), (\rho, 0, \ldots)]$ yields $(c^2_0, c^2_1, c^2_2, 0, \ldots)$ and $(\rho, a^2_2, 0, \ldots)$, and on the third round, $(c^3_0, c^3_1, c^3_2, c^3_3, 0, \ldots)$ and $(\rho, a^3_2, a^3_3, 0, \ldots)$. Iteratively applying the mapping $\tilde{T}$ generates $b^k_{ie} = c^k_0\Big(\frac{c^k_1}{1-\rho z} + \sum_{\ell=2}^k \frac{c^k_\ell}{1-a^k_\ell z}\Big)$, $k = 2, 3, \ldots$ with associated sequences

$$(\{a^1_1\}, \{a^2_1, a^2_2\}, \ldots, \{a^k_1, a^k_2, \ldots, a^k_{k-1}, a^k_k\}, \ldots), \;\text{ and } (\{c^1_0, c^1_1\}, \{c^2_0, c^2_1, c^2_2\}, \ldots, \{c^k_0, c^k_1, c^k_2, \ldots, c^k_{k-1}, c^k_k\}, \ldots).$$

We now show that iterating on $\tilde{T}$ preserves the first autoregressive term of an intermediate b_{ie} but changes its relative weight; subsequent terms are autoregressive with shrinking numerator coefficients (the c^k_ℓ) and shrinking autoregressive coefficients (the a^k_ℓ). We establish inductively the *key* properties that (i) the new coefficients a^k_ℓ lie in a bounded interval $(0, \rho)$, and (ii) $\sum_{\ell=1}^k c^k_\ell = 1$:

Lemma 4.1: *Take any sequence pair $(c^k_0, c^k_1, c^k_2, \ldots, c^k_k, 0, \ldots)$ and $(\rho, a^k_2, \ldots, a^k_k, 0, \ldots)$ such that $\sum_{\ell=1}^k c^k_\ell = 1$, with $|c^k_0| < \infty$, and such that $\rho = a^k_1 > a^k_2 > \ldots a^k_k > 0$. Then*

$$\tilde{T}[(c^k_0, c^k_1, c^k_2, \ldots, c^k_k, 0, \ldots), (\rho, a^k_2, \ldots, a^k_k, 0, \ldots)] = ((c^{k+1}_0, c^{k+1}_1, c^{k+1}_2, \ldots, c^{k+1}_{k+1}, 0, \ldots), (\rho, a^{k+1}_2, \ldots, a^{k+1}_{k+1}, 0, \ldots))$$

where (i) $\sum_{\ell=1}^{k+1} c^{k+1}_\ell = 1$, and (ii) $\rho = a^{k+1}_1 = a^k_1 > a^{k+1}_2 > \ldots > a^{k+1}_k > a^{k+1}_{k+1} \geq 0$.

Note that the characterization in Lemma 4.1 holds even for sequence pairs not generated by the $\tilde{T}$ mapping. Of course, it also holds for the sequence generated by the $\tilde{T}$ mapping that commences with the “naive” value of the private information trading-intensity filter, $b^1_{ie}(z) = \frac{1}{1-\rho z}$, that mimics the autoregressive structure of the value process:

We now prove that there is a unique stationary linear equilibrium.

Lemma 4.2: *Let*

$$b^k_{ie} = c^k_0\Big(\frac{c^k_1}{1-\rho z} + \sum_{\ell=2}^k \frac{c^k_\ell}{1-a^k_\ell z}\Big)$$

with $|a^k_\ell| < \beta^{-1/2}$, $\sum_{\ell=1}^k c^k_\ell = 1$, and $\rho = a^k_1 > a^k_2 > \ldots > a^k_k$. Then $T[b^k_{ie}] \to \mathcal{H}(\beta)$.

Proposition 4.3: *T has a fixed point $\tilde{b}$ in $\mathcal{H}(\beta)$.*

Proposition 4.4: *There is a unique stationary linear equilibrium.*

The proof uses a contraction mapping argument on the space of functions of a complex variable on the unit disk, which has a tractable structure—the frequency-domain methods prove extremely useful here. More specifically, we prove that a contraction argument applies to iterations of the J function, where we recall $J(L)w_t = \sum_{j=1}^{N} b_{je}(L)e_{jt} + u_t$ is the information-based portion of order flow left after subtracting out the portion based on the filtering of public information.

The contraction argument is easier to establish indirectly via J rather than b_{je} because the non-linearity of the spectral factorizations of J and g prevent a direct recursive formula for b_{je}. In a symmetric setting, substituting for b_{je} from (7) into JJ^* results in the term $(gg^*)^{-1}$, which can be translated into a term of J and ϕ, and in particular is not a function of b_{je}. We therefore have a recursive expression for J, and we show that the modulus of this recursion is a fraction, implying a contraction mapping.

Once we have the fixed point of this contraction mapping, we can solve for the primitive functions, b, γ, and μ. The equilibrium private trading intensity filter is characterized by an infinite sequence of numerator coefficients $\{\bar{c}_\ell\}$ and autoregressive coefficients $\{\bar{a}_\ell\}$ with the same characteristics as the iterated versions $\{c_\ell^k\}$ and $\{a_\ell^k\}$.

PROPOSITION 4.5: *Let* $((\bar{c}_0, \bar{c}_1, \bar{c}_2, \ldots), (\bar{a}_1, \bar{a}_2, \ldots)) = \lim_{k\to\infty} \tilde{T}^k[(1, 1, 0, \ldots), (\rho, 0, \ldots)]$. *Then*

$$\rho = \bar{a}_1 > \bar{a}_2 > \ldots \quad > 0, \qquad \textit{and} \qquad \sum_{\ell=1}^{\infty} \bar{c}_\ell = 1.$$

Proposition 4.5 shows that private information is used directly in that the first b_{ie} term matches the autoregressive structure of the value process, ϕ. Subsequent terms in the private-information trading intensity filter feature smaller autoregressive parameters, a_ℓ. Ultimately, $c_\ell > 0$, and this means that informed agents trade relatively less aggressively on older information—that is, their order flow on a piece of information decays *faster* than ρ. This is because agents' private information conditional on public information becomes increasingly negatively correlated with age.

With the basic properties of the trading intensity filter established, we now derive the functional forms of the ancillary functions. We first show that the autoregressive structure of the public information process $J(L)w_t$ is driven by the structure of the private-information trading intensity filters b_{ie}.

PROPOSITION 4.6: *The direct, informationally-based order flow process has the same autoregressive basis as the private trading intensity filter* $b_{ie}(z)$*:*

$$J(z) = \frac{j_1}{1 - \rho z} + \sum_{\ell=2}^{\infty} \frac{j_\ell}{1 - a_\ell z}.$$

The next result details that there is a common basis for the functions μ and γ_i.

PROPOSITION 4.7: *A trader's filter on the public information net order flow process*

$$\gamma_i(z) = \sum_{\ell=1}^{\infty} \frac{p_\ell}{1 - f_\ell z}$$

has the same basis elements as the market-maker's filter on net order flow,

$$\mu(z) = \sum_{\ell=1}^{\infty} \frac{d_\ell}{1 - f_\ell z}.$$

To understand why μ and γ_i have the same autoregressive basis note that μ is the market-maker's pricing function is a projection of the unobservable value process onto the net order flow history. So, too, informed trader i's γ_i function is a projection of his own private information onto the net order flow history. However, while μ and γ_i have the same basis elements, the two functions have different coefficients on those elements because informed agents have more information than the market maker.

Discussion. In the finite horizon settings of Bernhardt and Miao (2004) and Foster and Viswanathan (1996) there is a unique linear Markov equilibrium. The key feature underlying this uniqueness is that every possible order flow path can arise in equilibrium. Consequently, even if, off the equilibrium path, some trader i erred when submitting his order, the other agents continue to believe that the economy is on the equilibrium path with probability one. Trader i recognizes that he made a sub-optimal trade in some period t, and in subsequent periods it is equivalent to agent i observing a piece of liquidity trade in period t—trading one hundred shares more than was optimal is equivalent to knowing that there was a liquidity trade of one hundred shares—and i trades linearly on this "information" in the future. The other agents continue to believe that they are on the equilibrium path, and trade accordingly.

We have proved that in our infinite horizon setting there is a unique stationary linear Nash equilibrium. Further, this equilibrium should correspond to the equilibrium path of the unique stationary linear Markov equilibrium. Again this is because any possible order-flow path can arise in equilibrium so that if an agent has never erred then in that subgame, he believes that with probability one that other agents are following their equilibrium strategies. Our infinite horizon setting is also likely to have other non-stationary (e.g., more collusive) equilibria. However, our methodology does not extend to characterize non-stationary or non-linear equilibria.

5. TRADING INTENSITY PROPERTIES

We now characterize the properties of equilibrium trading strategies, pricing and the dynamics of information. We first derive how the amounts of noise trade and private information about the asset affect trading strategies, pricing and profits. Clearly, increasing the variance of noise trade, σ_u^2, will lead informed agents to increase their trading intensities—the informed agents can glean additional profits as they can "hide" their greater trading behind the higher noise trade that complicates the market maker's inference problem. We provide a much tighter characterization than just demonstrating this. What we prove is that the variance of noise trade and the variance of the innovations to the asset value affect equilibrium strategies in a particularly simple way: they scale the trading intensity with which informed agents trade on private information.

Fixing all other parameters, let $b_{ie}(\sigma_e^2, \sigma_u^2)$ be the equilibrium private trading intensity function as a function of the variance of innovations to the asset value, and the variance of noise trade. Define the equilibrium pricing function, $\lambda(\sigma_e^2, \sigma_u^2)$ and informed profit function, $\pi_i(\sigma_e^2, \sigma_u^2)$ analogously. Finally, let

$$b_{ie}(1,1) = c_0 \left(\frac{c_1}{1-\rho z} + \sum_{\ell=2}^{\infty} \frac{c_\ell}{1-a_\ell z} \right)$$

be the equilibrium private trading intensity function when $\sigma_e^2 = \sigma_u^2 = 1$. Then,

PROPOSITION 5.1: *The equilibrium private trading intensity function is proportional to $\frac{\sigma_u}{\sigma_e}$:*

$$b_{ie}(\sigma_e^2, \sigma_u^2) = \frac{\sigma_u}{\sigma_e} b_{ie}(1,1) \quad \textit{and} \quad \gamma_i(\sigma_e^2, \sigma_u^2) = \frac{\sigma_u}{\sigma_e} \gamma_i(1,1).$$

Proposition 5.1 details that the variances of noise trade and asset innovations do not affect the autoregressive structure of the equilibrium trading strategies. In turn, this means that the impacts of these variances on pricing and informed profits also take simple forms:

PROPOSITION 5.2: *The equilibrium pricing function is proportional to* $\frac{\sigma_e}{\sigma_u}$*:*

$$\lambda(\sigma_e^2, \sigma_u^2) = \frac{\sigma_e}{\sigma_u}\lambda(1, 1).$$

The equilibrium profit function is proportional to $\sigma_e \sigma_u$*:*

$$\pi_i(\sigma_e^2, \sigma_u^2) = \sigma_e \sigma_u \pi_i(1, 1).$$

Propositions 5.1 and 5.2 demonstrate that the qualitative impact of the variances of noise trade and asset innovations found in simpler settings by other researchers—static trading environments by Kyle (1985) and Admati and Pfleiderer (1989), and by Kyle (1985) and Back (1992) for a single informed agent who has a single piece of private information and can trade arbitrarily frequently—is actually a very general property that holds in rich dynamic environments.

6. NUMERICAL CHARACTERIZATIONS

In this section we provide quantitative characterizations of equilibrium outcomes. Specifically, we characterize how (1) competition amongst informed agents, (2) the persistence in asset valuations, and (3) the relative amount of noise trade affect equilibrium levels of (a) informed trading intensity on current and past information, (b) informed profit, (c) market maker pricing, and (d) the amount of information revealed through price about current and past innovations.

The algorithm is complicated by two factors. The first factor is that by the nature of the algorithm, which repeatedly finds common denominators, cancellation of numerator and denominator terms of the form $(1 - a_1 z)/(1 - a_2 z)$, in which a_1 and a_2 are theoretically equal but numerically unequal, must be performed. In the control and systems-engineering literature, this cancellation is appropriately performed using so-called state-space methods, and in particular a minimal-realization algorithm.

The second complicating factor is a consequence of the best-response behavior of the traders. The iteration calculates the best response of a trader, fixing the strategies of the other traders. In practical terms, this means that the polynomial order of the other traders' trading-intensity filter is held fixed, while the active trader's trading-intensity filter is optimized. This, in turn, raises the polynomial order k of the active trader's trading-intensity filter. In practice, this results in an over-reaction by the active trader, which destabilizes the iterative algorithm. The way we circumvent this over-reaction is to limit the polynomial order of the best response until convergence has occured for that k, and only then allow an increment of k. In practical terms, the algorithm produces $k + 1$ pole terms and we use state-space methods, in particular a balanced-realization algorithm, to choose the pole to discard.

Numerical results. We now illustrate the numerical algorithm. In our base-case parameterization (i) there are two informed traders with discount factor $\beta = 0.95$, (ii) each e_{jt} is independently, normally distributed with zero mean and variance $\frac{1}{2}$, (iii) the autoregressive parameter of the asset value process is $\rho = 0.97$, so that innovations have a persistent impact on the asset's value, and (iv) the variance of noise trade each period is one, which matches the total variance of the innovation to the asset's value.

