Stability region estimation for systems with unmodeled dynamics

Ufuk Topcu, Andrew Packard, Peter Seiler, and Gary Balas

***Abstract*— We propose a method to compute invariant subsets of the robust region-of-attraction for the asymptotically stable equilibrium points of systems with polynomial nominal vector fields and unmodeled dynamics. The effects of unmodeled dynamics are accounted for as systems satisfying certain gain relations or dissipation inequalities. The methodology is extended to systems with parametric uncertainties and an informal branch-and-bound type refinement procedure to reduce the conservatism is discussed. We demonstrate the method on a polynomial approximation of uncertain controlled short period aircraft dynamics.**

I. INTRODUCTION

We consider the problem of estimating the "robust" stability regions around stable equilibrium points of uncertain nonlinear dynamical systems. Two types of uncertainties are considered: (1) bounded uncertainties due to unmodeled dynamics in the feedback loop as shown in Figure 1, where $\mathbf{\Phi}$ is considered to be unknown with a known bound on the input-output gain and M is the known nonlinear part of the dynamics; (2) bounded parametric uncertainties in M. Krstic *et al.* demonstrated that existence of unmodeled dynamics may reduce the size of regions-of-attraction for nonlinear dynamical systems [1]. The approach for accounting for the unmodeled dynamics with bounded induced $\mathcal{L}_2 \rightarrow \mathcal{L}_2$ norms is based on separate analyses of input-output gain properties of M and $\mathbf{\Phi}$. A "local" small-gain type argument, which uses the certificates from separate input-output gain analyses, is used to estimate stability regions for the closed-loop dynamics. Following [2], [3], [4], we characterize upper bounds on "local" input-output gains for M due to bounded $\mathcal{L}_2$ disturbances by Lyapunov/storage functions which satisfy certain "local" dissipation inequalities [5]. We use polynomial Lyapunov/storage function candidates, simple generalizations of the S-procedure [6], and sum-of-squares (SOS) relaxations for polynomial nonnegativity [7] and compute upper bounds on the input-output gains via (bilinear) SOS programming problems.

The approach for the bounded parametric uncertainties is similar to that developed in [8], [9] in the context of region-of-attraction analysis for systems with parametric uncertainties (but with no unmodeled dynamics). Namely, a parameter-independent storage function is used to characterize input-output properties over the entire set of admissible values of uncertain parameters. The input-output relations characterized by a single parameter-independent certificate may be more conservative compared to those by parameter-dependent certificates [10]. This potential conservatism is reduced by partitioning the uncertainty set into subregions following an informal branch-and-bound type refinement procedure [11] and computing parameter-independent certificates for each subregion. Although it is simplistic (compared to theories based on parameter-dependent Lyapunov functions), this approach offers certain computational advantages as discussed in section III-C and in [9] in more detail.

Notation: For $x \in \mathcal{R}^n$, $x \succeq 0$ means that $x_k \geq 0$ for $k = 1, \cdots, n$. For $Q = Q^T \in \mathcal{R}^{n \times n}$, $Q \succeq 0$ $(Q \succ 0)$ means that $x^T Q x \geq 0$ (> 0) for all $x \in \mathcal{R}^x$. For $x_1 \in \mathcal{R}^{n_1}$ and $x_2 \in \mathcal{R}^{n_2}$, $[x_1; x_2] \in \mathcal{R}^{n_1+n_2}$ denotes the concatenation of x_1 and x_2. $\mathbb{R}[x]$ represents the set of polynomials in x with real coefficients. The subset $\Sigma[x] := \{\pi = \pi_1^2 + \pi_2^2 + \cdots + \pi_M^2 \ : \ \pi_1, \cdots, \pi_M \in \mathbb{R}[x]\}$ of $\mathbb{R}[x]$ is the set of SOS polynomials. For $\eta > 0$ and a function $g \ : \ \mathcal{R}^n \rightarrow \mathcal{R}$, define the η-sublevel set $\Omega_{g,\eta}$ of g as

$$\Omega_{g,\eta} := \{x \in \mathcal{R}^n \ : \ g(x) \leq \eta\}.$$

For a piecewise continuous map $u : [0, \infty) \rightarrow \mathcal{R}^m$, define the $\mathcal{L}_2$ norm as

$$\|u\|_2 := \sqrt{\int_0^\infty u(t)^T u(t) dt}.$$

In several places, a relationship between an algebraic condition on some real variables and state properties of a dynamical system is claimed, and same symbol for a particular real variable in the algebraic statement as well as the state of the dynamical system is used. This could be a source of confusion, so care on the reader's part is required. ◁

