Nonlinear Analysis of Adaptive Flight Control Laws

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Adaptive control algorithms have the potential to improve performance and reliability of flight control systems. The application of adaptive control on commercial or military aircraft will require validation and verification of the robustness of these algorithms to modeling errors and uncertainties. Currently, there is a lack of tools to rigorously analyze the performance and robustness of adaptive systems. This paper addresses the development of nonlinear robustness analysis tools for such systems. First a model-reference adaptive controller is derived for an aircraft short-period model. It is noted that the adaptive control law is a polynomial system. Polynomial optimization tools are applied to the closed loop model to assess the performance and robustness of the adaptive control law. Two sets of results are presented in this paper. First, input-output gains are calculated in the presence of model uncertainty to evaluate the performance of the adaptive law. Second, time delay margins are computed for varying parameters in the adaptive law, as well as in the presence of model uncertainty.

I. Introduction

Adaptive control laws have great potential to improve the performance and reliability of flight control systems. Adaptive control laws are nonlinear, time-varying, and few tools exist to rigorously analyze their robustness and performance. The lack of tools to validate and verify performance and robustness is a significant roadblock to implementation of adaptive controllers.

The objective of this paper is to demonstrate the suitability of sum-of-squares polynomial optimization tools for the analysis of adaptive control systems. There has recently been significant research on sum-of-squares optimization problems.[1–3] These optimization problems involve constraints on polynomial functions. Sum-of-squares optimizations can be used to analyze the performance and robustness of systems described by polynomial dynamics. Computational algorithms have been developed for estimating regions of attraction, reachability sets, input-output gains, robustness with respect to uncertainty, and time delay margins.[4–23] Moreover, there is freely available software to solve sum-of-squares optimizations, which allows easy application of these techniques to aerospace systems.[24–26]

This paper demonstrates that these sum-of-squares optimization tools can be applied to assess the performance and robustness of adaptive flight control laws. The analysis tools have been previously applied to simple one-state model-reference adaptive control systems.[20,27] This paper focuses on a more realistic flight control problem and the engineering insight that can be drawn from these nonlinear analyses. First a model-reference adaptive controller (MRAC) is developed for the short-period dynamics of an aircraft. It is noted that the MRAC is a polynomial system, hence the closed loop system can be modeled as a polynomial dynamical system. The sum-of-squares optimization tools are ideally suited to analyze MRAC algorithms because the MRAC is a polynomial system. A brief description of sum-of-squares optimization is provided, as well as an overview of the analyis tools. Finally, two applications of the analysis tools for polynomial systems are presented. First, a sufficient condition to compute an upper bound on input-output gain is given. This condition is used to analyze the reference to error gain for the closed-loop short-period dynamics with model uncertainty. Second, a sufficient condition for stability in the presence of time delay is given.

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This condition is used to compute time delay margins for varying parameters in the adaptive law and model uncertainty.

II. MRAC for Short Period Control

This section describes a simple model-reference adaptive controller for the short-period dynamics of the NASA X-15 hypersonic aircraft, taken from Ref. 28. The short-period model is given by:

$$\begin{aligned}\dot{x} &= A_\lambda x + Bu \\ y &= Cx\end{aligned} \tag{1}$$

where $x := [\alpha \text{ (deg)},\ q \text{ (deg/sec)}]^T$ and $u :=$ elevator deflection (deg). The state matrices are:

$$A_\lambda = \begin{bmatrix} -0.2950 & 1.0000 \\ -13.0798\lambda_\alpha & -0.2084\lambda_q \end{bmatrix} \tag{2}$$

$$B = \begin{bmatrix} 0 \\ -9.4725 \end{bmatrix} \tag{3}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{4}$$

where $\lambda := [\lambda_\alpha,\ \lambda_q]^T$ models the uncertainties in two of the aerodynamic coefficients. The nominal model corresponds to $\lambda_\alpha = 1$ and $\lambda_q = 1$, denoted as A_{nom}.