In the first iteration informed trading strategies on private information match the valuation process:

$$b_{je}^1(L)e_{jt} = \sum_{\tau=0}^{\infty} \rho^{\tau} e_{jt-\tau} \equiv \frac{1}{1-0.97L} e_{jt}.$$

The first few terms of the moving-average representation of this trading strategy are:

$$b_{je}^1(L)e_{jt} = 1.00e_{jt} + 0.970e_{jt-1} + 0.941e_{jt-2} + 0.913e_{jt-3} + 0.885e_{jt-4} + 0.859e_{jt-5} + 0.833e_{jt-6} + \ldots.$$

Then given the posited b_{je}^1, we use equation (9) to solve for

$$J^1(L)w_t = (.627 + \frac{.967}{1-.097L})w_t.$$

We next use b_{je}^1 and J^1 to solve for the trading intensity on net order flow using lemma 3.1,

$$\gamma_j^1(L)x_t = -\frac{.303}{1-.382L}x_t.$$

Finally, we use equation (11) to solve for

$$\mu^1 = \frac{0.606}{1-.382L}x_t.$$

In turn, using $\mu^1 = \lambda^1(1 + \sum_j \gamma_j^1)$, we solve for

$$\lambda^1(L)x_t = \frac{1.542}{1-0.97L}x_t \leftrightarrow p_t^1 = 0.97p_{t-1}^1 + 1.542x_t.$$

We then iterate, solving for how j should trade on his private information, using the first round iterations for pricing and filtering of public information as inputs:

$$b_{je}^2(L)e_{jt} = .609\left(\frac{.874}{1-0.97L}e_{jt} + \frac{.136}{1-0.297L}e_{jt}\right).$$

It is more helpful to compare the moving-average representation of this trading strategy,

$$b_{je}^2(L)e_{jt} = 0.609e_{jt} + 0.539e_{jt-1} + 0.508e_{jt-2} + 0.487e_{jt-3} + 0.457e_{jt-4} + 0.444e_{jt-5} + 0.430e_{jt-6} + \ldots +.$$

with that for the initial iteration, $b_{j\omega}^1(L)e_{jt}$. This reveals that in the second iteration, agent j reduces his trading intensity on *all* of his private information, curtailing especially sharply his trading intensity on older private information. Intuitively, the initial conjecture failed to incorporate the fact that private information was being revealed to the market through the price; the second iteration internalizes this, and, in particular, the more negative conditional correlations in the market maker's forecast errors of each informed agent's net private information at longer lags. Because of this negative conditional correlation j trades more aggressively on more recent private information relative to older information in the second iteration. Continuing, on successive iterations, the lag coefficients decline monotonically, with greater percentage reductions at longer lags. It takes six iterations for the algorithm to converge.

The market maker's equilibrium pricing function evolves according to an AR(1) with an autoregressive parameter that matches the valuation process,

$$\lambda^7(L)x_t = \frac{0.895}{1-0.97L}x_t \leftrightarrow p_t^7 = 0.97p_{t-1}^7 + .895x_t.$$

Finally, the moving average representation of an informed agent's equilibrium trading intensity on the public information net order flow is given by

Lag	0	1	2	3	4	5	6	7	8	9	10
$\gamma^7(L)x_t$	-0.101	-0.064	-0.043	-0.030	-0.021	-0.016	-0.012	-0.010	-0.008	-0.007	-0.006
$b^7_{je}(L)e_{jt}$	0.355	0.271	0.215	0.177	0.149	0.128	0.112	0.100	0.090	0.081	0.074

Contrasting the equilibrium trading intensities on private and public information reveals that trading intensities on the public information net order flow drop off more quickly at distant lags than do trading intensities on private information; the 10^{th} lag of the private information trading intensity filter is 21% of the 0^{th} lag, while the the 10^{th} lag of the net order flow trading intensity filter is only 6% of the 0^{th} lag. This reflects that accounting for j's trade on private information in the net order flow ceases to convey much information to j after only a few lags.

These findings lead to several questions: Why does j's trading intensity on older information drop off? how much private information remains at distant lags? and by how much does old private information contribute to agent j's period profit?

To address these questions, we decompose expected total informed agent period profit of 0.760 due to the first 10 lags of information by lag

Lag	0	1	2	3	4	5	6	7	8	9	10
$\sum_j \pi_{jt-\tau}$	0.359	0.172	0.090	0.051	0.031	0.020	0.013	0.010	0.007	0.005	0.004

Thus, relatively recent information contributes significantly to informed profits, but the contribution drops off sharply at distant lags. The most recent innovation contributes about 47% of the informed agent's profits; the three most recent innovations contribute 82% of the informed agent's expected profits; and his six most recent pieces of information contribute about 95% of his profit.

Two factors underlie j's decreasing ability to profit on older information: (i) some of j's private information is revealed over time through trade, so there is "less of it", and (ii) j's remaining net private information—the market maker's forecast error of the asset's valuation given j's information—becomes increasingly negatively correlated at distant lags with that of other traders.

To illustrate how j's private information declines at distant lags, we decompose by lag the contribution of past information to the variance of the market maker's forecast error of the asset's value. The total forecast error variance from the first 10 lags is $\sigma^2_{FE} = 3.47$, and the decomposition by lag is

Lag	0	1	2	3	4	5	6	7	8	9	10
σ^{2FE}_m	1.095	0.674	0.451	0.323	0.243	0.189	0.152	0.125	0.104	0.088	0.075

The forecast error variance associated with the 10^{th} lag is about 7% of that associated with current period innovations. Thus, distant lags still contribute to the market maker's forecast error.

We now compute the correlation in agents' order flows, and thus of net private information, by lag.

Lag	0	1	2	3	4	5	6	7	8	9	10
Correlation	-0.020	-0.233	-0.425	-0.580	-0.694	-0.775	-0.831	-0.871	-0.898	-0.918	-0.933

Because agents see the current period price, the conditional correlations between informed agents' private information are negative at all lags, even the 0^{th}. Further, agents's private information quickly becomes *very* negatively correlated at distant lags. It is this high negative correlation that underlies both j's inability to profit on forecast errors on $e_{jt-\tau}$, and the reduced trading intensity by agent j on older information, as a positive forecast error on one agent's net private information is associated with a largely offsetting negative forecast error on other agents' net private information.

<u>Competition.</u> We now explore how increased competition affects equilibrium outcomes when signals are independently distributed across agents. Table 2 presents equilibrium outcomes for $N = 2, 4$, and 32 agents. In each period t, informed agent j sees e_{jt}, an independently, normally distributed innovation with zero mean and variance $\frac{1}{N}$, so that the variance of total private information is fixed at one. The first panel reveals that aggregate variables—pricing, total informed profit and market maker forecast error—are remarkably insensitive to how information is divided among agents. In a static environment, when information is independently distributed across agents, aggregate equilibrium outcomes are completely unaffected by the division of information. In a dynamic framework, the division of information across agents matters, but not by very much. What underlies this result is that the division of information among informed agents affects informed trading strategies according to $\frac{N-1}{N}$. Going from two informed agents to 32, total informed profits fall by about 2%; and there is a similar percentage decline in the price impact of order flow. The market maker's forecast error—a measure of the information revealed through trade—rises by a little bit more with increased competition—about 5%—reflecting that greater competition causes agents to trade less aggressively on their information.

The second panel decomposes equilibrium variables by lag for different numbers of informed agents. This decomposition reveals that the impact of increased competition is more subtle than the first panel might suggest. The MA decomposition of b_{je}^N reveals that the increased competition causes agents to trade 7% more aggressively on current private information, but to reduce their trading intensity on older private information (so that when $N = 32$, agents trade more than 10% less aggressively on their private information at lags exceeding 2 than when $N = 2$). Underlying this result is that increased competition causes net private information to become far more negatively correlated at all lags. In turn, this higher negative correlation causes informed agents to trade less aggressively on older information. Anticipating this increased negative correlation at lags when information is divided among more agents, the agents choose to trade more aggressively on current information.

Similarly, increased competition alters the source of informed profits: informed profits from the current innovation are higher, but these profits come at the expense of reduced future trading profits. So, too, a decomposition of the variance of the market maker's forecast error reveals that as competition rises, trade reveals more information about the current innovation, but less about past innovations. Consequently, increased competition leads to more residual unrevealed private information at long lags.

<u>Persistence in valuation process.</u> Table 3 contrasts equilibrium outcomes for $\rho = 0.97, 0.75, 0.5$ in our base-case parameterization with two traders. As ρ falls, the contribution of the lagged innovations $e_{jt-\tau}$ to the asset's period t value, $\rho^\tau e_{jt-\tau}$ falls. Reducing ρ effectively reduces the total private information in the economy, as the variance of the valuation process is $\frac{N\sigma_e^2}{1-\beta\rho^2}$. Panel 1 illustrates that as the persistence in the valuation process is reduced, price becomes far more sensitive to current period order flow. Indeed, the price impact of order flow quickly approaches what it would be in a static environment ($p_t = X_t + u_t$ when $\rho = 0$). As ρ declines, informed profits fall, but by far less than proportionately to the reduction in the amount of information. This is because informed agents increase their trading intensity on newer private information (see the lag decomposition of b_{je}), while reducing their trading

intensity on older information (both because older information has a smaller impact on asset values and because information is conditionally more negatively correlated due to the greater trading intensity on the information when it was newer). In turn, this increased trading intensity due to a decline in ρ causes the market maker's forecast error to fall even more rapidly than informed profits.

1. The lag decomposition of γ_j reveals that as ρ is reduced, agents use their larger own current period own order flow to extract more information about current innovations from net order flow, while the projection of distantly-lagged private information onto net order flow is almost zero.
2. The lag decomposition of profits reveals that as ρ is reduced, agents extract increasing amounts of profits from current information (both in absolute and relative terms).
3. The lag decomposition of the market maker's forecast error variance reveals that reducing ρ raises informed trading intensities on newer information, reducing the contribution of current information to the forecast error variance. At lags this is reinforced by the reduced contribution of lagged innovations to the asset's value.
4. Agents' net private information becomes almost perfectly negatively correlated by the third lag.

Noise trade. Our analytical results imply that increasing σ_u^2 by a factor of four doubles informed trading intensities, halves the price sensitivity to net order flow and quadruples informed profits.

Correlated information. If traders' information is correlated, then the same frequency-domain methods can be used to solve the model, but the vector version of the methods set out at the end of Appendix A must be employed. Bernhardt, Seiler and Taub (2003) develop the vector methods in a multi-asset contexts. The vector methods demand a modification of the numerical analysis, and, in general, it is far more difficult to solve the pole-zero cancellation and spectral factorization problems. The demands of the numerical methods require that we consider parameterizations in which the persistence of the value process ρ, the degree of correlation of information, and the number of traders are limited.

We set $\rho = 0.5$, and consider the interaction between correlation and competition. Fixing $N = 2$ traders, increasing the correlation parameter modestly from $\theta = 0$ to $\theta = 0.025$, causes trading intensities on current information rise by about 6.5 percent, but trading intensities at other lags are only marginally affected, and informed profits only fall by about 1.5 percent. Increasing correlation to a high level, $\theta = 0.5$, causes trading intensities on recent information to rise sharply. In particular, relative to the uncorrelated case, trading intensities on current, first and second lags are 50, 40 and 33 percent higher, respectively. Intuituively, it is the race to trade on common information ahead of the other agent leads to far higher trading intensities. Still, relative to when signals are uncorrelated, total informed profits profits fall by only about 7 percent.

However, when we then maintain the high correlation, $\theta = 0.5$, and double the number of agents to $N = 4$, we see dramatic impacts. When signals were uncorrelated, increasing the number of agents had minimal effects. However, when signals are correlated, doubling the number of agents dramatically raises trading intensity on recent information by 40 percent, with far smaller trading intensity increases at longer lags. What drives this result is that each speculator has an incentive to trade on common information before the other speculators. Because trading intensities on recent information are so much higher, the conditional correlation in their information is reduced sharply at longer lags, so their incentive to trade on that information is far less. The increased trading intensity on recent information sharply reduces total informed profit by about 14 percent. Thus, we see that in dynamic contexts, what reduces informed profits is the combination of signal correlation and high numbers of informed agents.

7. CONCLUSION

How information is dispersed through prices has long been a central question in economics. We provide the foundation to explore this question. Our model of speculative informed trade in stock markets resides in an infinite horizon setting, so that our findings are unimpeded by finite horizon boundaries. We characterize precisely how information in one time period interacts with information from other periods. We show that the use of private information and its revelation through price never ends: each new realization of private information leads agents to re-interpret the history of private and public information.

We characterize analytically how the primitives of the model affect equilibrium outcomes—trading strategies, pricing, profit and information transmission. We prove that noise trade and private information proportionately scale trading strategies, pricing, profit and information transmission. We also show that competition slows the transmission of information, but that the quantitative impacts of competition are slight unless the signals that speculators receive are correlated, so that agents compete over trading on common information. The methods developed here extend to multi-asset settings. In ongoing work, Bernhardt, Seiler and Taub (2003) are deriving how speculators allocate their trades across assets, and how information in the price of one asset affects the prices and dynamics of order flows in other assets.

More generally, standard linear-quadratic models do not feature any interaction between private and public information filters. The methods we develop enable us to handle this interaction. Our iterative best-response algorithm is also new and provides a practical method for characterizing equilibrium outcomes. We anticipate the wide applicability of these techniques to economic models in which information is embodied in prices and agents strategically consider how their actions affect information flows.

Appendix A: z-Transform Methods

Consider a serially-correlated stochastic process a_t that can be expressed as a weighted sum of i.i.d. innovations:

$$a_t = \sum_{k=0}^{\infty} A_k e_{t-k}.$$

While the innovations change through time, the weights A_k remain fixed. The stochastic process can therefore be written succinctly as a function of the lag operator, L: $a_t = A(L)e_t$. The list of weights $\{A_k\}$ can be viewed as a sequence, and by a fundamental theorem of analysis (Riesz-Fischer theorem, see Rudin (1974), pp. 86-90), are equivalent to functions of a complex variable z. The function of the lag operator $A(L)$ is then mathematically equivalent to a function $A(z)$ of a complex variable z. The function $A(z)$ can be analyzed with the rules of complex analysis, and this, in turn, fully characterizes the stochastic process a_t.