II. PRELIMINARIES

Consider the autonomous nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the state vector and f is an n-vector with entries in $\mathbb{R}[x]$ satisfying $f(0) = 0$, i.e., the origin is an equilibrium point of (1). Let $\phi(t; \mathbf{x}_0)$ denote the solution to (1) at time t with the initial condition $x(0) = \mathbf{x}_0$. The region-of-attraction (ROA) of the origin for the system (1) is

$$\left\{\mathbf{x}_0 \in \mathcal{R}^n : \lim_{t \rightarrow \infty} \phi(t; \mathbf{x}_0) = 0\right\}.$$

U. Topcu is with Control and Dynamical Systems at California Institute of Technology, Pasadena, CA, 91125, USA. utopcu@cds.caltech.edu

A. Packard is with the Department of Mechanical Engineering, The University of California, Berkeley, 94720-1740, USA. pack@jagger.me.berkeley.edu

P. Seiler and G. Balas are with the Department of Aerospace Engineering and Mechanics, The University of Minnesota, Minneapolis, MN, 55455, USA. peter.j.seiler@gmail.com and balas@aem.umn.edu

A modification of a similar result in [12] provides a characterization of invariant subsets of the ROA in terms of sublevel sets of appropriately chosen Lyapunov functions.

Lemma II.1. *Let $\alpha \in \mathcal{R}$ be positive. If there exists a continuously differentiable function $V : \mathcal{R}^n \to \mathcal{R}$ such that*

$$\Omega_{V,\alpha} \text{ is bounded, and} \quad (2)$$
$$V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \in \mathcal{R}^n \quad (3)$$
$$\Omega_{V,\alpha} \setminus \{0\} \subset \{x \in \mathcal{R}^n \ : \ \nabla V(x) f(x) < 0\}, \quad (4)$$

then for all $\mathbf{x}_0 \in \Omega_{V,\alpha}$, the solution of (1) exists, satisfies $\phi(t;\mathbf{x}_0) \in \Omega_{V,\alpha}$ for all $t \geq 0$, and $\lim_{t\to\infty} \phi(t;\mathbf{x}_0) = 0$, i.e., $\Omega_{V,\alpha}$ is an invariant subset of the ROA. ◁

Our goal is to construct Lyapunov functions that estimate the regions-of-attraction for the closed-loop dynamics in Figure 1 based on the input-output properties of M and $\mathbf{\Phi}$. To this end, consider the input-output system governed by

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t)) \\ z(t) &= h(x(t)), \end{aligned} \quad (5)$$

where $x(t) \in \mathcal{R}^n$, $w(t) \in \mathcal{R}^{n_w}$, and f is a n-vector with elements in $\mathbb{R}[(x,w)]$ such that $f(0,0) = 0$ and h is an n_z-vector with elements in $\mathbb{R}[x]$ such that $h(0) = 0$. Let $\phi(t;\mathbf{x}_0, w)$ denote the solution to (5) at time t with the initial condition $x(0) = \mathbf{x}_0$ driven by the input/disturbance w.

Lemma II.2. *[3] If there exists a real scalar $\gamma > 0$ and a continuously differentiable function V such that*

$$V(0) = 0 \text{ and } V(x) \geq 0 \quad \forall x \in \mathcal{R}^n, \quad (6)$$
$$\nabla V f(x,w) \leq w^T w - \gamma^{-2} z^T z \quad (7)$$
$$\forall x \in \Omega_{V,R^2} \text{ and } \forall w \in \mathcal{R}^{n_w},$$

then it holds that for the system in (5)

$$x(0) = 0 \text{ and } \|w\|_2 \leq R \quad \Rightarrow \quad \|y\|_2 \leq \gamma \|w\|_2. \quad (8)$$

◁

In other words, γ is a *local* upper bound for the input-output gain for the system in (5). We call γ to be a *local* upper bound because the upper bound on the norm of the output z is only supposed to hold whenever the norm of the input is bounded by R. This is unlike the input-output gains for linear systems which hold for all values of input norms.

Finally, the following lemma is a straightforward generalization of the S-procedure [6] and is used to obtain algebraic sufficient conditions for certain set containment constraints. See [8] for a proof.