The control law consists of a state feedback K_x, a reference feedforward K_r, and an adaptive augmentation term u_{ad}:

$$u = K_x x + K_r r + u_{ad} \tag{5}$$

where r is the angle of attack reference command. K_x and K_r are chosen to provide the desired system response on the nominal model. Specifically, the reference model is:

$$\dot{x}_m = A_m x_m + B_m r \tag{6}$$

$$y = C_m x_m \tag{7}$$

where $A_m := A_{nom} + BK_x$, $B_m := BK_r$, and $C_m := C$. K_r is chosen as $K_r := (-CA_m^{-1}B)^{-1}$ so that the reference model has a steady state gain of one. It is noted that the reference model is equivalent to the nominal open loop plant model augmented with the state feedback term.

The adaptive term in the control signal is given by:

$$u_{ad} = \theta^T x \tag{8}$$

where θ is updated based on the parameter update law described below. It is assumed that for all λ there exists an ideal gain θ^* such that $A_\lambda + B(K_x + \theta^*) = A_m$. In other words, there exists a gain θ^* such that the uncertain plant matches the reference model. The value of θ^* depends on λ, i.e. the optimal gain depends on the particular values of the plant uncertainties. Specifically, $B\theta^*$ is a solution to the following:

$$B\theta^* = A_m - A_\lambda - BK_x = A_{nom} - A_\lambda \tag{9}$$

The term $B\theta^*$ is a measure of the deviation between the nominal and the uncertain state matrix. For the nominal case, i.e. $A_\lambda = A_{nom}$, it is clear that θ^* is zero. Hence parameter adaptation is not required.

Define the tracking error $e := x - x_m$ and the parameter error $\tilde{\theta} := \theta - \theta^*$. It can be shown that the error dynamics are expressed by Equation 10.

$$\dot{e} = A_m e + B\tilde{\theta}^T x \tag{10}$$

Let $P > 0$ solve the Lyapunov equation $A_m^T P + PA_m = -Q$ for some $Q > 0$. Consider the Lyapunov function candidate in Equation 11 where $\kappa > 0$ is a scalar constant:

$$V = e^T Pe + \frac{1}{\kappa}\tilde{\theta}^T\tilde{\theta} \tag{11}$$

An adaptive control law with sigma modification is proposed in Equation 12.

$$\dot{\theta} = -\kappa x e^T P B - \sigma\theta \tag{12}$$

Using Equations 10 through 12, it can be shown that $\dot{V}$ is the following:

$$\dot{V} = -e^T Q e - \frac{2\sigma}{\kappa}\tilde{\theta}^T\tilde{\theta} \tag{13}$$

In general, Equation 13 is not negative semidefinite because the second term is not sign definite. However, if sigma modification is eliminated, i.e. $\sigma = 0$, then $\dot{V} \leq 0$. Convergence of $e \to 0$ and boundedness of the estimation parameters follows from Lyapunov theory and Barbalat's Lemma.[29] No conclusion is drawn about convergence of the adaptive parameters. The sigma modification is required, with $\sigma > 0$, to increase robustness and to ensure that the adaptive parameters remain bounded. As such, the Lyapunov funcion in Equation 11 can no longer be used to prove convergence of $e \to 0$. It is a known result that sigma modification increases robustness at the expense of the precise convergence of e to the origin.[30] However, all signals remain bounded, and e converges to a small closed region near the origin.[30]

The control law is defined by Equations 5, 8 and 12. The reference model (Equations 6 and 7) is also part of the control law since the reference model state x_m appears in the definition of the tracking error. The adaptive law in Equation 12 is a polynomial function of x and $x - x_m$. Equation 8 is a polynomial function of x and θ. All other terms in the controller are linear. Thus, the key observation is that the MRAC is a polynomial system. In other words the MRAC system is of the form

$$\begin{aligned} \dot{z} &= f(z, y) \\ u &= h(z, y) \end{aligned} \tag{14}$$

where $z := [\theta,\ x_m]^T$ is the MRAC state, $y := [x,\ r]^T$ are the measurements available to the MRAC, and f and h are polynomial functions.