An important feature of complex analysis is that the properties of a function are characterized by the domain over which they are specified—the unit disk, or sets that are topologically equivalent to the unit disk, are often the domains of interest. If a complex function on the disk can be expressed as a Taylor expansion—an infinite series where the powers of the independent variable, z, range from zero to infinity—then the function is said to be *analytic* on the disk. However, some functions, termed nonanalytic functions, when expressed as a generalized Taylor expansion—a Laurent expansion—have both positive and negative powers of z. This implies that they correspond to functions containing negative powers of the lag operator, which means that they operate on future values of a variable. If a variable is stochastic, this is not permissible, as it would mean that the future is predictable, contradicting its stochastic aspect. In particular, solutions to an agent's optimization problem cannot be forward-looking.

To eliminate negative powers of z in a posited solution to an agent's optimization problem, we use the *annihilator operator*, $[\cdot]_+$. The annihilator operator sets the coefficients of negative powers of z in the Laurent expansion to zero, while preserving all coefficients on the non-negative powers of z. This leaves a permissible, backward-looking solution to an agent's optimization problem.

A second property of a function concerns its invertibility. If a serially-correlated stochastic process can be represented by an invertible operator, the innovations of the process can be completely and exactly recovered by observing the history of the process. That is, the inverse of the operator applied to the vector of realizations of the process yields the vector of innovations, exactly as it would if a finite vector of innovations were converted into a finite vector of realizations by an invertible matrix. An analytic function is invertible on its domain if it does not take on a value of zero at a point inside the domain. If, instead, an analytic function takes on a value of zero at a point inside the domain, then it is *noninvertible*. The inverse of a noninvertible function is not analytic. Hence, one cannot recover the vector of innovations by observing the vector of realizations, because inverting a function with a zero results in a function with negative powers of z. Recovery of the innovations would then depend on knowledge of future realizations. The *factorization* theorem of Rozanov ensures that any process described by a z-transform with either negative powers of z or zeroes can be converted into an observationally-equivalent process that is characterized by an operator that is invertible and has only non-negative powers of z, so that it is backward-looking.

To illustrate the variational method that we employ in the frequency domain, we present a simple consumer optimization problem. Consider an individual whose earnings evolve stochastically according to $y_t = A(L)e_t$, where e_t is an i.i.d., zero mean, "white noise" period innovation to earnings. The consumer's problem is to adjust bond holdings $\{b_t\}_{t=0}^{\infty}$ to maximize quadratic utility,

$$\max_{\{b_t\}} -E\sum_{t=0}^{\infty}\beta^t(y_t + rb_{t-1} - b_t)^2, \tag{A.1}$$

where r is the gross interest rate satisfying $\beta r > 1$. [1] The decision problem is to choose not just the initial value of b_t, but the entire sequence $\{b_t\}_{t=0}^{\infty}$. This problem implicitly requires the choice of *functions* that react to current and possibly past states. Stationarity results in the same function applying each period.

The stochastic component of a quadratic utility function is essentially a conditional variance. If innovations are i.i.d., then the expectation of cross-products of random variables yields the sum of variances. For *white-noise* innovations, for $k > s$, $k > r$,

$$E_{t-k}\left[e_{t-r}e_{t-s}\right] = \begin{cases} 0, & r \neq s \\ \sigma_e^2, & r = s, \end{cases} \tag{A.2}$$

because of the independence of the innovations. Expressed in lag operator notation, this is

$$E_{t-k}\left[(L^r e_t)(L^s e_t)\right] = \begin{cases} 0, & r \neq s \\ \sigma_e^2, & r = s. \end{cases}$$

Notice that the "action" is in the exponents of the lag operators. From Cauchy's theorem (Conway, 1978), it is equivalent to write

$$\sigma_e^2\frac{1}{2\pi i}\oint z^r z^{-s}\,\frac{dz}{z} = \begin{cases} 0, & r \neq s \\ \sigma_e^2, & r = s, \end{cases}$$

where the integration is counterclockwise around the unit circle. In Cauchy's theorem, z, which is a complex number with unit radius (it is on the boundary of the disk), is represented in polar form: $z = e^{-i\theta}$. Now a more conventional integral can be undertaken, integrating over $\theta \in [0, 2\pi]$. Using Euler's theorem, which represents complex numbers in trigonometric form, $e^{-i\theta} = \cos\theta - i\sin\theta$, gives θ the interpretation of a frequency, so that z and functions of z are in the *frequency domain*.

Whiteman (1985) demonstrated that a discounted conditional covariance involving complicated lags can be succinctly expressed as a convolution. Consider two serially-correlated processes, a_t and b_t, where

$$a_t = \sum_{k=0}^{\infty} A_k e_{t-k} \qquad \text{and} \qquad b_t = \sum_{k=0}^{\infty} B_k e_{t-k}.$$

The discounted conditional covariance as of time t, setting realized innovations to zero, is

$$E_t\left[\sum_{s=1}^{\infty}\beta^s a_{t+s}b_{t+s}\right] = E_t\left[\sum_{s=1}^{\infty}\beta^s\left(\sum_{k=0}^{\infty}A_k e_{t+s-k}\right)\left(\sum_{k=0}^{\infty}B_k e_{t+s-k}\right)\right]. \tag{A.3}$$

[1] To make the problem well-defined a (small) adjustment cost must also be included, but we suppress it here because the net effect of the adjustment cost is just to make the solution stationary. Alternatively, one could simply impose the requirement that any solution be stationary.

Because cross-product terms drop out, coefficients of like lags of e_t can be grouped:

$$\begin{aligned}&\beta[A_0B_0+\beta A_1B_1+\beta^2A_2B_2+\ldots]E_t[e_{t+1}^2]+\beta^2[A_0B_0+\beta A_1B_1+\beta^2A_2B_2+\ldots]E_t[e_{t+2}^2]+\ldots\\&=\beta[A_0B_0+\beta A_1B_1+\beta^2A_2B_2+\ldots]\sigma_e^2+\beta^2[A_0B_0+\beta A_1B_1+\beta^2A_2B_2+\ldots]\sigma_e^2+\ldots\\&=\frac{\beta\sigma_e^2}{1-\beta}\sum_{s=0}^{\infty}\beta^kA_kB_k=\frac{\beta\sigma_e^2}{1-\beta}\frac{1}{2\pi i}\oint A(z)B(\beta z^{-1})\,\frac{dz}{z}.\end{aligned}\tag{A.4}$$

This is a useful transformation because the integrand is a product. Since the optimal policy for an optimization problem in which the objective is an expected value like that in (A.3), the representation in (A.4) permits a direct variational approach. Equation (A.4) is an instance of Parseval's formula, which states that the inner product of analytic functions is the sum of the products of the coefficients of their power series expansions.

Optimization in the frequency domain. We apply these insights to the consumer's optimization problem. Hansen and Sargent (1978, 1979) showed that the first-order conditions of linear-quadratic stochastic optimization problems could be expressed in lag-operator notation, z-transformed, and solved. Whiteman noticed that the z-transformation could be performed on the objective function itself, skipping the step of finding the time-domain version of the Euler condition. The objective is then a functional, i.e., a mapping of functions into the real line. One can then use the calculus of variations to find the optimal policy function.

The first step is to conjecture that the solution to the agent's optimization problem must be an analytic function of the fundamental process e_t:

$$b_t=B(L)e_t.$$

The agent's objective can then be restated in terms of the functions A and B, and the innovations:

$$\max_{B(\cdot)}-E\left[\sum_{t=0}^{\infty}\beta^t\Big((A(L)-(1-rL)B(L))e_t\Big)^2\right].$$

Expressing the objective in frequency-domain form, using the equivalence established in (A.4), the agent's objective can be written as

$$\max_{B(\cdot)}-\frac{\beta\sigma_e^2}{1-\beta}\frac{1}{2\pi i}\oint(A(z)-(1-rz)B(z))(A(\beta z^{-1})-(1-r\beta z^{-1})B(\beta z^{-1}))\,\frac{dz}{z}.$$

The variational method. Let $\zeta(z)$ be an arbitrary analytic function on the domain $\{z:|z|\le\beta^{\frac{1}{2}}\}$, and let a be a real number. Let $B(z)$ be the agent's optimal choice. His objective can be restated as

$$J(a)=\max_{a}-\frac{\beta\sigma_e^2}{1-\beta}\frac{1}{2\pi i}\oint(A(z)-(1-rz)(B(z)+a\zeta(z)))(A(\beta z^{-1})-(1-r\beta z^{-1})B(\beta z^{-1})+a\zeta(\beta z^{-1})))\,\frac{dz}{z}.$$

This is a conventional problem. Differentiating with respect to a and setting $a=0$ yields the first-order condition describing the agent's optimal choice of $B(\cdot)$:

$$\begin{aligned}J'(0)=0=&-\frac{\beta\sigma_e^2}{1-\beta}\frac{1}{2\pi i}\oint\zeta(z)(1-rz)(A(\beta z^{-1})-(1-r\beta z^{-1})B(\beta z^{-1}))\,\frac{dz}{z}\\&-\frac{\beta\sigma_e^2}{1-\beta}\frac{1}{2\pi i}\oint\zeta(\beta z^{-1})(1-r\beta z^{-1})(A(z)-(1-rz)B(z))\,\frac{dz}{z}.\end{aligned}$$

Observe the symmetry between the two integrals—everywhere βz^{-1} appears in the first integral, z appears in the second, and conversely. Whiteman establishes that the two integrals are in fact equal; we refer to this property as "β-symmetry". Therefore, the first-order condition simplifies to

$$0 = -\frac{1}{2\pi i}\oint (A(z) - (1 - rz)B(z))(1 - r\beta z^{-1})\zeta(\beta z^{-1})\,\frac{dz}{z}, \tag{A.5}$$

where we have dropped the constant $\frac{\beta\sigma_e^2}{1-\beta}$.

Clearly, the solution to the agent's optimization problem cannot depend on ζ: the integral in first-order condition (A.5) must be zero for arbitrary analytic functions ζ. By Cauchy's integral theorem, a contour integral around a singularity—a function of z that has no component that can be represented as a convergent power series expansion within a domain like the unit disk—is zero. Thus, all that is needed to make the integral in (A.5) zero is to make the integrand singular inside the unit disk.

Recall that a solution to the agent's optimization problem must be an analytic function. The next step in the solution is to separate the forward-looking components in (A.5) from the backward-looking components, so that we can then eliminate the non-analytic portion from our solution. Examining equation (A.5), note that by construction ζ is analytic, so that it can be represented by a power series,

$$\zeta(z) = \sum_{j=0}^{\infty} \zeta_j z^j.$$

This means that $\zeta(\beta z^{-1})$ has an expansion of the form

$$\zeta(\beta z^{-1}) = \sum_{j=0}^{\infty} \zeta_j \beta^j z^{-j},$$

which has only nonpositive powers of z. The negative powers of z—all but the first term—define singularities at $z = 0$, which is an element of the unit disk. However, the rest of the integrand in (A.5), $(1 - r\beta z^{-1})(A(z) - (1 - rz)B(z))$, can have both positive and negative powers of z in its power series expansion. If it were possible to guarantee that only negative powers of z appeared in $(1-r\beta z^{-1})(A(z)-(1 - rz)B(z))$, then its expansion would take the form

$$(1 - r\beta z^{-1})(A(z) - (1 - rz)B(z)) = \sum_{j=1}^{\infty} f_j \beta^j z^{-j},$$

for some $\{f_j\}$, and the product of this with $\zeta(\beta z^{-1})$ would take the form

$$\zeta(\beta z^{-1})(1 - r\beta z^{-1})(A(z) - (1 - rz)B(z)) = \sum_{j=1}^{\infty} g_j \beta^j z^{-j}.$$

for some $\{g_j\}$. Every term in the sum is a singularity, and the integral of the sum is therefore zero.

The first-order condition (A.5) can now be broken out of the integral and stated as follows:

$$(1 - r\beta z^{-1})(A(z) - (1 - rz)B(z)) = \sum_{-\infty}^{-1}, \tag{A.6}$$

where $\sum_{-\infty}^{-1}$ is shorthand for an arbitrary function that has only negative powers of z, and hence cannot be part of the solution to the agent's optimization problem. This type of equation is known as a *Wiener-Hopf equation.*

<u>Factorization</u>. To solve the Wiener-Hopf equation of a stochastic linear-quadratic optimization problem, we must factor the equation to separate the nonanalytic parts from the analytic parts. The factorization problem is a generalization of the problem of solving a quadratic equation, but there is no general formula for the solution. However, if a candidate factorization can be found, then even if it is not analytic and invertible, there is a general formula for converting that solution into an analytic and invertible factorization (Ball, Gohberg, Rodman (1990)).

The Wiener-Hopf equation (A.6) can be restated as:

$$(1 - r\beta z^{-1})(1 - rz)B(z) = (1 - r\beta z^{-1})A(z) + \sum_{-\infty}^{-1}.$$

It is tempting to solve for $B(z)$ by dividing the left-hand side by the coefficient of $B(z)$, $(1 - r\beta z^{-1})(1 - rz)$. However, this would multiply the $\sum_{-\infty}^{-1}$ term by positive powers of z, making it impossible to establish the coefficients of the positive powers of z in the solution.