Lemma II.3. *Given $g_0, g_1, \cdots, g_m \in \mathbb{R}[x]$, if there exist $s_1, \cdots, s_m \in \Sigma[x]$ such that*

$$g_0 - \sum_{i=1}^{m} s_i g_i \in \Sigma[x],$$

then

$$\{x \in \mathcal{R}^n \ : g_1(x), \ldots, g_m(x) \geq 0\} \subseteq \{x \in \mathcal{R}^n : g_0(x) \geq 0\}.$$

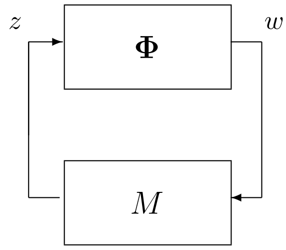

Fig. 1. Feedback interconnection of $\mathbf{\Phi}$ and M.

III. ROA ANALYSIS FOR SYSTEMS WITH UNMODELED DYNAMICS

Consider the system interconnection in Figure 1. Let

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t)) \\ z(t) &= h(x(t)) \end{aligned} \quad (9)$$

be a realization of M, where $x \in \mathcal{R}^n$, f and h are vectors of polynomials satisfying $f(0,0) = 0$ and $h(0) = 0$.

A. Local "small-gain" type theorems

Proposition III.1. *Consider the system interconnection in Figure 1 with $\mathbf{\Phi}$ stable linear time-invariant system satisfying $\|\mathbf{\Phi}\|_\infty < 1$. Let $0 < \gamma < 1$, $R > 0$, and l be a positive definite function with $l(0) = 0$. If there exists a continuously differentiable positive definite function V satisfying $V(0) = 0$ and*

$$\begin{aligned} \nabla V f(x,w) \leq w^T w - \tfrac{1}{\gamma^2} z^T z - l(x) \\ \text{for all } w \in \mathcal{R}^{n_w} \text{ and } x \in \Omega_{V,R^2}, \end{aligned} \quad (10)$$

and Ω_{V,R^2} is bounded, then, for all $x(0) \in \Omega_{V,R^2}$ and $\mathbf{\Phi}$ starting from rest, $x(t) \in \Omega_{V,R^2}$ for all $t \geq 0$ and $x(t) \to 0$ as $t \to \infty$. ◁

Proof: Let

$$\begin{aligned} \dot{\xi}(t) = A\xi(t) + Bz(t) \\ w(t) = C\xi(t) + Dz(t) \end{aligned}$$

be a realization of $\mathbf{\Phi}$ with $\xi \in \mathcal{R}^{n_\xi}$. By Bounded Real Lemma [6], there exist $\tilde{Q} \succ 0$ and $\epsilon > 0$ such that

$$\begin{bmatrix} A^T\tilde{Q} + \tilde{Q}A + C^TC & \tilde{Q}B + C^TD \\ B^T\tilde{Q} + D^TC & -I + D^TD \end{bmatrix} + \epsilon I \preceq 0 \quad (11)$$

Let $Q(\xi) = \xi^T\tilde{Q}\xi$, then, for all $\xi \in \mathcal{R}^{n_\xi}$ and $z \in \mathcal{R}^{n_z}$, the inequality (11) implies that

$$\begin{aligned} \tfrac{d}{dt}Q(\xi) &= \tfrac{\partial Q(\xi)}{\partial \xi}(A\xi + Bz) \\ &= -\epsilon\xi^T\xi - \epsilon z^T z - w^T w + z^T z \\ &\leq z^T z - w^T w - \epsilon\xi^T\xi. \end{aligned} \quad (12)$$

Now, let $(x,\xi) \in \Omega_{V+Q,R^2}$. By (10) and (12),

$$\begin{aligned} \tfrac{d}{dt}\left(V(x) + Q(\xi)\right) &\leq \left(1 - \tfrac{1}{\gamma^2}\right) z^T z - l(x) - \epsilon\xi^T\xi \\ &\leq -l(x) - \epsilon\xi^T\xi. \end{aligned} \quad (13)$$

Integrate (13) from 0 to $T \geq 0$ to get

$$\begin{aligned} V(x(T)) &\leq V(x(T)) + Q(\xi(T)) \\ &\leq -\int_0^T (l(x) + \epsilon\xi^T\xi)dt + V(x(0)) \\ &\leq V(x(0)) \leq R^2, \end{aligned}$$

which implies that, for all $t \geq 0$, $x(t) \in \Omega_{V,R^2}$ whenever $x(0) \in \Omega_{V,R^2}$ and $\xi(0) = 0$. Convergence of (x, ξ) and, in particular of x, follows from Lemma II.1 using the inequality in (13) (note that $S := V + Q$ is a Lyapunov function for the closed-loop system). □