III. Sum-of-Squares Programs

This section provides a brief review of sum-of-squares optimizations. Additional details can be found in Refs. 1–3. A polynomial p is a *sum of squares* (SOS) if there exist polynomials $\{f_i\}_{i=1}^m$ such that $p = \sum_{i=1}^m f_i^2$. For example, $p = x^2 - 4xy + 7y^2$ is a sum of squares since $p = f_1^2 + f_2^2$ where $f_1 = (x - 2y)^2$ and $f_2 = 3y^2$. Note that if p is a sum-of-squares, then $p(x) \geq 0\ \forall x \in \mathbb{R}^n$.

Quadratic forms can be expressed as $p(x) = x^T Q x$ where Q is a symmetric matrix. Similarly, polynomials of degree $\leq 2d$ can be expressed as $p(x) = z(x)^T Q z(x)$ where the vector z contains all monomials of degree $\leq d$. This is known as the Gram matrix form. An important fact is that p is SOS if and only if there exists $Q \succeq 0$ such that $p(x) = z(x)^T Q z(x)$. This provides a connection between SOS polynomials and positive semidefinite matrices.

A sum-of-squares program is an optimization problem with a linear cost and SOS constraints on the decision variables:[24]

$$\begin{aligned} &\min_{u \in \mathbb{R}^n} c^T u \\ &a_{k,0}(x) + a_{k,1}(x)u_1 + \cdots + a_{k,n}(x)u_n \in \text{ SOS } (k = 1, \ldots N_s) \end{aligned} \tag{15}$$

The vector $c \in \mathbb{R}^n$ and polynomials $\{a_{k,j}\}$ are given as part of the optimization data. $u \in \mathbb{R}^n$ are decision variables. SOS programs can be converted to semidefinite programs (SDP) using the connection between SOS polynomials and positive semidefinite matrices. SOSTOOLS,[24] Yalmip,[25] and SOSOPT[26] are freely available MATLAB toolboxes for solving SOS optimizations. These packages allow the user to specify the polynomial constraints using a symbolic toolbox. Then they convert the SOS optimization into an SDP which is solved with SeDuMi[31,32] or another SDP solver. Finally the solution of the SDP is converted back to a polynomial solution.

A drawback is that the size of the resulting SDP grows rapidly in the number of variables and polynomial degree. For a generic degree $2d$ polynomial p in n variables, the Gram matrix representation involves $\binom{n+d}{d}$ monomials. An SOS constraint on p is enforced via the constraint $Q \succeq 0$ on the $l_z \times l_z$ Gram matrix.

For example, Q has dimension $l_z = 495$ for a generic degree 8 polynomial in 8 variables. The size of this matrix constraint is near the limits of current SDP solvers. The problem structure can be exploited[33] but this computational growth is a generic trend in SOS optimizations. For analysis of polynomial systems, this roughly limits the approach to systems with fewer than 8-10 states and cubic degree models. Polynomial models of higher degree can be handled if there are fewer states.

By definition, SOS polynomials are globally positive definite. Lyapunov based analysis is generally centered around satisfying non-negativity conditions on storage functions and their derivatives, which involve the governing system dynamics. Constraining storage functions to be SOS polynomials, Lyapunov type stability certificates can be obtained using SOS optimization. This is only possible if the control systems are governed by polynomial dynamics.

IV. Analysis of Polynomial Systems

This section describes methods to compute input-output gains and time delay margins using SOS techniques. Many other nonlinear analysis problems (estimating regions of attraction, reachability sets, local input-output gains, input-output gains with other signal norms, and robustness with respect to uncertainty) can be formulated as polynomial optimization problems with sum-of-squares constraints.[4–21]

A. SOS Input-Output Gain Analysis

Consider nonlinear dynamical systems of the form:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{16}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the input, and $y \in \mathbb{R}^{n_y}$ is the output. Assume that f is an $n_x \times 1$ polynomial function of x and u such that $f(0,0) = 0$. Also assume that h is an $n_y \times 1$ polynomial function of x such that $h(0) = 0$. Denote this system by $\mathcal{S}$.