We must first *factor* the coefficient of $B(z)$ into the product of analytic and non-analytic functions:

$$(1 - r\beta z^{-1})(1 - rz) = \beta r^2 (1 - (\beta r)^{-1}\beta z^{-1})(1 - (\beta r)^{-1} z).$$

Because by assumption $\frac{1}{\beta r} < 1$, the first factor on the right-hand side, $(1 - (\beta r)^{-1}\beta z^{-1})$, when inverted has a convergent power series (on the unit disk) in negative powers of z. Hence, we can divide through by this factor to rewrite the Wiener-Hopf equation as

$$\beta r^2 (1 - (\beta r)^{-1} z)B(z) = \frac{(1 - r\beta z^{-1})}{1 - (\beta r)^{-1}\beta z^{-1}} A(z) + \sum_{-\infty}^{-1}, \tag{A.7}$$

where we use the fact that

$$\frac{1}{(1 - (\beta r)^{-1}\beta z^{-1}} \sum_{-\infty}^{-1}$$

has only negative powers of z. Since the left-hand side of (A.7) is the product of analytic functions, applying the annihilator to (A.7) yields

$$\beta r^2 (1 - (\beta r)^{-1} z)B(z) = [\frac{(1 - r\beta z^{-1})}{(1 - (\beta r)^{-1}\beta z^{-1})} A(z)]_+.$$

Since $(\beta r)^{-1} < 1$, it follows that the inverse of $(1 - (\beta r)^{-1} z)$ is also analytic, so that we can divide by $(1 - (\beta r)^{-1} z)$ to solve for the optimal $B(z)$,

$$B(z) = \frac{[(1 - (\beta r)^{-1}\beta z^{-1})^{-1}(1 - r\beta z^{-1})A(z)]_+}{[(\beta r^2)(1 - (\beta r)^{-1} z)]}.$$

A more explicit solution for $B(z)$ obtains if the endowment process is AR(1), so that

$$A(z) = \frac{1}{1 - \rho z}.$$

Proposition A.1 establishes a key result that is used repeatedly: the annihilate when there is an AR(1) construct can be simply calculated—if $A(z)$ is an AR(1), then $[f(\beta z^{-1})A(z)]_+ = f(\beta\rho)A(z)$.

PROPOSITION A.1: *If f is analytic on $\beta^{-1/2}$ and $\rho < \beta^{-1/2}$, then $[f^*(1-\rho z)^{-1}]_+ = f(\beta\rho)(1-\rho z)^{-1}$.*

PROOF: Direct computation (see, *e.g.*, Taub (1986)). ■

Proposition A.2 shows that the proposition about annihilates of first-order AR functions must be used with caution. If there is a zero in the annihiland, the proposition changes.

PROPOSITION A.2: *Let* $a < \beta^{-1/2}$. *Then* $[f^* \frac{1-\frac{1}{a}z^{-1}}{1-az}]_+ = 0$.

PROOF:

$$[f^* \frac{1-\frac{1}{a}z^{-1}}{1-az}]_+ = \frac{1}{a}[z^{-1}f^*\frac{az-1}{1-az}]_+ = \frac{1}{a}[-f^*z^{-1}]_+ = 0.$$

■

Using Proposition A.1, it follows that

$$B(z) = \frac{(1-r\beta)A(z)}{[(\beta r^2)(1-(\beta r)^{-1}\beta\rho)(1-(\beta r)^{-1}z)]}.$$

This formula has a simple "permanent income" interpretation: the agent applies the filter

$$\frac{1-r\beta}{[(\beta r^2)(1-(\beta r)^{-1}\beta\rho)(1-(\beta r)^{-1}L)]}$$

to the endowment process $A(L)e_t$ in order to smooth consumption.

Vector formulation. The ideas presented above apply with little change to our multi-agent model of strategic informed stock trading. The primary difference is that our economy has multiple informed traders, so there is a vector of fundamental processes. Because of this, a vector formulation of the translation to the frequency domain and manipulations within the frequency domain must be used.

Consider a vector model with objective

$$\max_{B(\cdot)} -E\left[\sum_{t=0}^{\infty}\beta^t\Big((A(L)-B(L)F(L))e_t\Big)^2\right],$$

where e_t is now a vector of fundamental processes with covariance matrix S. Using the trace operator, the objective can be written as

$$\max_{B(\cdot)} -E\left[\sum_{t=0}^{\infty}\beta^t \mathrm{tr}\Big((A(L)-B(L)F(L))e_t\Big)^2\right].$$

Commuting under the trace, taking the expectation, and transforming to the frequency-domain yields the objective

$$\max_{B(z)} \frac{1}{2\pi i}\oint \mathrm{tr}[(A(\beta z^{-1})-B(\beta z^{-1})F(\beta z^{-1}))S(A(z)-B(z)F(z))']\,\frac{dz}{z}, \tag{A.8}$$

where $F(z)$ is a function analogous to the net bond trade $\rho - rz$ in a vector setting and $'$ denotes the transpose.

As in the scalar case, a variational procedure is used to solve (A.8). The variation is $B(z)+a\zeta(z)$. When the derivative is taken inside the integral and trace, the first-order condition can be stated:

$$0 = -\frac{1}{2\pi i}\oint \mathrm{tr}[(A(\beta z^{-1})-B(\beta z^{-1})F(\beta z^{-1}))SF(z)'\zeta(z)']\frac{dz}{z} - \frac{1}{2\pi i}\oint \mathrm{tr}[\zeta(\beta z^{-1})F(\beta z^{-1})S(A(z)-B(z)F(z))']\frac{dz}{z}$$

Exploiting β-symmetry (the integrals are equal), the Wiener-Hopf equation simplifies to

$$(A(z) - B(z)F(z))SF(\beta z^{-1})' = \sum_{-\infty}^{-1},$$

where $\sum_{-\infty}^{-1}$ is a vector of functions of strictly negative powers of z. Taking transposes, the equation can be rewritten as

$$B(z)F(z)SF(\beta z^{-1})' = A(z)SF(\beta z^{-1})' - \sum_{-\infty}^{-1}.$$

The factorization theorem applies here as well: $F(z)SF(\beta z^{-1})$ can be factored into the product of two matrixes, $H(z)$ and $H(\beta z^{-1})$,

$$H(z)H(\beta z^{-1}) = F(z)SF(\beta z^{-1})',$$

where every entry in the matrix H is an analytic function and its determinant has no zeroes. The solution is then

$$B(z) = [A(z)SF(\beta z^{-1})'H(\beta z^{-1})^{-1}]_{+}H(z)^{-1}.$$

The factorization problem can be far more difficult in a matrix setting than in a scalar setting; the methods set out in Ball and Taub (1991) typically must be used. Solving the annihilate remains possible if the functional forms are tractable; if $A(z)$ has an AR structure then explicit solutions again obtain.

Appendix B: Proofs of Results

Proof (Lemma 2.1): Equation (1) can be written as

$$\max_{\{x_{iT}\}_{T\geq t}} E_t\left[\sum_{T=t}^{\infty}\pi(1-\pi)^{T-t}\left[\sum_{\tau\leq T}(\tilde{\beta}^{T-t}\tilde{v}_T-\tilde{\beta}^{\tau-t}p_\tau)x_{i\tau}\right]\mid \mathbf{\tilde{e}_{it}},\mathbf{\Omega_t},\mathbf{x_{it-1}}\right] \tag{1'}$$

$$= E_t\left[\sum_{T=t}^{\infty}\pi(1-\pi)^{T-t}\left[\sum_{\tau< t}(\tilde{\beta}^{T-t}\tilde{v}_T-\tilde{\beta}^{\tau-t}p_\tau)x_{i\tau}\right]\mid \mathbf{\tilde{e}_{it}},\mathbf{\Omega_t},\mathbf{x_{it-1}}\right]$$
$$+\max_{\{x_{iT}\}_{T\geq t}} E_t\left[\sum_{T=t}^{\infty}\pi(1-\pi)^{T-t}\left[\sum_{t\leq\tau\leq T}(\tilde{\beta}^{T-t}\tilde{v}_T-\tilde{\beta}^{\tau-t}p_\tau)x_{i\tau}\right]\mid \mathbf{\tilde{e}_{it}},\mathbf{\Omega_t},\mathbf{x_{it-1}}\right].$$

To write agent i's optimization problem so that the dating on prices, orders and expected contributions to the value process correspond, expand expected lifetime profits from *future* trading as:

$$E_t\left[\pi x_{it}\left[\sum_{T=t}^{\infty}[\tilde{\beta}(1-\pi)]^{T-t}\tilde{v}_T-\sum_{T=t}^{\infty}(1-\pi)^{T-t}p_t\right]+\pi x_{it+1}\left[\sum_{T=t+1}^{\infty}[\tilde{\beta}(1-\pi)]^{T-t}\tilde{v}_T-\sum_{T=t+1}^{\infty}(1-\pi)^{T-t}\tilde{\beta}p_{t+1}\right]\right.$$
$$\left.+\pi x_{it+2}\left[\sum_{T=t+2}^{\infty}[\tilde{\beta}(1-\pi)]^{T-t}\tilde{v}_T-\sum_{T=t+2}^{\infty}(1-\pi)^{T-t}\tilde{\beta}^2p_{t+2}\right]+\ldots\mid \mathbf{\tilde{e}_{it}},\mathbf{\Omega_t}\right]$$
$$=E_t\left[\pi x_{it}\left[\sum_{T=t}^{\infty}[\tilde{\beta}(1-\pi)]^{T-t}\tilde{v}_T-\frac{1}{\pi}p_t\right]+\pi x_{it+1}\left[\sum_{T=t+1}^{\infty}[\tilde{\beta}(1-\pi)]^{T-t}\tilde{v}_T-\frac{\tilde{\beta}(1-\pi)}{\pi}p_{t+1}\right]\right.$$
$$\left.+\pi x_{it+2}\left[\sum_{T=t+2}^{\infty}[\tilde{\beta}(1-\pi)]^{T-t}\tilde{v}_T-\frac{[\tilde{\beta}(1-\pi)]^2}{\pi}p_{t+2}\right]+\ldots\mid \mathbf{\tilde{e}_{it}},\mathbf{\Omega_t}\right].$$

Again using summation notation, we write the agent's objective as:

$$\max_{\{x_{i\tau}\}_{\tau\geq t}} E_t\left[\sum_{\tau=t}^{\infty}\pi x_{i\tau}\left[\sum_{T=\tau}^{\infty}[\tilde{\beta}(1-\pi)]^{T-\tau}\tilde{v}_T-[\tilde{\beta}(1-\pi)]^{\tau-t}\frac{p_\tau}{\pi}\right]\mid \mathbf{\tilde{e}_{it}},\mathbf{\Omega_t}\right]$$
$$=\max_{\{x_{i\tau}\}_{\tau\geq t}} E_t\left[\sum_{\tau=t}^{\infty}x_{i\tau}[\tilde{\beta}(1-\pi)]^{\tau-t}\left(\sum_{T=\tau}^{\infty}[\tilde{\beta}(1-\pi)]^{T-\tau}\pi\tilde{v}_T-p_\tau\right)\mid \mathbf{\tilde{e}_{it}},\mathbf{\Omega_t}\right].$$

We now iterate on the expectation operator to obtain:

$$\max_{\{x_{i\tau}\}_{\tau\geq t}} E_t\left[E_\tau\left[\sum_{\tau=t}^{\infty}x_{i\tau}[\tilde{\beta}(1-\pi)]^{\tau-t}\left(\sum_{T=\tau}^{\infty}[\tilde{\beta}(1-\pi)]^{T-\tau}\pi\tilde{v}_T-p_\tau\right)\mid \mathbf{\tilde{e}_{i\tau}},\mathbf{\Omega_\tau}\right]\mid \mathbf{\tilde{e}_{it}},\mathbf{\Omega_t}\right].$$

Since at date τ, $x_{i\tau}$ is a deterministic function of date τ information (solving agent i's optimization problem), we pass the date τ expectation operator through to obtain:

$$\max_{\{x_{i\tau}\}_{\tau\geq t}} E_t\left[\sum_{\tau=t}^{\infty}x_{i\tau}[\tilde{\beta}(1-\pi)]^{\tau-t}E_\tau\left[\left(\sum_{T=\tau}^{\infty}[\tilde{\beta}(1-\pi)]^{T-\tau}\pi\tilde{v}_T-p_\tau\right)\mid \mathbf{\tilde{e}_{i\tau}},\mathbf{\Omega_\tau}\right]\mid \mathbf{\tilde{e}_{it}},\mathbf{\Omega_t}\right].$$

Using the AR(1) structure of $\tilde{v}_T$, $E_\tau\left[\tilde{v}_T|\mathbf{\tilde{e}_{i\tau}},\mathbf{\Omega_\tau}\right]=\rho^{T-\tau}E_\tau\left[\tilde{v}_\tau|\mathbf{\tilde{e}_{i\tau}},\mathbf{\Omega_\tau}\right]$, we simplify the objective to

$$\max_{\{x_{i\tau}\}_{\tau\geq t}} E_t\left[\sum_{\tau=t}^{\infty}x_{i\tau}[\tilde{\beta}(1-\pi)]^{\tau-t}E_\tau\left[\left(\sum_{T=\tau}^{\infty}[\rho\tilde{\beta}(1-\pi)]^{T-\tau}\pi\tilde{v}_\tau-p_\tau\right)\mid \mathbf{\tilde{e}_{i\tau}},\mathbf{\Omega_\tau}\right]\mid \mathbf{\tilde{e}_{it}},\mathbf{\Omega_t}\right]$$
$$=\max_{\{x_{i\tau}\}_{\tau\geq t}} E_t\left[\sum_{\tau=t}^{\infty}x_{i\tau}[\tilde{\beta}(1-\pi)]^{\tau-t}E_\tau\left[\left(\frac{\pi}{1-\rho\tilde{\beta}(1-\pi)}\tilde{v}_\tau-p_\tau\right)\mid \mathbf{\tilde{e}_{i\tau}},\mathbf{\Omega_\tau}\right]\mid \mathbf{\tilde{e}_{it}},\mathbf{\Omega_t}\right].$$

Integrating out and rearranging slightly, we finally write agent i's objective as

$$\max_{\{x_{i\tau}\}_{\tau\geq t}} E_t\left[\sum_{\tau=t}^{\infty}[\tilde{\beta}(1-\pi)]^{\tau-t}\left(\frac{\pi}{1-\rho\tilde{\beta}(1-\pi)}\tilde{v}_\tau - p_\tau\right)x_{i\tau} \mid \tilde{\mathbf{e}}_{\mathbf{it}}, \mathbf{\Omega_t}\right]. \quad \blacksquare$$