Remarks III.1.
Note that Proposition III.1 only states that Ω_{V,R^2} is invariant but does not assure the invariance of the sublevel sets Ω_{V,r^2} for any $0 < r^2 < R^2$. Hence, V may increase along the trajectories of the closed-loop system. Nevertheless, if $V(x(0)) \leq R^2$ and $\mathbf{\Phi}$ starts from rest, then V cannot exceed R^2 and converges to the origin along every trajectory with $x(0) \in \Omega_{V,R^2}$ and $\xi(0) = 0$. ◁

In the proof Proposition III.1, the linear time-invariance property of $\mathbf{\Phi}$ is only used to establish the existence of a storage function Q satisfying the inequality (12). Therefore, if $\mathbf{\Phi}$ is a dynamical system known to satisfy

$$\frac{d}{dt}Q(\xi) \leq z^T z - w^T w - \epsilon \xi^T \xi$$

for all $\xi \in \mathcal{R}^{n_\xi}$ and $z \in \mathcal{R}^{n_z}$ with $\epsilon > 0$ where Q is a continuously differentiable, positive definite function with bounded sublevel sets and $Q(0) = 0$, then the conclusion of Proposition III.1 still applies.

Proposition III.2. *Consider the system interconnection in Figure 1. Let $0 < \gamma < 1$, $R > 0$, and $l : \mathcal{R}^n \to \mathcal{R}$ and $\tilde{l} : \mathcal{R}^{n_\xi} \to \mathcal{R}$ be positive definite functions with $l(0) =$ and $\tilde{l}(0) = 0$, and let*

$$\begin{array}{l} \dot{\xi}(t) = f_2(\xi(t), z(t)) \\ w(t) = h_2(\xi(t)) \end{array} \quad (14)$$

be a realization of $\mathbf{\Phi}$ with $\xi \in \mathcal{R}^{n_\xi}$ such that there exists a continuously differentiable positive definite function Q that has bounded sublevel sets and satisfies $Q(0) = 0$ and

$$\begin{array}{r} \nabla Q f_2(\xi, z) \leq z^T z - w^T w - \tilde{l}(\xi) \\ \text{for all } z \in \mathcal{R}^{n_z} \text{ and } \xi \in \mathcal{R}^{n_\xi}. \end{array} \quad (15)$$

If there exists a continuously differentiable positive definite function V satisfying $V(0) = 0$ and (10), and Ω_{V,R^2} is bounded, then, for $\xi(0) = 0$ and all $x(0) \in \Omega_{V,R^2}$, $(x(t), \xi(t)) \in \Omega_{V+Q,R^2}$ for all $t \geq 0$ and $\lim_{t\to\infty} (x(t), \xi(t)) = (0, 0)$, in particular, $x(t) \in \Omega_{V,R^2}$ for all $t \geq 0$ and $\lim_{t\to\infty} x(t) = 0$. ◁

Conditions $\mathbf{\Phi}$ in Proposition III.2 can be relaxed to obtain the following result.

Proposition III.3. *Consider the system interconnection in Figure 1. Let $0 < \gamma < 1$, $R > \tilde{R} > 0$, and $l : \mathcal{R}^n \to \mathcal{R}$ be a positive definite function with $l(0) = 0$. Let (14) be a realization of $\mathbf{\Phi}$ such that there exists a continuously differentiable positive definite function Q that has bounded sublevel sets and satisfies $Q(0) = 0$ and*

$$\nabla Q f_2(\xi, z) \leq z^T z - w^T w \quad \text{for all } z \in \mathcal{R}^{n_z} \text{ and } \xi \in \mathcal{R}^{n_\xi}. \quad (16)$$

If there exists a continuously differentiable positive definite function V satisfying $V(0) = 0$ and (10), and Ω_{V,R^2} is bounded, then, for $\xi(0) = 0$ and all $x(0) \in \Omega_{V,\tilde{R}^2}$, $(x(t), \xi(t)) \in \Omega_{V+Q,\tilde{R}^2}$ for all $t \geq 0$ and, in particular, $x(t) \in \Omega_{V,\tilde{R}^2}$ for all $t \geq 0$. Moreover, $\lim_{t\to\infty} x(t) = 0$. ◁