Define the L_2 norm of a signal as $\|u\| := \left[\int_0^\infty u^T(t)u(t)dt\right]^{0.5}$. u is called an L_2 signal if this integral is finite, and it is assumed that all inputs to $\mathcal{S}$ are L_2 signals. The L_2-L_2 input-output gain of the system is defined as:

$$\|\mathcal{S}\| := \sup_{\|u\| \neq 0} \frac{\|y\|}{\|u\|} \tag{17}$$

Lemma 1 provides a sufficient condition for the L_2-L_2 input-output gain to be less than γ. This is a standard result which can be found in textbooks.[34,35]

Lemma 1 *If there exists a $\gamma > 0$ and a polynomial $V : \mathbb{R}^n \to \mathbb{R}$ such that:*

1) $V(0) = 0$ and $V(x) \geq 0 \ \ \forall x \in \mathbb{R}^{n_x}$

2) $\gamma^2 u^T u - y^T y - \frac{\partial V}{\partial x} f(x,u) \geq 0 \ \forall x \in \mathbb{R}^{n_u} \ \forall u \in \mathbb{R}^{n_u}$

then $x(0) = 0$ implies $\|y\| \leq \gamma \|u\|$.

Lemma 1 provides a sufficient condition to prove $\|\mathcal{S}\| \leq \gamma$ in terms of a storage function, V. This lemma involves two polynomial non-negativity conditions. Bounds on the system gain can be computed by minimizing γ while searching over a class of polynomial storage functions, e.g. all polynomials V of a specified degree, that satisfy the non-negativity constraints. Constraining a polynomial to be a sum-of-squares is sufficient to ensure that it is globally non-negative. Thus the non-negativity constraints in the system gain calculation can be relaxed to sum-of-squares constraints. This leads to an SOS program that can be used to compute a bound on the system gain. Further details for solving the L_2-L_2 gain problem are discussed in Refs. 14, 15, 18, 20, 27.

B. SOS Time Delay Analysis

An approach for calculating time delay margins using SOS optimization was proposed in Ref. 22. This section reviews the results contained in Ref. 22. The complete derivation for this approach, along with proofs, can be found in Refs. 22 and 23.

Consider nonlinear time delay dynamical systems of the form:

$$\dot{x} = f(x(t), x(t-r)) \tag{18}$$

where $x(t)$ is the current state, and $x(t-r)$ is the delayed state vector. Assume that f is an $n_x \times 1$ polynomial function of $x(t)$ and $x(t-r)$ such that $f(0,0) = 0$. Implicitly, this system is infinite dimensional. The current derivative depends on the current state and the delayed state. However, predicting future states requires knowledge of the entire time history of the state vector on the interval of the time delay. Denote this time history of the state vector as $x(m)$, where $m \in [t-r, t]$. r is the maximum time delay for which stability is shown, hence the time delay margin.

If stability of a system in the presence of time delay is independent of the size of the time delay, then the system is delay-independent stable. Enforcing such stability is conservative for typical engineering applications. Instead, delay-dependent stability is considered where particular time delay margins are found. The time delay is treated as a parameter altering the dynamics of the system. A brief derivation of delay-dependent SOS conditions for stabilty is provided in this section.

The dynamics in Eq. 18 are nonlinear, hence quadratic form Lyapunov functions used in linear analysis will not be sufficient in proving stability. A more complex Lyapunov function structure is required. Consider the function V:

$$V(x(m)) = V_0(x(t)) + \int_{-r}^{0} V_1(\tau, x(t), x(t+\tau))\, d\tau + \int_{-r}^{0}\int_{t+\tau}^{t} V_2(x(\xi))\, d\xi, d\tau \tag{19}$$

where V_0, V_1, and V_2 are polynomials. The polynomial function V depends on the current state and the entire interval of the state trajectory inside the window of the time delay margin. Instead of constraining each term to be non-negative, the kernels of the integrals are constrained to be SOS polynomials. Using this structure for the storage function V, a set of sufficient conditions for global stability for the time delayed system are formulated.