Proof (Lemma 3.1): The first order condition for γ_i can be written as

$$[\lambda(\sum_{j=1}^{N}\gamma_j) + \lambda^*\gamma_i]JJ^* = \phi\sum_{j=1}^{N} b_{je}^*\sigma_{je}^2 - \lambda^* b_{ie}b_{ie}^*\sigma_{ie}^2 - \lambda JJ^* + \sum_{-\infty}^{-1}.$$

We combine this equation with those of the other traders to obtain the matrix equation

$$JJ^*\begin{pmatrix}\lambda+\lambda^* & \lambda & \dots & \lambda\\ \vdots & & & \vdots\\ \lambda & \lambda & \dots & \lambda+\lambda^*\end{pmatrix}\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_N\end{pmatrix} = \begin{pmatrix}\phi\sum_{j=1}^{N} b_{je}^*\sigma_{je}^2 - \lambda^* b_{1e}b_{1e}^*\sigma_{1e}^2 - \lambda JJ^*\\ \vdots\\ \phi\sum_{j=1}^{N} b_{je}^*\sigma_{je}^2 - \lambda^* b_{Ne}b_{Ne}^*\sigma_{Ne}^2 - \lambda JJ^*\end{pmatrix} + \sum_{-\infty}^{-1}. \quad (B.1)$$

Next, substitute for $JJ^* = \sum_{j=1}^{N} b_{je}b_{je}^*\sigma_{je}^2 + \sigma_u^2$ into the first order condition for λ to obtain

$$\lambda JJ^*(1+\sum_{j=1}^{N}\gamma_j)(1+\sum_{j=1}^{N}\gamma_j^*) = \phi(1+\sum_{j=1}^{N}\gamma_j^*)(\sum_{k=1}^{N} b_{ke}^*\sigma_{ke}^2) + \sum_{-\infty}^{-1}.$$

Dividing $(1+\sum_{j=1}^{N}\gamma_j^*)$ out of the λ first-order condition yields

$$\lambda JJ^*(1+\sum_{j=1}^{N}\gamma_j) = \phi(\sum_{k=1}^{N} b_{ke}^*\sigma_{ke}^2) + \sum_{-\infty}^{-1}. \quad (B.2)$$

We use (B.2) to simplify γ_i. Writing equation (B.2) as

$$\lambda JJ^*\,(\,\lambda \quad \dots \quad \lambda\,)\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_N\end{pmatrix} = \phi(\sum_{k=1}^{N} b_{ke}^*\sigma_{ke}^2) - \lambda JJ^* + \sum_{-\infty}^{-1},$$

we subtract it from each row of (B.1) to obtain

$$JJ^*\begin{pmatrix}\lambda^* & 0 & \dots & 0\\ \vdots & & & \vdots\\ 0 & 0 & \dots & \lambda^*\end{pmatrix}\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_N\end{pmatrix} = -\lambda^*\begin{pmatrix}b_{1e}b_{1e}^*\sigma_{1e}^2\\ \vdots\\ b_{Ne}b_{Ne}^*\sigma_{Ne}^2\end{pmatrix} + \sum_{-\infty}^{-1}$$

Canceling the λ^* terms and computing the annihilate yields

$$\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_N\end{pmatrix} = -J^{-1}[\begin{pmatrix}J^{*-1}b_{1e}b_{1e}^*\sigma_{1e}^2\\ \vdots\\ J^{*-1}b_{Ne}b_{Ne}^*\sigma_{Ne}^2\end{pmatrix}]_+. \quad \blacksquare$$

Proof (Lemma 3.2): Using the definition of J in (9), we can solve for

$$\sum_{j=1}^{N}\gamma_j = -J^{-1}[J^{*-1}(JJ^* - \sigma_u^2)]_+.$$

Cancelling terms yields

$$\sum_{j=1}^{N} \gamma_j = -1 + J^{-1}[J^{*-1}\sigma_u^2]_+ \Rightarrow 1 + \sum_{j=1}^{N} \gamma_j = J^{-1}[J^{*-1}\sigma_u^2]_+.$$

Since $[J^{*-1}\sigma_u^2]_+$ is a scalar and J^{-1} is invertible by construction, the result follows. ∎

LEMMA B.1: *Let $0 < a < 1$. Define $f(z)$ by $f(z)f(z^{-1}) \equiv (1-az)(1-az^{-1}) + \sigma^2$. Then $f(z) = f_0(1 - f_1 z)$ and $f_1 < a$.*

PROOF: The equations for f_0 and f_1 are

$$f_0^2 f_1 = a, \qquad \text{and} \qquad \sigma^2 + 1 + a^2 = f_0^2(1 + f_1^2).$$

The solution for f_1 is

$$f_1 = \frac{\frac{\sigma^2+1+a^2}{a} \pm \sqrt{\left(\frac{\sigma^2+1+a^2}{a}\right)^2 - 4}}{2}.$$

Because the root must be fractional, the smaller root must be chosen, so the radical is subtracted. Routine algebra then reveals that

$$\frac{\frac{\sigma^2+1+a^2}{a} - \sqrt{\left(\frac{\sigma^2+1+a^2}{a}\right)^2 - 4}}{2} < a. \quad \blacksquare$$

Lemma B.1 shows that adding an MA process to an independent i.i.d. process results in a moving-average component of the joint process that has a smaller MA parameter than the initial MA parameter. The next lemma establishes a similar result: adding an AR process to an independent i.i.d. process results in a moving-average component of the joint process that has a smaller MA parameter than the initial AR parameter. We use this result to establish the characteristics of the function $J(z)$. In Lemma B.2 the function $f(z)$ differs from $J^k(z)$ only in that it omits the leading constant $J^k(0)$.

LEMMA B.2: *Let $1 > a_1 > a_2 > \ldots > a_k > 0$. Define $f(z)$ by*

$$f(z)f(z^{-1}) \equiv \left(\frac{c_1}{1-a_1 z} + \frac{c_2}{1-a_2 z} + \ldots + \frac{c_k}{1-a_k z}\right)\left(\frac{c_1}{1-a_1 z^{-1}} + \frac{c_2}{1-a_2 z^{-1}} + \ldots + \frac{c_k}{1-a_k z^{-1}}\right) + \sigma^2.$$

Then

$$f(z) = f_0 \prod_{i=1}^{k} \frac{1-f_i z}{1-a_i z}$$

where $1 > a_1 > f_1 > a_2 > f_2 > \ldots > a_k > f_k > 0$, and $f_0 = \sigma\left(\prod_{i=1}^{k} \frac{a_i}{f_i}\right)^{1/2} > \sigma$. Further, each f_i is increasing in σ.

PROOF: Consider a candidate root $\overline{z}_1$ with $\frac{1}{a_1} < \overline{z}_1 < \frac{1}{a_2}$. For $i \geq 2$,

$$\frac{c_i}{1 - a_i\overline{z}_1} > 0 \qquad \text{and} \qquad \frac{1}{1 - a_i\overline{z}_1^{-1}} > 0.$$

However, $\frac{1}{1-a_1\overline{z}_1} < 0$, and indeed becomes arbitrarily negative as $\overline{z}_1$ approaches a_1^{-1}. Conversely,

$$\left(\frac{c_1}{1-a_1 z} + \frac{c_2}{1-a_2 z} + \ldots + \frac{c_k}{1-a_k z}\right)\left(\frac{c_1}{1-a_1 z^{-1}} + \frac{c_2}{1-a_2 z^{-1}} + \ldots + \frac{c_k}{1-a_k z^{-1}}\right), \qquad (B.3)$$

becomes arbitrarily positive as $\overline{z}_1$ approaches a_2^{-1}. Hence, there is a crossing point where $f(\overline{z}_1) = 0$. Because $(1 - a_i z)$ is a monotone function, $\overline{z}_1$ is the only such solution in the interval (a_1^{-1}, a_2^{-1}). Define $f_{11} \equiv \overline{z}_1^{-1}$. Similar reasoning yields $f_{12}, \ldots, f_{1,k-1}$. Now consider $\overline{z}_k > a_k^{-1}$. As $\overline{z}$ approaches a_k^{-1}, (B.3) becomes arbitrarily negative; conversely, as $\overline{z}_k \to \infty$, (B.3) shrinks to zero. In that case, $f(z)f(z^{-1}) \to \sigma^2 > 0$. Hence, there is a unique crossing point in the interval (a_k^{-1}, ∞) where $f(z)f(z^{-1}) = 0$.

To obtain f_0, evaluate $f(z)f(z^{-1})$ at $z = 0$:

$$f(z)f(z^{-1})\Big|_{z=0} = f_0^2 \prod_{i=1}^{k} \frac{f_i}{a_i} = \sigma^2,$$

and solve for f_0. Finally, if σ is increased, then $1/\overline{z}_\ell$ must be increased toward a_ℓ in order to make the term $c_\ell/(1 - a_\ell z^{-1})$ more negative. ∎

A corollary of Lemma B.2 is that if an additional term $c_{k+1}/(1 - a_{k+1} z)$ is added to f, then f_0 increases in a continuous fashion.

PROPOSITION B.3: *Let* $h(z) = \prod_{i=1}^{k-1}(1 - f_i z) / \prod_{i=1}^{k}(1 - a_i z)$ *with* $1 > a_1 > f_1 > \ldots > a_k > 0$. *The partial fractions representation of* $h(z)$ *is*

$$h(z) = \sum_{i=1}^{k} \frac{c_i}{1 - a_i z} \qquad \textit{where} \quad \sum_{i=1}^{k} c_i = 1.$$

PROOF: The second assertion of the proposition follows from expanding $\prod_i^{k-1}(1 - f_i z)/\prod_i^k(1 - a_i z)$ into partial fraction form recursively. First consider the partial fractions expansion

$$\frac{1 - f_1 z}{(1 - a_1 z)(1 - a_2 z)} = \frac{\tilde{c}_1}{1 - a_1 z} + \frac{\tilde{c}_2}{1 - a_2 z}.$$

Then

$$(i)\;\; \tilde{c}_1 = \frac{a_1 - f_1}{a_1 - a_2} \in (0,1); \qquad (ii)\;\; \tilde{c}_2 = \frac{f_1 - a_2}{a_1 - a_2} \in (0,1); \qquad \text{and}\;\; (iii)\;\; \tilde{c}_1 + \tilde{c}_2 = 1.$$

Now consider the recursive case:

$$\frac{\prod_{i=1}^{k}(1 - f_i z)}{\prod_{i=1}^{k+1}(1 - a_i z)} = \frac{\prod_{i=1}^{k-1}(1 - f_i z)}{\prod_{i=1}^{k}(1 - a_i z)} \frac{(1 - f_k z)}{(1 - a_{k+1} z)} = \frac{\sum_{i=1}^{k-1} c_i^k}{\prod_{i=1}^{k}(1 - a_i z)} \frac{(1 - f_k z)}{(1 - a_{k+1} z)}$$
$$= \sum_{i=1}^{k-1} \frac{c_i^k (1 - f_k z)}{(1 - a_i z)(1 - a_{k+1} z)} = \sum_{i=1}^{k-1} \frac{c_i^k \tilde{c}_{i1}^k}{(1 - a_i z)} + \sum_{i=1}^{k-1} \frac{c_i^k \tilde{c}_{i2}^k}{(1 - a_{k+1} z)} = \sum_{i=1}^{k-1} \frac{c_i^k \tilde{c}_{i1}^k}{(1 - a_i z)} + \frac{\sum_{i=1}^{k-1} c_i^k \tilde{c}_{i2}^k}{(1 - a_{k+1} z)}$$

Observe that because $\tilde{c}_{ij}^k < 1$ and $\tilde{c}_{i1}^k + \tilde{c}_{i2}^k = 1$, and because, by induction, $\sum_{i=1}^{k-1} c_i^k = 1$, then

$$\sum_{i=1}^{k-1} c_i^k (\tilde{c}_{i1}^k + \tilde{c}_{i2}^k) = \sum_{i=1}^{k-1} c_i^k = 1. \quad ∎$$

We now turn to the proof of Lemma 4.1 in the main text. We begin with a preliminary lemma.

LEMMA B.4: $1 + \gamma_i = J^{-1}[J^{*-1}(\sigma_u^2 + (N-1) b_{ie} b_{ie}^* \sigma_{ie}^2)]_+$.

PROOF: In this proof we assume symmetry across agents, writing b instead of b_{ie} and σ^2 instead of σ_{ie}^2. By definition, $JJ^* = Nbb^*\sigma_e^2 + \sigma_u^2$, so that $JJ^* - bb^*\sigma_e^2 = (N-1)bb^*\sigma_e^2 + \sigma_u^2$. Dividing both sides by J^*, taking the annihilate and then dividing by J yields

$$1 - J^{-1}[J^{*^{-1}}bb^*\sigma_e^2]_+ = J^{-1}[J^{*^{-1}}((N-1)bb^*\sigma_e^2 + \sigma_u^2)]_+ = J^{-1}[J^{*^{-1}}((N-1)bb^*\sigma_e^2 + \sigma_u^2)]_+.$$

Finally, Lemma 3.1 reveals that the left-hand-side, $1 - J^{-1}[J^{*^{-1}}bb^*\sigma_e^2]_+$, is $(1+\gamma)$. ■

PROOF (Lemma 4.1): In this proof we use i to index i^{th} element of a series, not the i^{th} trader, so that we write $b(z)$ in lieu of $b_{ie}(z)$, and so on. The proof employs an inductive argument. We first conjecture a preliminary trading intensity filter b^k of the appropriate form: the finite sum of AR terms $\frac{c_i^k}{1-a_i^k z}$. This preliminary value is then mapped into a new trading-intensity filter $b^{k+1}(z)$. We demonstrate that the appropriate properties are preserved by this mapping. These properties are characterized by finding the zeroes of the appropriate functions—the inverses of these zeroes are the autoregressive coefficients.