Proof: Define $S := V + Q$.

$$\begin{array}{rl} \frac{d}{dt}S(x,\xi) & = \frac{d}{dt}\left(V(x) + Q(\xi)\right) \\ & \leq -l(x) \ \ \forall x \in \Omega_{V,R^2}, \ \ \forall \xi \in \mathcal{R}^{n_\xi}. \end{array}$$

Let $x(0) \in \Omega_{V,\tilde{R}^2}$ and $\xi(0) = 0$, and integrate this last inequality to get

$$\begin{array}{l} S(x(t), \xi(t)) - S(x(0), \xi(0)) \\ \quad \leq V(x(t)) + Q(\xi(t)) - V(x(0)) \\ \quad \leq -\int_0^t p(x(\tau))\tau, \end{array}$$

and $x(t) \in \Omega_{V,\tilde{R}^2}$ and $(x(t), \xi(t)) \in \Omega_{S,\tilde{R}^2}$ follow. With $x(0) \in \Omega_{V,\tilde{R}^2}$ and $\xi(0) = 0$, S is monotonically non-increasing and bounded below (by zero). Therefore, $\lim_{t\to\infty} \int_0^t \dot{S}(\tau)d\tau$ exists and is finite. Since S is bounded from below and positive definite, $S(x(t), \xi(t))$ is uniformly bounded for $t \geq 0$, and $(\dot{x}(t), \dot{\xi}(t)) = (f(x(t), w(\xi(t))), f_2(z(x(t)), \xi(t)))$ is uniformly bounded for all $t \geq 0$. Hence $x(t), \xi(t)$ is uniformly continuous on $[0, \infty)$. S is uniformly continuous in t on $[0, \infty)$ because $S(x, \xi)$ is uniformly continuous in (x, ξ) on the compact set $\Omega_{S,\tilde{R}^2}$. Therefore, by Barbalat's Lemma [13], $\dot{S}(t) \to 0$ as $t \to \infty$. Consequently, $l(x(t)) \to 0$ and $x(t) \to 0$ as $t \to \infty$. □

B. Estimating the region-of-attraction in the presence of unmodeled dynamics

By Proposition III.1, for positive scalars μ and γ with $\gamma \leq 1$, and linear time-invariant $\mathbf{\Phi}$ with $\|\mathbf{\Phi}\|_\infty < 1$ starting from rest, if there exists positive definite V such that $V(0) = 0$, Ω_{V,R^2} is bounded, and

$$\begin{array}{r} \nabla V f(x,w) \leq w^T w - \frac{1}{\gamma^2} z^T z - \mu V(x) \\ \text{for all } w \in \mathcal{R}^{n_w} \text{ and } x \in \Omega_{V,R^2}, \end{array} \quad (17)$$

then Ω_{V,R^2} is invariant and all trajectories of the closed-loop system with $x(0) \in \Omega_{V,R^2}$ converge to the origin. In order to enlarge Ω_{V,R^2} by choice of V, we define a variable sized region $\Omega_{p,\beta} = \{x \in \mathcal{R}^n \ : \ l(x) \leq \beta\}$, where $p \in \mathbb{R}[x]$ is a fixed positive definite polynomial, and maximize β while imposing the constraint $\Omega_{p,\beta} \subseteq \Omega_{V,R^2}$ along with the constraints $V(0) = 0$, that Ω_{V,R^2} is bounded, and (17), i.e.,

$$\begin{array}{c} \max\limits_{V \in \mathcal{V}, \beta \geq 0, R \geq 0} \quad \beta \quad \text{subject to} \\ V(x) > 0 \text{ for all } x \neq 0, \quad V(0) = 0, \\ \Omega_{p,\beta} \subseteq \Omega_{V,R^2}, \\ \Omega_{V,R^2} \text{ is bounded}, \\ \nabla V f(x,w) \leq w^T w - \frac{1}{\gamma^2} z^T z - \mu V(x) \\ \forall \ x \in \Omega_{V,R^2}, \ \ \forall \ w \in \mathcal{R}^{n_w}, \end{array} \quad (18)$$

where $\mathcal{V}$ is the family of functions over which the maximum in (18) is computed. Now, let l be a positive definite polynomial (typically $l(x) = \epsilon x^T x$ with a small positive scalar ϵ), μ be positive and $0 < \gamma \leq 1$, and $\mathcal{V}_{poly} \subseteq \mathcal{V}$ be a prescribed subset of $\mathbb{R}[x]$. Define