Lemma 2 *Assume the origin is an equilibrium point for the system in Eq. 18, polynomials V_0, V_1, and V_2 exist, and that $\psi(x(t))$ is a positive definite polynomial function such that:*

1) $V_0(x(t)) - \psi(x(t)) \geq 0$
2) $V_1(\tau, x(t), x(t+\tau)) \geq 0 \quad \forall \tau \in [-r, 0]$
3) $V_2(x(\xi)) \geq 0$
4) $r\frac{\partial V_1}{\partial x(t)} f + \frac{dV_o}{dx(t)} f - r\frac{\partial V_1}{\partial \tau} + rV_2(x(t)) - rV_2(x(t+\tau)) + V_1(0, x(t), x(t)) - V_1(-r, x(t), x(t-r)) \leq 0 \quad \forall \tau \in [-r, 0]$

then the origin is a globally stable equilibrium for time delays up to size r.

The first condition ensures that V_0 is positive definite. The second and third conditions ensure that the kernels of the integral terms in V are positive semi-definite. Finally, the fourth condition ensures that the derivative of V is negative semi-definite. Conditions 2 and 4 depend on the time delay τ, hence their respective inqequalities must hold for all time delays on the interval $[-r, 0]$. Together, these constraints are sufficient to prove global stability of the origin.

To translate the conditions in Lemma 2 into SOS polynomial constraints, the time delay interval restriction $\tau \in [-r, 0]$ must enter as a polynomial function in conditions 2 and 4. A variant of the S-procedure is used to augment the interval into the constraints. In condition 2, V_1 is required be positive semi-definite only on the interval $\tau \in [-r, 0]$. Defining a separate polynomial function $h(\tau) = \tau(\tau + r) = \tau^2 + \tau r$, this function is negative only on the given interval. Augmenting $h(\tau)$ to the condition 2, and using an SOS multiplier p_1, the following constraint is derived:

$$V_1(\tau, x(t), x(t+\tau)) + p_1(\tau, x(t), x(t+\tau))\ h(\tau) \in SOS \tag{20}$$

With the same approach, the fourth condition in Lemma 2 can be augmented and converted into an SOS constraint using multiplier p_2. Using this variant of the S-procedure ensures that the required conditions hold on the desired time delay interval, and relaxes the constraints otherwise.

The SOS conditions in Lemma 2 are sufficient to prove global stability. In the case of many nonlinear systems, this is conservative. It may even be infeasibile, as nonlinear systems can have multiple equilibria. To limit the SOS conditions to a local domain around the equilibrim point, a variant of the S-procedure is used. The desired local domain for each state in the system is constrained as:

$$|x_i| \leq \zeta_i \tag{21}$$

To obtain local SOS conditions, polynomial functions h_{ji} are defined as negative in the local region, and positive elsewhere. These functions augment V and $\dot{V}$, hence conditions 1 and 4 must be changed. Since $x(t)$, $x(t+\tau)$, and $x(t-r)$ are treated as separate sets of state variables, three sets of h_{ji} functions are defined. Each state variable set is denoted with the j index. The i index is reserved for the individual state in the set. Consider the following structure of h_{ji} functions:

$$h_{1i} = (x(t) - \zeta_i)\ (x(t) + \zeta_i) \tag{22}$$

$$h_{2i} = (x(t+\tau) - \zeta_i)\ (x(t+\tau) + \zeta_i) \tag{23}$$

$$h_{3i} = (x(t-r) - \zeta_i)\ (x(t-r) + \zeta_i) \tag{24}$$

The h_{ji} polynomials enter the conditons with SOS multiplier functions q_{ji}. The final result is a set of sufficient conditions that prove local stability for a range of time delays. The final SOS conditions are summarized with the following:

Lemma 3 *Assume that the origin is an equilibrium point for the system in 18, that polynomials V_0, V_1, and V_2 exist, and that $\psi(x(t))$, p_1, p_2, and q_{ji} are SOS polynomials such that:*

1) $V_0(x(t)) - \psi(x(t)) + \sum_{i=1}^{n} q_{1i}h_{1i}$ *is SOS*

2) $V_1(\tau, x(t), x(t+\tau)) + p_1(\tau, x(t), x(t+\tau))h(\tau)$ *is SOS*

3) $V_2(x(\xi))$ *is SOS*

4) $-r\frac{\partial V_1}{\partial x(t)}f - \frac{dV_o}{dx(t)}f + r\frac{\partial V_1}{\partial \tau} - rV_2(x(t)) + rV_2(x(t+\tau)) - V_1(0, x(t), x(t)) + V_1(-r, x(t), x(t-r)) + p_2(\tau, x(t), x(t+\tau))h(\tau) + \sum_{i=1}^{n}(q_{1i}h_{1i} + q_{2i}h_{2i} + q_{3i}h_{3i})$ *is SOS*

then the origin is a locally stable equilibrium for time delays up to size r.