Let a preliminary value of $b(z)$ be $b^k(z) = \sum_{i=1}^k \frac{c_i^k}{1-a_i^k z}$, where $\rho = a_1^k > a_2^k > \ldots a_k^k$, and with $c_i^k > 0$. Using $b^k(z)$ to construct the $J^k(z)$ function, we write $J^k(z)$ as

$$J^k(z) = J_0^k \frac{\prod_{i=1}^k (1-f_i^k z)}{\prod_{i=1}^k (1-a_i^k z)}, \tag{B.4}$$

where by Lemma B.2, $a_1 > f_1 > \ldots > a_k > f_k$. It is important to note that the numerator is of the same polynomial order as the denominator. Next recall the definition of the function g:

$$gg^* \equiv \lambda(1 + \sum_{j=1}^N \gamma_j)(1+\gamma_i^*) + \lambda^*(1 + \sum_{j=1}^N \gamma_j^*)(1+\gamma_i) = \mu(1+\gamma_i^*) + \mu^*(1+\gamma_i).$$

Using Lemma B.4, the solutions for for μ and γ are given by

$$\mu^k(z) = J^k(z)^{-1} J^k(\beta\rho)^{-1} \left(N \sum_{i=1}^k \frac{c_i^k}{1-a_i^k\beta\rho} \right) \frac{1}{1-\rho z}\sigma_e^2$$

$$1+\gamma_j^k = J^k(z)^{-1}[J^{k^{*^{-1}}}(\sigma_u^2 + (N-1)b^k b^{k^*}\sigma_e^2)]_+ = J^k(z)^{-1}\left(J^k(0)^{-1}\sigma_u^2 + (N-1)\sigma_e^2 \sum_{i=1}^k J^k(\beta a_i^k)^{-1} b^k(a_i^k\beta) \frac{c_i^k}{1-a_i^k z} \right).$$

The $J^k(\beta\rho)^{-1}$ and $J^k(\beta a_i^k)^{-1}$ terms appear by an application of Proposition A.1, because the annihilands are products of functions of z^{-1} with autoregressive terms $(1-a_i^k z)^{-1}$ for each value of i.

Substituting for μ^k and γ^k into $g^k g^{k^*}$ yields:

$$\begin{aligned} g^k g^{k^*} = J^{-1}J^{*^{-1}}J(\beta\rho)^{-1}\sigma_e^2 \Bigg(&\left(N\sum_{i=1}^k \frac{c_i^k}{1-a_i^k\beta\rho}\frac{1}{1-\rho z}\right)\left(J(0)^{-1}\sigma_u^2 + (N-1)\sigma_e^2 \sum_{i=1}^k J(\beta a_i^k)^{-1} b^k(a_i^k\beta)\frac{c_i^k}{1-a_i^k\beta z^{-1}}\right) \\ &+ \left(N\sum_{i=1}^k \frac{c_i^k}{1-a_i^k\beta\rho}\frac{1}{1-\rho z^{-1}}\right)\left(J(0)^{-1}\sigma_u^2 + (N-1)\sigma_e^2 \sum_{i=1}^k J(\beta a_i^k)^{-1} b^k(a_i^k\beta)\frac{c_i^k}{1-a_i^k z}\right)\Bigg), \end{aligned}$$

where we omit the superscript k from the J functions. Rewrite $g^k g^{k*}$ as

$$\begin{aligned} g^k g^{k^*} =& J^{-1}J^{*^{-1}}J(\beta\rho)^{-1}\phi\phi^* \left(N\sum_{i=1}^k \frac{c_i^k}{1-a_i^k\beta\rho}\right)\sigma_e^2 \\ &\times \Bigg(\left(J(0)^{-1}\sigma_u^2 + \sigma_e^2(N-1)\sum_{i=1}^k J(\beta a_i^k)^{-1} b(a_i^k\beta)\frac{c_i^k}{1-a_i^k\beta z^{-1}}\right)(1-\rho\beta z^{-1}) \\ &+ \left(J(0)^{-1}\sigma_u^2 + \sigma_e^2(N-1)\sum_{i=1}^k J(\beta a_i^k)^{-1} b(a_i^k\beta)\frac{c_i^k}{1-a_i^k z}\right)(1-\rho z)\Bigg) \equiv J^{-1}J^{*^{-1}}\phi\phi^* h^k h^{k^*}, \end{aligned} \tag{B.5}$$

where h^k is implicitly defined by

$$\begin{aligned} h^k h^{k^*} &\equiv J(\beta\rho)^{-1}\left(N\sum_{i=1}^{k}\frac{c_i^k}{1-a_i^k\beta\rho}\right)\sigma_e^2 \\ &\times\Bigg(\left(J(0)^{-1}\sigma_u^2+\sigma_e^2(N-1)\sum_{i=1}^{k}J(\beta a_i^k)^{-1}b(a_i^k\beta)\frac{c_i^k}{1-a_i^k\beta z^{-1}}\right)(1-\rho\beta z^{-1}) \qquad (B.6)\\ &+\left(J(0)^{-1}\sigma_u^2+\sigma_e^2(N-1)\sum_{i=1}^{k}J(\beta a_i)^{-1}b(a_i^k\beta)\frac{c_i^k}{1-a_i^k z}\right)(1-\rho z)\Bigg). \end{aligned}$$

The expression $h^k h^{k^*}$ can be written in factored form as

$$h^k(z)h^k(\beta z^{-1}) = (h_0^k)^2\frac{(1-\rho z)(1-\rho\beta z^{-1})\prod_{j=2}^{k+1}(1-m_j^k z)(1-m_j^k\beta z^{-1})}{\prod_{j=1}^{k}(1-a_j^k z)(1-a_j^k\beta z^{-1})},$$

where

$$h^k(z) = h_0^k\frac{(1-\rho z)\prod_{j=2}^{k+1}(1-m_j^k z)}{\prod_{j=1}^{k}(1-a_j^k z)}, \qquad (B.7)$$

and $\{m_i^k\}$ are as yet undetermined. Note that the polynomial order of the numerator terms in h is $k+1$, while that of the denominator is k. The term $(1-\rho z)(1-\rho\beta z^{-1})$ in hh^* follows from $a_1 = \rho$; when the common denominator is created the coefficient $1-\rho\beta z^{-1}$ is multiplied by $1-\rho z$ and vice versa.

From equation (7) for b, the iterated version of b^{k+1} is

$$b^{k+1}(z) = g^k(z)^{-1}[g^k(\beta z^{-1})^{-1}(1+\gamma^k(\beta z^{-1}))\phi(z)]_+.$$

Substituting for g^k and then applying Proposition A.1 to the annihilator term yields

$$\begin{aligned} b^{k+1}(z) &= J^k(\beta\rho)J^k(z)\phi(z)^{-1}h^k(z)^{-1}J^k(\beta\rho)h^k(\beta\rho)^{-1}(1+\gamma^k(\beta\rho))\phi(\beta\rho)^{-1}\phi(z)\\ &= C^k(\beta\rho)\phi(z)^{-1}\phi(z)J^k(z)\frac{\prod_{j=1}^{k}(1-a_j^k z)}{(1-\rho z)\prod_{j=2}^{k+1}(1-m_j^k z)}. \end{aligned}$$

Proposition A.2 does not apply because ϕ^{*-1} cancels with the ϕ^* in h^{k^*-1}. Also note that $(h_0^k)^2$ has been subsumed into the constant $C^k(\beta\rho)$. Next, use (B.4) to substitute for $J^k(z)$ into $b^{k+1}(z)$ to obtain

$$b^{k+1}(z) = J_0^k C^k(\beta\rho)\frac{\prod_{j=1}^{k}(1-f_j^k z)}{\prod_{j=1}^{k}(1-a_j^k z)}\frac{\prod_{j=1}^{k}(1-a_j^k z)}{\prod_{j=1}^{k+1}(1-m_j^k z)} = J_0^k C^k(\beta\rho)\frac{\prod_{j=1}^{k}(1-f_j^k z)}{\prod_{j=1}^{k+1}(1-m_j^k z)}.$$

The polynomial order of the denominator is $k+1$, while the order of the numerator is k. Therefore, there is a partial fractions expansion of $b^{k+1}(z)$,

$$b^{k+1}(z) \equiv \sum_{i=1}^{k+1}\frac{c_i^{k+1}}{1-a_i^{k+1}z},$$

in which the numerator coefficients are all scalars, and where $a_1^{k+1} = \rho$ and $a_i^{k+1} = m_i^k$ for $i = 2,\ldots,k$.

Note that since the a_j^{k+1} derive from the definition of b in equation (7), which is a function of g^{-1}, and g derives from the factorization in gg^*, we are guaranteed by the factorization theorem that the a_i^{k+1} are real. Thus, it is just a matter of verifying that $a_i^{k+1} < \rho$, for $i = 2,\ldots,k+1$.

Our strategy is to locate the zeroes of the appropriate functions and to relate the zeroes to the m_i^k. We first derive properties of the $h^k(z)h^k(\beta z^{-1})$ factorization in (B.6). Computing the common denominator of

$$J(0)^{-1}\sigma_u^2 + \sigma_e^2(N-1)\sum_{i=1}^{k} J(\beta a_i^k)^{-1} b(a_i^k\beta)\frac{c_i^k}{1-a_i^k z},$$

and its complement that appear in equation (B.6); and then finding the common denominator of the entire expression simplifies the $h^k(z)h^k(\beta z^{-1})$ factorization to:

$$\begin{aligned} h^k(z)h^k(\beta z^{-1}) &= J(\beta\rho)^{-1}\left(N\sum_{i=1}^{k}\frac{c_i^k}{1-a_i^k\beta\rho}\right)\sigma_e^2\frac{(1-\rho z)(1-\rho\beta z^{-1})}{\prod_{i=1}^{k}(1-a_i^k z)\prod_{i=1}^{k}(1-a_i^k\beta z^{-1})} \\ \times\Bigg(&\Big(J(0)^{-1}\sigma_u^2\prod_{i=2}^{k}(1-a_i^k z)(1-a_i^k\beta z^{-1})(1-\rho z+1-\rho\beta z^{-1}) \\ &+\sigma_e^2(N-1)\sum_{i=2}^{k} J(\beta a_i^k)^{-1}b(a_i^k\beta)c_i^k\prod_{j=2,j\neq i}^{k}(1-a_j^k z)(1-a_j^k\beta z^{-1})\big((1-\rho\beta z^{-1})(1-a_i^k z)+(1-\rho z)(1-a_i^k\beta z^{-1})\big) \\ &+2\sigma_e^2(N-1)J(\beta a_1^k)^{-1}b(a_1^k\beta)c_1^k\prod_{j=2}^{k}(1-a_j^k z)(1-a_j^k\beta z^{-1})\Big)\Bigg), \end{aligned} \tag{B.8}$$

where we omit the k superscripts on J and b. We decompose the numerator of $h^k(z)h^k(\beta z^{-1})$ into the leading constant, $J(\beta\rho)^{-1}\left(N\sum_{i=1}^{k}\frac{c_i^k}{1-a_i^k\beta\rho}\right)\sigma_e^2$, times $M^k(z)$, where $M^k(z)$ is implicitly defined by

$$h^k(z)h^k(\beta z^{-1}) = J(\beta\rho)^{-1}\left(N\sum_{i=1}^{k}\frac{c_i^k}{1-a_i^k\beta\rho}\right)\sigma_e^2\frac{M^k(z)}{\prod_{i=1}^{k}(1-a_i^k z)\prod_{i=1}^{k}(1-a_i^k\beta z^{-1})}.$$

We can write $M^k(z)$ more compactly in factored form as

$$M^k(z) \equiv (m_0^k)^2(1-\rho z)(1-\rho\beta z^{-1})\prod_{i=2}^{k+1}(1-m_i^k z)(1-m_i^k\beta z^{-1}),$$

where $m_0^k = h_0^k$ (see equation (B.7)) and the m_i^k coefficients will now be characterized.

We next prove that the coefficients $\{m_i^k\}$ of $M^k(z)$ lie in the interval $(0,\rho)$ for $k>1$. There are two cases: in the first case all the c_i^k are positive; in the second case the c_i^k can be negative.

In the first case, where $c_i^k > 0$, the strategy is to show that $M^k(1/a_\ell^k) \neq 0$ for each value of ℓ, and that the sign of $M^\ell(1/a_{\ell+1}^k)$ is opposite that of $M^k(1/a_\ell^k)$. This guarantees the existence of a value $\tilde{z}_\ell^k \in (1/a_\ell^k, 1/a_{\ell+1}^k)$ such that $M^k(\tilde{z}_\ell^k)=0$. There are three cases: (i) $\ell=1$, (ii) $2\le\ell\le k$, and (iii) $\ell=k+1$.

(i) For $\ell=1$, $m_1^k=\rho$ automatically holds.

(ii) m_ℓ^k, $2\le\ell\le k$. Examining $M^k(1/a_\ell^k)$, observe that most of the terms in $M^k(1/a_\ell^k)$ are zero because $1-a_\ell^k\frac{1}{a_\ell^k}=0$. However, there is a non-zero term,

$$M^k(1/a_\ell^k) = (N-1)J(\beta a_\ell^k)^{-1}b(a_\ell^k\beta)c_\ell^k\prod_{j=1,j\neq\ell}^{k}(1-a_j^k/a_\ell^k)(1-a_j^k\beta a_\ell^k)(1-\rho/a_\ell^k)(1-a_\ell^k\beta a_\ell^k).$$

To sign $M^k(1/a_\ell^k)$ note that $b(a_\ell^k\beta)>0$ and

$$\begin{aligned} 1-\rho/a_\ell^k<0; &\qquad 1-a_j^k/a_\ell^k<0, \quad j=2,\ldots,\ell-1 \\ &\qquad 1-a_j^k/a_\ell^k>0, \quad j=\ell+1,\ldots,k \\ 1-\rho\beta a_\ell^k>0; &\qquad 1-a_j^k\beta a_\ell^k>0, \quad j=1,\ldots,k \end{aligned}$$

Thus, $M^k(1/a_\ell^k)$ is the product of positive terms and $\ell-1$ negative terms. Hence, $M^k(1/a_\ell^k)$ has sign $(-1)^{\ell-1}$.