$$\begin{array}{c}\beta^* := \max_{V\in\mathcal{V}_{poly},\beta\geq 0,R\geq 0,s_1\in\mathcal{S}_1,s_2\in\mathcal{S}_2} \beta \text{ subject to} \\ V - l \in \Sigma[x], \\ (R^2 - V) - s_1(\beta - p) \in \Sigma[x], \\ -\nabla V f + w^T w - \frac{1}{\gamma^2} z^T z - \mu V - s_2(R^2 - V) \\ \in \Sigma[(x,w)],\end{array} \quad (19)$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are prescribed subsets of $\mathbb{R}[x]$ and $\mathbb{R}[(x,w)]$, respectively. Then, by Lemma II.3, β^* is a lower bound for the maximum value of β in (18).

Remark III.1. *Note from (18) that computation of the set Ω_{V,R^2} does not require the knowledge of the storage function for $\mathbf{\Phi}$ but requires the extra assumption that $\mathbf{\Phi}$ starts from rest. When a storage function for $\mathbf{\Phi}$ is known, this extra assumption can be removed. Furthermore, even when only an upper bound $\bar{Q} \geq 0$ on the value of Q at the nonzero initial condition of $\mathbf{\Phi}$ is known such that $V(0) + \bar{Q} \leq R^2$, then the conclusion of Proposition III.1 can be modified such that $x(t) \in \Omega_{V,R^2-\bar{Q}}$ for all $t \geq 0$ whenever $x(0) \in \Omega_{V,R^2-\bar{Q}}$ and $x(t) \to 0$ as $t \to \infty$. By Willems [5], such bounds on the value of the storage functions at the initial value of $\mathbf{\Phi}$ may theoretically be established using input-output tests, i.e., does not require the knowledge of specific realization of $\mathbf{\Phi}$. Nevertheless, these bounds may not be easily determined by numerical computation in practice.* ◁

C. ROA analysis in the presence of parametric uncertainties

We now generalize the development in section III-B to the case where M is governed by ordinary differential equations that contain unknown but fixed and bounded parameters. Following the methodology discussed in [9] in the context of robust stability analysis, we first restrict our attention to

$$\begin{array}{rl}\dot{x}(t) & = f(x(t), w(t), \delta) \\ & := f_0(x(t), w(t)) + \sum_{i=1}^m \delta_i f_i(x(t), w(t)) \\ z(t) & = h(x(t)),\end{array} \quad (20)$$

where $f_0, f_1, \ldots, f_m$ are n-vectors with elements in $\mathbb{R}[(x,w)]$ such that $f_0(0,0,\delta) = f_1(0,0,\delta) = \ldots = f_m(0,0,\delta) = 0$, for all $\delta \in \mathbf{\Delta} \subset \mathcal{R}^m$, and $\mathbf{\Delta}$ is a known bounded polytope. Let $\phi(t; \mathbf{x}_0, w, \delta)$ denote the solution of (20) for δ at time t with the initial condition $x(0) = \mathbf{x}_0$ driven by the input/disturbance w and $\mathcal{E}_{\mathbf{\Delta}}$ denote the set of vertices of $\mathbf{\Delta}$.

Proposition III.4. *If, for some $V : \mathcal{R}^n \to \mathcal{R}$, and scalars γ, μ, and R,*

$$\begin{array}{r}\nabla V f(x, w, \delta) \leq w^T w - \frac{1}{\gamma^2} z^T z - \mu V \\ \forall x \in \Omega_{V,R^2}, \ \forall w \in \mathcal{R}^{n_w}\end{array} \quad (21)$$

holds for each $\delta \in \mathcal{E}_{\mathbf{\Delta}}$, then (21) holds for each $\delta \in \mathbf{\Delta}$.