The conditions in Lemma 3 can be used with a bisection algorithm to calculate the time delay margin for the system. Time delay margins are crucial in nonlinear performance analysis since classical notions of gain and phase margins do not exist.

V. Results

SOS methods described in the previous section are used to analyze the MRAC system. This section summarizes the findings and insightful conclusions that can be drawn from the analysis. First, input-output gains are calculated to aid in the understanding of the interaction between adaptation and plant uncertainty. Then, time delay margins are found to analyze how the system robustness depends on the sigma modification parameter and plant model uncertainty.

A. Input-Output Gain Analysis

Input-output L_2 gains for the MRAC system described in Section II are caluclated using the proposed SOS methods from Section IV. Figure 1 provides a graphical interpretation of the system architecture. The gain considered in this analysis computes bounds on the L_2 gain from input reference r to error signal e. Note that signal e is the error between the reference model state x_m and the aircraft state x. The aircraft model is denoted by P_λ, specifying that it is an uncertain model. Recall that uncertainty enters the state matrix directly through scaling parameters λ_α and λ_q on the (2,1) and (2,2) entries, respectively. The MRAC architecture consists of a state feedback, a reference feedforward, and a parameter update law term. The components pertaining to the conroller are contained in the dashed box seen in Figure 1.

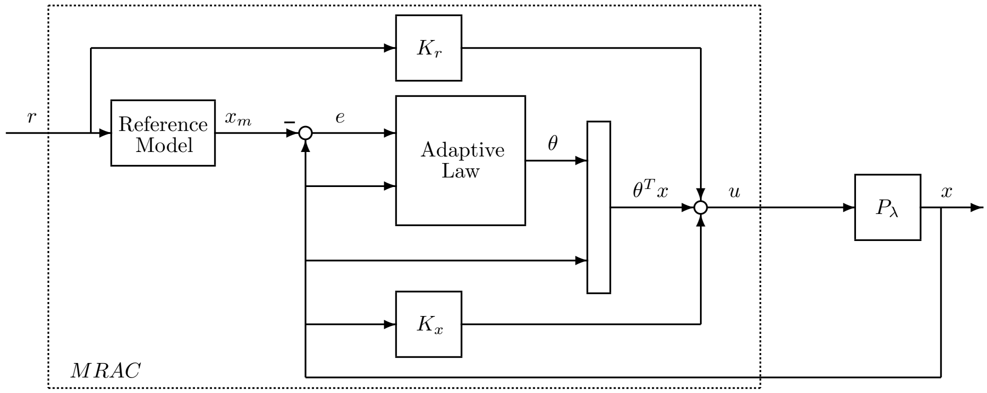


Figure 1: Architecture for MRAC gain analysis

The adaptive law in Figure 1 is defined with Equation 12. Although sigma modification is used, the parameter update law is mainly driven by error signal e. This error represents deviation of the true aircraft state from the reference model state. Using the nominal condition, i.e. λ_α=λ_q= 1, the reference model is defined as the nominal model $P_{\lambda=1}$ augmented with state feedback term K_x. Recall from Section II that the gain K_r is built into the reference model's input matrix. In the nominal case, with $\theta^* = 0$, the error dynamics in Equation 10 simplify to the following:

$$\dot{e} = A_m e + B\theta^T x \tag{25}$$

Note that the error dynamics now depend only on θ. Given this result and the parameter update law in Equation 12, $e(t) = 0$ and $\theta(t) = 0$ are valid solutions given any reference signal. Hence in the nominal case without uncertainty, the gain from reference to error is zero. The true aircraft state is always equal to the reference model state. As a result, the adaptive law is not activated.

When uncertainty is considered, the models no longer match exactly, and a transient occurs in the error signal. The transient is used by the adaptive law to estimate the uncertainty and drive the true aircraft state to the reference model state. The goal of the analysis is to gain engineering insight into the interaction of model uncertainty with the error transients pertaining to the adaptive law.