By similar reasoning, the sign of $M^k(1/a_{\ell+1}^k)$ switches to $(-1)^\ell$. Because $M^k(\cdot)$ is continuous, there is at least one zero $\tilde{z}_\ell^k \in (1/a_\ell^k, 1/a_{\ell+1}^k)$ such that $M^k(\tilde{z}_\ell^k)=0$. In fact, there is only one zero in each interval—were there more than one zero in an interval, there would have to be an odd number of them. However, we have proven that there are at most $k+1$ such zeroes; this bound would be violated were there three or more zeroes in some interval. It follows that there are $k-1$ zeroes $\tilde{z}_\ell^k$ corresponding to coefficients m_ℓ^k such that $\rho > a_2^k > m_2^k > \ldots > a_{k-1}^k > m_{k-1}^k > a_k^k$.

(iii) m_{k+1}^k. It remains to show that $a_k^k > m_{k+1}^k$. Suppose instead that $1 > m_{k+1}^k > \rho$. Then $\tilde{z}_{k+1}^k = 1/m_{k+1}^k$ and clearly $M^k(\tilde{z}_k) > 0$ because all terms are now positive. Hence, m_{k+1}^k cannot exceed ρ. Because the other intervals $a_i^k > m_i^k > a_{i+1}^k$ have been accounted for uniquely, it must be that $a_k^k > m_{k+1}^k > 0$.

Now consider the second case, in which it is possible for the c_i^k to be negative. Suppose that $c_k^k < 0$. In that instance, the previous reasoning leads to the conclusion that in the interval $(1/a_\ell^k, 1/a_{\ell+1}^k)$, there are two roots instead of one root, because the sign of $M(1/a_\ell^k)$ is the same as the sign of $M(1/a_{\ell+1}^k)$. Complex roots are ruled out by the existence of a sufficient number of roots in the factorization of gg^*. Therefore there are two real roots in the interval.

We have therefore established that $m_1^k, \ldots, m_{k+1}^k$ lie in the interval $(0, \rho)$.

As all of the above arguments hold for $k=1$, the induction argument is complete.

Assertion (i) of Lemma 4.1 follows from Proposition B.3. ■

Proof (Lemma 4.2): Because T is equivalent to $\tilde{T}$, the result follows from Lemma 4.1, which proves that $\tilde{T}$ maps the sequence $((c_0, c_1, c_2, \ldots), (a_1, a_2, \ldots))$ into a sequence with the same properties. Specifically, a b_{ie}^k with k AR-basis elements is mapped into $\mathcal{H}(\beta)$ with $k+1$ basis elements. ■

Proof (Proposition 4.3): We use the equations for b, J, μ, γ, and g (i.e., equations (7), (9), (11), Lemma 3.1, and (6), respectively) and combine them into a single equation in JJ^*. We have

$$JJ^* = Nbb^*\sigma_e^2 + \sigma_u^2$$

from equation (9). Substituting for b from (7) yields:

$$b = g^{-1}[g^{*-1}(1+\gamma^*)\phi]_+ = g^{-1}A_b\phi$$

where the constant A_b comes from applying the annihilator lemma, Proposition A.1. Thus

$$bb^* = A_b^2 \frac{\phi\phi^*}{\mu(1+\gamma^*) + \mu^*(1+\gamma)}$$

Substituting into the expression for JJ^*,

$$JJ^* = NA_b^2 \frac{\phi\phi^*}{\mu(1+\gamma^*) + \mu^*(1+\gamma)}\sigma_e^2 + \sigma_u^2$$

Again applying the annihilator lemma,

$$\mu = J^{-1}A_\mu\phi,$$

where A_μ is another constant; we will detail its structure below. Also,

$$\gamma = -J^{-1}[J^{*-1}b^*b\sigma_e^2]_+ = -J^{-1}[J^{*-1}\frac{J^*J-\sigma_u^2}{N}]_+ = -\frac{1}{N} + J^{-1}J(0)^{-1}\frac{\sigma_u^2}{N},$$

so

$$1+\gamma = \frac{N-1}{N} + J^{-1}J(0)^{-1}\frac{\sigma_u^2}{N}.$$

Substitute these values in the JJ^* equation:

$$JJ^* = NA_b^2\frac{\phi\phi^*}{J^{-1}A_\mu\phi(\frac{N-1}{N}+J^{*-1}J(0)^{-1}\frac{\sigma_u^2}{N}) + J^{*-1}A_\mu\phi^*(\frac{N-1}{N}+J^{-1}J(0)^{-1}\frac{\sigma_u^2}{N})}\sigma_e^2+\sigma_u^2. \qquad (B.9)$$

This is a nonlinear functional equation in J. It is highly algebraic in character, that is, if J is a rational function then it reduces to finding the roots of a polynomial of finite or possibly infinite order.

Denote the mapping implicit in the recursion equation (B.9) by T^*, with $T^*: L^2(D_\beta)\to L^2(D_\beta)$, where $L^2(D_\beta)$ is the analytic square-integrable functions on the β-disk, $D_\beta = \{z\big||z|\le\beta^{1/2}\}$. The factorization step that recovers J from JJ^* adds the additional mapping $U: L_2(D_\beta)\to H^2(D_\beta)$. The expression for JJ^* can be integrated around the unit circle to calculate the norm. Therefore the following norm bounds hold:

$$\|J\|^2 \le \|\nu\phi J\|^2\sigma_e^2+\sigma_u^2, \qquad (B.10)$$

where ν is defined by the denominator term

$$\nu^{-1}\nu^{*-1} \equiv J^*A_\mu\phi(\frac{N-1}{N}+J^{*-1}J(0)^{-1}\frac{\sigma_u^2}{N}) + JA_\mu\phi^*(\frac{N-1}{N}+J^{-1}J(0)^{-1}\frac{\sigma_u^2}{N})\sigma_e^2.$$

We now establish that there is a fixed point of T^*. We need the following two lemmas, which are used at the end of the proof.

Lemma B.5: *Let $0<f<\rho$. Then*

$$inf_{\{|z|=1\}}2\,Re\frac{1-fz}{1-f\rho} > 1.$$

Proof: The argument works for arbitrary moving-average terms as well. Using the polar form $z=e^{-i\theta}=\cos(\theta)+i\sin(\theta)$,

$$\inf_{\{|z|=1\}}\mathrm{Re}\,(1-fz) = \inf_{\{|z|=1\}}(1-f\cos(\theta)), \qquad (B.11)$$

with the infimum clearly attained at $\theta=0$, or $z=1$. ■

Lemma B.6: *Let $\alpha>0$ and either $\rho>f_1>a_2>f_2>0$ or $\rho>f_1>f_2>a_2>0$. Then*

$$\inf_{\{|z|=1\}}2\mathrm{Re}\,\frac{\frac{1}{\frac{\alpha}{1-f_1z}+\frac{1-\alpha}{1-f_2z}}}{\frac{1}{\frac{\alpha}{1-f_1\rho}+\frac{1-\alpha}{1-f_2\rho}}} > 1.$$

Proof: If $0<\alpha<1$, then there exists a positive real fraction $\beta(z)$ such that

$$\frac{1}{\frac{\alpha}{1-f_1z}+\frac{1-\alpha}{1-f_2z}} = \beta(z)(1-f_1z)+(1-\beta(z))(1-f_2z).$$

Therefore

$$\mathrm{Re}\,(1 - f_1 z) < \beta(z)\mathrm{Re}(1 - f_1 z) + (1 - \beta(z))\mathrm{Re}(1 - f_2 z) < \mathrm{Re}(1 - f_2 z),$$

or

$$\mathrm{Re}\,(1 - f_1 z) > \beta(z)\mathrm{Re}(1 - f_1 z) + (1 - \beta(z))\mathrm{Re}(1 - f_2 z) > \mathrm{Re}(1 - f_2 z).$$

Because the appropriate infimum holds for each of the two terms $1 - f_1 z$ and $1 - f_2 z$ separately by Lemma B.5, the convex combination is bounded between them and we are essentially done.

In the non-interleaved case, $\rho > f_1 > f_2 > a$, and $\alpha > 1$. In that case, $\beta(z) > 1$ as well. Notice that

$$\inf 2\,\mathrm{Re}\,\frac{1 - fz}{1 - \rho f} = \inf 2\,\frac{1 - f}{1 - \rho f}$$

decreases in f. Therefore, raising f_2 toward f_1, possibly to the point where $f_2 > a_2$ and so $\alpha > 1$, can only raise the overall infimum. We have already suitably bounded the infimum of the constituents, so we are done. ■

Proposition B.7:

$$\|\nu\phi\| < 1.$$

Therefore T^* is a contraction mapping and a unique fixed point of T^* exists.

Proof: The proof works by first working out the algebraic expression in (B.9), including the constants, and then establishing a norm bound for a contraction mapping argument. We assume a normalization of $\beta = 1$ in the proof.

Constants. We first calculate the constants. Applying the annihilator lemma yields:

$$A_b = \frac{1 + \gamma(\beta\rho)}{g(\beta\rho)} \qquad A_\mu = \frac{Nb(\beta\rho)}{J(\beta\rho)}\sigma_e^2.$$

We can write (B.9) as

$$JJ^* = N\frac{A_b^2}{A_\mu}\frac{\phi\phi^* JJ^*}{\phi(\frac{N-1}{N}J^* + J(0)^{-1}\frac{\sigma_u^2}{N}) + \phi^*(\frac{N-1}{N}J + J(0)^{-1}\frac{\sigma_u^2}{N})}\sigma_e^2 + \sigma_u^2.$$

Now consider the ratio A_b^2/A_μ:

$$\frac{A_b^2}{A_\mu} = \left(\frac{1 + \gamma(\beta\rho)}{g(\beta\rho)}\right)^2 \frac{J(\beta\rho)}{Nb(\beta\rho)\sigma_e^2}.$$

The constituents of the constant are:

$$b(\beta\rho) = \frac{1 + \gamma(\beta\rho)}{g(\beta\rho)^2}\phi(\beta\rho)$$

and

$$1 + \gamma(\beta\rho) = \frac{N-1}{N} + J(\beta\rho)^{-1}J(0)^{-1}\frac{\sigma_u^2}{N}.$$

After algebraic manipulation the recursion reduces to

$$JJ^* = \frac{1}{\frac{\phi^{*-1}(\frac{N-1}{N}J(0)J^* + \frac{\sigma_u^2}{N})}{\phi(\beta\rho)^{-1}(\frac{N-1}{N}J(0)J(\beta\rho) + \frac{\sigma_u^2}{N})} + \frac{\phi^{-1}(\frac{N-1}{N}J(0)J + \frac{\sigma_u^2}{N})}{\phi(\beta\rho)^{-1}(\frac{N-1}{N}J(0)J(\beta\rho) + \frac{\sigma_u^2}{N})}}JJ^* + \sigma_u^2.$$

We express this succinctly as

$$JJ^* = \frac{1}{H + H^*} JJ^* + \sigma_u^2 \qquad = \frac{1}{2\,\mathrm{Re}\,(H)} JJ^* + \sigma_u^2. \tag{B.12}$$

We can bound the denominator of (B.12). We cannot compute the norm of the denominator directly so we use an indirect argument. The indirect argument is to bound the infimum of the denominator below by one, and this in turn generates a bound on the norm of $1/2\mathrm{Re}(H)$ below one. That is,

$$\frac{1}{2\pi i} \oint_{\{|z=1|\}} \frac{1}{2\,\mathrm{Re}\,(H)} \frac{dz}{z} \le \frac{1}{\inf_{\{|z|=1\}} 2\,Re\,(H)}.$$

We now bound this expression.

Recall that the structure of J is

$$J = J(0) \frac{\prod_1^k (1 - f_i z)}{\prod_1^k (1 - a_i z)} \equiv J(0)\tilde{J},$$

with $a_i \le \rho$ and $f_i \le \rho$. We can write the numerator of H as

$$\frac{N-1}{N} J(0)^2 \frac{\prod_1^k (1 - f_i z)}{\prod_2^k (1 - a_i z)} + \frac{\sigma_u^2}{N} (1 - \rho z),$$

where we note that because of the product $\phi^{-1} J$ in the numerator, there is one more f_i term than a_i term. In common denominator form, the numerator becomes

$$\frac{\frac{N-1}{N} J(0)^2 \prod_1^k (1 - f_i z) + \frac{\sigma_u^2}{N} (1 - \rho z) \prod_2^k (1 - a_i z)}{\prod_2^k (1 - a_i z)},$$

which, using a direct extension of Lemma B.1, can be written as

$$C \frac{\prod_1^k (1 - \tilde{f}_i z)}{\prod_2^k (1 - a_i z)}$$

where C is a constant, and $\rho > \tilde{f} > f_k$. Therefore, we just need to demonstrate that

$$2\inf_{\{|z|=1\}} \mathrm{Re} \left(\frac{\frac{\prod_1^k (1-\tilde{f}_i z)}{\prod_2^k (1-a_i z)}}{\frac{\prod_1^k (1-\tilde{f}_i \rho)}{\prod_2^k (1-a_i \rho)}} \right) > 1.$$

But this follows immediately as an extension of Lemma B.6. Thus the modulus of T^* is a fraction, and we have a contraction. Since the space of analytic functions H^2 (not to be confused with our function H!) is complete, there is a unique fixed point. ■

Manipulating the recursion (B.12) reveals more about the structure of J:

PROPOSITION B.8: *The dynamic structure of J is determined entirely by the structure of ϕ and N, and is independent of σ_e^2 and σ_u^2.*

PROOF: We can make the recursion dimensionless by dividing by σ_u^2/N:

$$\overline{JJ}^* = \frac{1}{\frac{\phi^{*-1}((N-1)\overline{J}(0)\overline{J}^*+1)}{\phi(\beta\rho)^{-1}((N-1)\overline{J}(0)\overline{J}(\beta\rho)+1)} + \frac{\phi^{-1}((N-1)\overline{J}(0)\overline{J}+1)}{\phi(\beta\rho)^{-1}((N-1)\overline{J}(0)\overline{J}(\beta\rho)+1)}} \overline{JJ}^* + 1,$$

which is indepdendent of σ_e^2 and σ_u^2. ■

Corollary B.9: *$J \propto \sigma_u$ and $b \propto \frac{\sigma_u}{\sigma_e}$.*

Proof: The result for J follows immediately from inspecting the JJ^* recursion. The result for b follows from the identity

$$bb^* = \frac{JJ^* - \sigma_u^2}{N\sigma_e^2}.$$

■

Proof (Proposition 4.4): This is an immediate consequence of the previous proposition.