Proposition III.4 follows from the affine δ dependence in (21) and the restriction that $\mathbf{\Delta}$ is a bounded polytope. By Proposition III.4, for positive scalar μ, positive scalar γ with $\gamma \leq 1$, and linear time-invariant $\mathbf{\Phi}$ that starts from rest and satisfies $\|\mathbf{\Phi}\|_\infty < 1$, if there exists a positive definite V such that $V(0) = 0$, Ω_{V,R^2} is bounded, and

$$\begin{array}{r}\nabla V f(x, w, \delta) \leq w^T w - \frac{1}{\gamma^2} z^T z - \mu V(x) \\ \text{for all } w \in \mathcal{R}^{n_w}, \ x \in \Omega_{V,R^2}, \text{ and } \delta \in \mathcal{E}_{\mathbf{\Delta}}\end{array} \quad (22)$$

then all trajectories of the closed-loop system in (20) with $x(0) \in \Omega_{V,R^2}$ converge to the origin. Furthermore, SOS based sufficient conditions for those in (22) can be obtained using Lemma II.3 and SOS relaxations.

The approach proposed here to account for parametric uncertainties is restrictive: (1) only affine dependence on δ and polytopic $\mathbf{\Delta}$ are allowed; (2) SOS relaxations for the conditions in (22) may include a large number of semidefinite programming (SDP) constraints - one for each $\delta \in \mathcal{E}_{\mathbf{\Delta}}$; (3) single ($\delta$-independent) Lyapunov/storage function is to certify properties for an entire family of systems. These limitations can be partially alleviated using techniques proposed in [9]. For example, polynomial dependence on δ in the vector field and the output map can be handled by replacing non-affine appearances of δ by artificial parameters and covering the graph of non-affine functions of δ (in the conditions in (22)) by bounded polytopes in the lifted uncertain parameter space. Furthermore, the fact that constraints in the SOS relaxations for the conditions in (22) are only coupled through the Lyapunov/storage functions (which include relatively small portion of all decision variables in the associated SDPs [1].) can be exploited through a suboptimal two-step procedure, which effectively decouples the large number of constraints enabling the use of trivial parallelization. Finally, conservatism (due to using a single parameter-independent Lyapunov function and due to the suboptimal two-step procedure) can be reduced by an informal branch-and-bound type refinement procedure where $\mathbf{\Delta}$ is partitioned into smaller subregions and a different Lyapunov/storage function is computed for each subregion. See [9] for details.

D. Implementation issues

The SOS relaxations in (19) lead to bilinear SDPs due to the multiplication between the decision variables in V and the multipliers. Therefore, solution techniques for these problems are usually limited local search schemes such as PENBMI [14] or coordinate-wise affine search based on the observation that, for given V and R, constraints in these problems are affine in the decision variables in the multipliers. For example, one can obtain a suboptimal solution for the problem in (19) by alternating between solving *Problems 1* and *2* (stated below) until a maximum number of iterations or an increase in the value of certified R smaller than a pre-specified tolerance is reached.

Problem 1: For given $\bar{V}$ (known to be) feasible for (19), solve

$$\begin{array}{c}\max_{R\geq 0,s_2\in\mathcal{S}_2} R^2 \text{ subject to} \\ -\nabla \bar{V} f + w^T w - \frac{1}{\gamma^2} z^T z - \mu \bar{V} - s_2(R^2 - \bar{V}) \\ \in \Sigma[(x,w)],\end{array} \quad (23)$$

[1]Note that the SOS relaxation for each constraint in (22) contains decision variables in the S-procedure multipliers and those introduced in the corresponding SDP to certify the SOS property [7]

and, with the optimal value $\bar{R}$ of R in (23), solve

$$\begin{array}{c} \max_{\beta \geq 0, s_1 \in \mathcal{S}_1} \beta \text{ subject to} \\ (\bar{R}^2 - V) - s_1(\beta - p) \in \Sigma[x], \end{array} \tag{24}$$

Problem 2: For given $\bar{s}_1$ and $\bar{s}_2$ (known to be) feasible for (19), solve

$$\begin{array}{c} \max_{V \in \mathcal{V}_{poly}, \beta \geq 0, R \geq 0} \beta \text{ subject to} \\ V - l \in \Sigma[x], \\ (R^2 - V) - \bar{s}_1(\beta - p) \in \Sigma[x], \\ -\nabla V f + w^T w - \frac{1}{\gamma^2} z^T z - \mu V - \bar{s}_2(R^2 - V) \\ \in \Sigma[(x, w)], \end{array} \tag{25}$$

Note that *Problems 1* and *2* can be solved using linear SDP solvers, such as SeDuMi [15], through a line search on R and β in (23) and (24), respectively.