Proof (Proposition 4.5): We have already established that a fixed point $(\bar{c}, \bar{a}) = ((\bar{c}_0, \bar{c}_1, \ldots), (\bar{a}_1, \bar{a}_2, \ldots))$ exists and that the $\bar{a}_\ell$ are ordered; in the limit it is possible that the ordering is only weak. We now show that the ordering is strict and that there are infinitely many $\bar{a}_\ell$ in the interval $(\underline{a}, \rho)$. Suppose there were an m such that there are only finitely many $\bar{a}_\ell$ between ρ and $\underline{a}$ (which by Lemma 4.1 is zero), and that for all $\ell > m$, $\bar{a}_\ell = \underline{a}$. Then there would effectively only be $m+1$ values of $\bar{a}_\ell$, with the $(m+1)^{th}$ value of the numerator coefficient $\bar{c}_\ell$ equal to the sum $\sum_{m+1}^{\infty} c_\ell$. Applying the operator $\tilde{T}$ to this finite list of a_ℓ and c_ℓ terms using Proposition 7.2 generates new values of the a_ℓ, $\{a_\ell{}^*\}$, where

$$\rho = a_1^* = a_1 > a_2^* > a_2 > a_3^* > a_3 > \ldots > a_m > a_{m+1}^* > a_{m+1} = \underline{a} = 0.$$

This contradicts the fixed point property of the original $\{a_\ell\}$. ■

Proof (Proposition 4.6): This is a direct result of factoring in the definition of J from JJ^* in (13) and finding the partial fractions representation. ■

Proof (Proposition 4.7): This follows directly from the equations defining μ and γ. ■

<u>Proportionality results</u>. We now establish the proportionality results in section 5. We already established that $J \propto \sigma_u$ and $b \propto \frac{\sigma_u}{\sigma_e}$ in Corollary B.9. Therefore

$$\gamma(z) \propto \frac{b(0)^2}{J(0)^2}\sigma_e^2 \propto \frac{\frac{\sigma_u^2}{\sigma_e^2}}{\sigma_u^2}\sigma_e^2 \propto 1.$$

Therefore,

$$1 + \sum \gamma_i \propto 1.$$

Now recall that

$$\mu(z) = J^{-1}[J^{*-1}Nb^*\sigma_e^2\phi]_+ \propto \frac{\sigma_e^2}{\sigma_u^2}[b^*\phi]_+ \propto \frac{\sigma_e^2}{\sigma_u^2}\frac{\sigma_u}{\sigma_e}\phi \propto \frac{\sigma_e}{\sigma_u}.$$

Recall that the pricing filter λ is defined as $\mu/(1+\sum\gamma_i)$. Therefore,

$$\lambda \propto \frac{\sigma_e}{\sigma_u}.$$

The proportionality result for profit is similar. From the objective (9), an informed trader's expected profit is:

$$\pi = N(\phi - \mu b)\gamma b\sigma_e^2 + (\phi - \mu b)b\sigma_e^2 + \mu\sigma_u^2.$$

Using the proportionality results for b, μ, and γ, we have

$$\mu b \propto 1; \qquad \gamma b\sigma_e^2 \propto \sigma_u\sigma_e; \qquad b\sigma_e^2 \propto \sigma_u\sigma_e; \qquad \mu\sigma_u^2 \propto \gamma\sigma_e;$$

and therefore

$$\pi \propto \sigma_e\sigma_u.$$

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Table 1: MA expansion of b_{je} for successive iterations

MA expansion	e_{jt}	e_{jt-1}	e_{jt-2}	e_{jt-3}	e_{jt-4}	e_{jt-5}	e_{jt-6}	e_{jt-7}	e_{jt-8}
b_{je}^1	1.000	0.970	0.941	0.913	0.885	0.859	0.833	0.808	0.784
b_{je}^2	0.609	0.539	0.508	0.487	0.457	0.444	0.430	0.417	0.405
b_{je}^3	0.435	0.355	0.313	0.290	0.274	0.263	0.254	0.246	0.238
b_{je}^4	0.377	0.292	0.241	0.211	0.191	0.178	0.168	0.160	0.154
b_{je}^5	0.359	0.274	0.220	0.184	0.160	0.143	0.131	0.122	0.115
b_{je}^6	0.355	0.271	0.215	0.177	0.149	0.124	0.112	0.100	0.090
b_{je}^7	0.355	0.271	0.215	0.177	0.149	0.125	0.112	0.100	0.090

FIGURE 7.1

Table 2: Competition and Equilibrium Outcomes

	Aggregate Variables		
	N=2	N=4	N=32
Price (p_t)	$0.97p_{t-1} + .895x_t$	$0.97p_{t-1} + .889x_t$	$0.97p_{t-1} + .886x_t$
Informed Profit ($\sum_j \pi_j$)	0.760	0.752	0.747
Forecast error (σ^{2FE})	3.87	4.00	4.07

	Lag										
Variable	0	1	2	3	4	5	6	7	8	9	10
b_{je}^2	0.355	0.271	0.215	0.177	0.149	0.128	0.112	0.100	0.090	0.081	0.074
b_{je}^4	0.371	0.267	0.205	0.165	0.137	0.118	0.103	0.091	0.082	0.075	0.068
b_{je}^{32}	0.382	0.264	0.198	0.158	0.131	0.112	0.098	0.087	0.079	0.071	0.065
γ_j^2	-0.101	-0.064	-0.043	-0.030	-0.021	-0.016	-0.012	-0.010	-0.008	-0.006	-0.005
γ_j^4	-0.051	-0.031	-0.020	-0.014	-0.010	-0.008	-0.006	-0.005	-0.004	-0.003	-0.003
γ_j^{32}	-0.006	-0.004	-0.002	-0.002	-0.001	-0.001	-0.001	-0.001	-0.000	-0.000	-0.000
$\sum_j \pi_j^2$	0.359	0.172	0.090	0.051	0.031	0.020	0.013	0.010	0.007	0.005	0.004
$\sum_j \pi_j^4$	0.365	0.166	0.085	0.048	0.029	0.019	0.013	0.009	0.007	0.005	0.004
$\sum_j \pi_j^{32}$	0.370	0.162	0.082	0.046	0.029	0.019	0.013	0.009	0.007	0.005	0.004
σ^{2FE2}	1.095	0.674	0.451	0.323	0.243	0.189	0.152	0.125	0.104	0.088	0.075
σ^{2FE4}	1.073	0.668	0.458	0.335	0.257	0.205	0.167	0.138	0.116	0.099	0.085
σ^{2FE32}	1.059	0.663	0.460	0.341	0.265	0.212	0.174	0.145	0.122	0.104	0.090
corr^2	-0.020	-0.233	-0.425	-0.580	-0.694	-0.775	-0.831	-0.871	-0.898	-0.918	-0.933
corr^4	-0.049	-0.329	-0.536	-0.677	-0.770	-0.830	-0.871	-0.899	-0.919	-0.933	-0.944
corr^{32}	-0.078	-0.400	-0.606	-0.733	-0.810	-0.860	-0.892	-0.915	-0.931	-0.943	-0.952

FIGURE 7.2

Table 3: Persistence and Equilibrium Outcomes

	Aggregate Variables			
	$\rho = 0.97$	$\rho = 0.75$	$\rho = 0.50$	$\rho = 0.00$
Price (p_t)	$0.97p_{t-1} + .895x_t$	$0.75p_{t-1} + .935x_t$	$0.50p_{t-1} + .978x_t$	$1.000x_t$
Informed Profit ($\sum_j \pi_j$)	0.760	0.599	0.542	0.500
Forecast error (σ^{2FE})	3.87	1.017	0.673	0.500
Amount of information ($\frac{1}{1-\beta\rho^2}$)	9.42	2.14	1.31	1.00

	Lag										
Variable	0	1	2	3	4	5	6	7	8	9	10
$b_{je}^{0.97}$	0.355	0.271	0.215	0.177	0.149	0.128	0.112	0.100	0.090	0.081	0.074
$b_{je}^{0.75}$	0.681	0.390	0.238	0.152	0.100	0.068	0.046	0.032	0.022	0.015	0.011
$b_{je}^{0.50}$	0.831	0.330	0.141	0.062	0.028	0.012	0.005	0.003	0.001	0.001	0.000
$\gamma_j^{0.97}$	-0.101	-0.064	-0.043	-0.030	-0.021	-0.016	-0.012	-0.010	-0.008	-0.006	-0.005
$\gamma_j^{0.75}$	-0.206	-0.075	-0.030	-0.013	-0.006	-0.003	-0.002	-0.001	-0.001	-0.000	-0.000
$\gamma_j^{0.50}$	-0.253	-0.060	-0.014	-0.003	-0.001	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000
$\sum_j \pi_j^{0.97}$	0.359	0.172	0.090	0.051	0.031	0.020	0.013	0.010	0.007	0.005	0.004
$\sum_j \pi_j^{0.75}$	0.490	0.083	0.018	0.004	0.001	0.000	0.000	0.000	0.000	0.000	0.000
$\sum_j \pi_j^{0.50}$	0.508	0.031	0.002	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$\sigma^{2FE0.97}$	1.095	0.674	0.451	0.323	0.243	0.189	0.152	0.125	0.104	0.088	0.075
$\sigma^{2FE0.75}$	0.731	0.188	0.060	0.021	0.009	0.004	0.001	0.001	0.000	0.000	0.000
$\sigma^{2FE0.50}$	0.609	0.056	0.006	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
$\mathrm{corr}^{0.97}$	-0.020	-0.233	-0.425	-0.580	-0.694	-0.775	-0.831	-0.871	-0.898	-0.918	-0.933
$\mathrm{corr}^{0.75}$	-0.168	-0.593	-0.829	-0.928	-0.966	-0.981	-0.988	-0.991	-0.992	-0.993	-0.993
$\mathrm{corr}^{0.50}$	-0.238	-0.710	-0.919	-0.977	-0.989	-0.992	-0.992	-0.993	-0.993	-0.993	-0.994

FIGURE 7.3

Table 2: Correlation and Equilibrium Zero-Correlation Benchmarks: $\rho = 0.5$

	Aggregate Variables	
	N=2	N=4
Correlation θ	0.50	0.50
Covariance matrix Σ_e	$\begin{pmatrix} .5 & 0 \\ .5 & 0 \end{pmatrix}$	$\begin{pmatrix} .25 & 0 & 0 & 0 \\ 0 & .25 & 0 & 0 \\ 0 & 0 & .25 & 0 \\ 0 & 0 & 0 & .25 \end{pmatrix}$
Informed Profit ($\sum_j \pi_j$)	0.533189	0.532196

	Lag										
Variable	0	1	2	3	4	5	6	7	8	9	10
b^2_{je} $(\theta = 0)$	.8718	.326	.131	.056	0.024	0.011	0.005	.002	0.001	.000	.000
b^4_{je} $(\theta = 0)$	.8754	.306	.120	.050	.022.010	.004	0.002	0.001	0.000	.000	.000
γ^2_j $(\theta = 0)$	-0.226	-0.047	-0.011	-0.003	-0.001	-0.000	-0.000	-0.000	-0.000	-0.000	.000
γ^4_j $(\theta = 0)$	-0.113	-0.022	-0.005	-0.001	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	.000
πj^2 $(\theta = 0)$	0.496	0.033	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000	.000
πj^4 $(\theta = 0)$	0.496	0.032	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000	.000

FIGURE 7.4

Table 2: Correlation and Equilibrium Outcomes: $\rho = 0.5$

	Aggregate Variables		
	N=2	N=2	N=4
Correlation θ	0.025	0.50	0.50
Covariance matrix Σ_e	$\begin{pmatrix} .487805 & .0121951 \\ .0121951 & .487805 \end{pmatrix}$	$\begin{pmatrix} .333333 & .166667 \\ .333333 & .166667 \end{pmatrix}$	$\begin{pmatrix} .1 & .05 & .05 & .05 \\ .05 & .1 & .05 & .05 \\ .05 & .05 & .1 & .05 \\ .05 & .05 & .05 & .1 \end{pmatrix}$
Informed Profit ($\sum_j \pi_j$)	0.533442	0.505867	0.437422

	Lag										
Variable	0	1	2	3	4	5	6	7	8	9	10
b_{je}^2 ($\theta = .025$)	0.8887	0.333	0.135	0.058	0.026	0.012	0.006	0.003	0.001	0.000	
b_{je}^2 ($\theta = .5$)	1.2574	0.465	0.187	0.081	0.036	0.017	0.008	0.002	0.	0.001	
b_{je}^4 ($\theta = .5$)	1.7676	0.538	0.203	0.085	0.038	0.018	0.008	0.004	0.002	0.001	
γ_j^2 ($\theta = .025$)	-0.231	-0.047	-0.011	-0.003	-0.001	-0.000	-0.000	-0.000	-0.000	-0.000	
γ_j^2 ($\theta = .5$)	-0.311	-0.044	-0.008	-0.002	-0.001	-0.000	-0.000	-0.000	-0.000	-0.000	
γ_j^4 ($\theta = .5$)	-0.190	-0.014	-0.002	-0.001	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	
πj^2 ($\theta = .025$)	0.498	0.032	0.003	0.003	0.000	0.000	0.000	0.000	0.000	0.000	
πj^2 ($\theta = .5$)	0.488	0.018	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
πj^4 ($\theta = .5$)	0.429	0.017	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	

FIGURE 7.5