IV. Examples

Consider the controlled short period aircraft dynamics shown in Figure 2 with

$$\dot{x}_p = \begin{bmatrix} c_{01}(x_p) + \delta_1 c_{11}(x_p) + \delta_1^2 q_{31}(x_p) \\ q_{02}(x_p) + \delta_1 \ell_{12}^T x_p + \delta_2 q_{22}(x_p) \\ x_1 \end{bmatrix} + \begin{bmatrix} \ell_b^T x_p + b_{11} + b_{12}\delta_1 \\ b_{21} + b_{22}\delta_2 \\ 0 \end{bmatrix} u, \tag{26}$$

where $x_p := [x_1 \; x_2 \; x_3]^T$, x_1, x_2, and x_3 denote the pitch rate, the angle of attack, and the pitch angle, respectively. Here, $\delta_1 \in [0.99, 2.05]$ models variations in the center of gravity in the longitudinal direction and $\delta_2 \in [-0.1, 0.1]$ models variations in the mass. Here, c_{01} and c_{11} are cubic polynomials, q_{02}, q_{22}, and q_{31} are quadratic polynomials, ℓ_{12} and ℓ_b are vectors in $\mathcal{R}^3$, b_{11}, b_{12}, b_{21}, and b_{22} are real scalars (see [9] for the values of the missing parameters). The controller output v, the elevator deflection, is determined by

$$\begin{array}{ll} \dot{x}_4 & = -0.864 y_1 - 0.321 y_2 \\ v & = 2x_4, \end{array} \tag{27}$$

where x_4 is the controller state and the plant output $y = [x_1 \; x_3]^T$. Define $x := \begin{bmatrix} x_p^T & x_4 \end{bmatrix}^T$ and let the system from the input w to the output z be governed by $\dot{x} = f(x, w, \delta)$ and $z = h_z(x)$. Using the procedure explained in section III with $p(x) = x^T x$, $\gamma = 0.99$, and $\mu = 10^{-6}$, we computed estimates of the ROA for two cases:

- (uncertain parameters set to the nominal values $\delta_1 = 1.52$ and $\delta_2 = 0$) $\Omega_{p,4.24}$ and $\Omega_{p,6.67}$ are certified to be in the robust region-of-attraction with $\partial(V) = 2$ and $\partial(V) = 4$, respectively.
- (with parametric uncertainty in the plant $\delta_1 \in [0.99, 2.05]$ and $\delta_2 \in [-0.1, 0.1]$) We implemented a branch-and-bound type refinement procedure coupled with a generalization of the sequential suboptimal solution technique (as mentioned in section III-C and detailed in [9]). $\Omega_{p,2.39}$ and $\Omega_{p,4.14}$ are certified to be in the robust region-of-attraction with $\partial(V) = 2$ and $\partial(V) = 4$, respectively.

On the other hand, in case there is no unmodeled dynamics, the following was proven in [16].

- (uncertain parameters set to the nominal values) $\Omega_{p,9.38}$ and $\Omega_{p,16.11}$ are certified to be in the region-of-attraction with $\partial(V) = 2$ and $\partial(V) = 4$, respectively.
- (with parametric uncertainty in the plant) $\Omega_{p,5.45}$ and $\Omega_{p,7.93}$ are certified to be in the region-of-attraction with $\partial(V) = 2$ and $\partial(V) = 4$, respectively.

V. Conclusions

We proposed a method to compute invariant subsets of the robust region-of-attraction for the asymptotically stable equilibrium points of systems with polynomial nominal vector fields and unmodeled dynamics. The effects of unmodeled dynamics were accounted for as systems satisfying certain gain relations or dissipation inequalities. The methodology was extended to systems with parametric uncertainties and an informal branch-and-bound type refinement procedure to reduce the conservatism is discussed. We demonstrated the method on a polynomial approximation of uncertain controlled short period aircraft dynamics.

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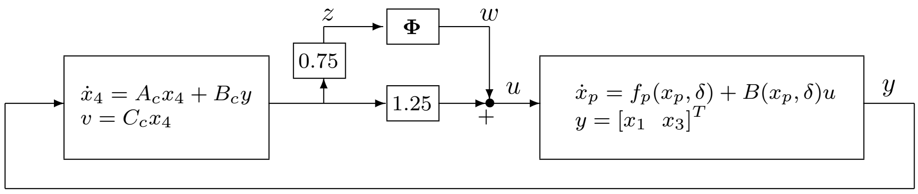

Fig. 2. Controlled short period aircraft dynamics with unmodeled dynamics ($\delta = (\delta_1, \delta_2)$).

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