For the controller design, the following MRAC parameters are used:

$$K_x = \begin{bmatrix} 0.0577 & 0.9843 \end{bmatrix} \tag{26}$$
$$Q = 2I_2 \tag{27}$$
$$\kappa = 1 \tag{28}$$
$$\sigma = 1 \tag{29}$$

The state feedback term K_x is used to place the poles of the referece model. The short-period model is very lightly damped in the open loop, with damping ratio 0.069. K_x is used as a stability augmentation system to move the poles to the real axis, eliminating any oscillation.

The short period-model of the X-15 depends on uncertainties λ_α and λ_q. Thus the closed loop L_2 gain from r to e is also a function of these uncertainties, denoted $\gamma(\lambda_\alpha, \lambda_q)$. The uncertain parameters enter the state matrix by multiplying the (2,1) and (2,2) entires, hence considering values ranging from 0 to 2 represents 100 percent uncertainty in the respective aerodynamic coefficients.

As noted above, the nominal model perfectly matches the reference model for any input r and thus the gain from r to e is zero for the nominal model, $\gamma(1, 1) = 0$. The gain is nonzero for other values of the plant uncertainty. A grid was created for the uncertain parameters, and the gain was calculated using SOSOPT.[26] A 25x25 grid on λ_α and λ_q took 104 seconds to calculate. Figure 2 illustrates the gain as a function of uncertainty in the plant model. Values of λ_α and λ_q ranging from 0 to 2 were selected.

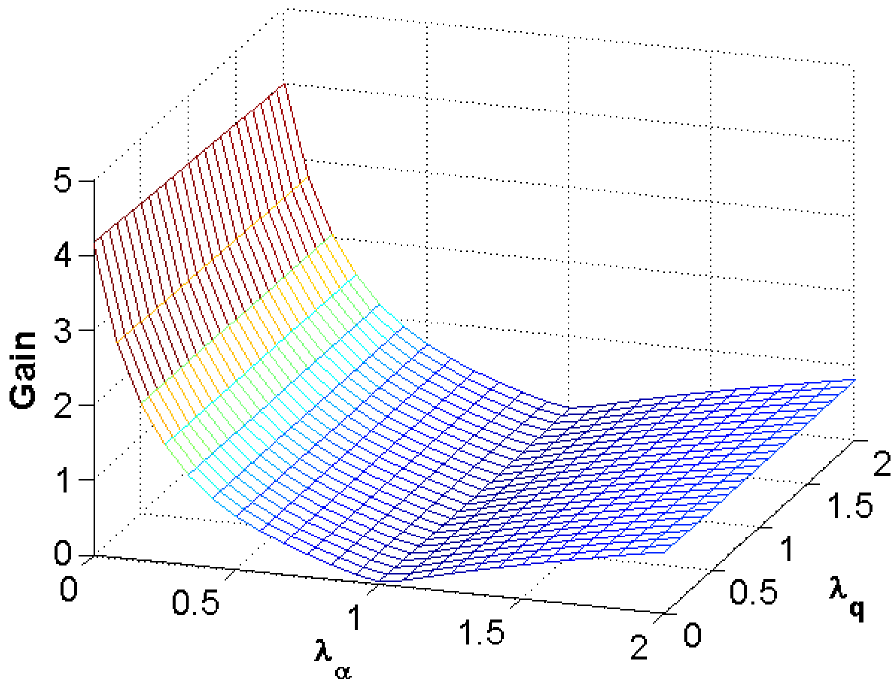


Figure 2: Reference to error gain versus plant uncertainty

The input-output gain analysis provides several insightful conclusions. As expected, the gain for the nominal condition is zero. The results in Figure 2 also indicate that the system is more sensitive to uncertainty entering through λ_α than through λ_q. The system can tolerate 100 percent uncertainty through λ_q without significant effects in the closed loop dynamics. Equivalent uncertainty through λ_α induces larger variations in the reference to error gain.

It is important to note that when the uncertainty reaches a certain level, e.g. $\lambda_\alpha \leq -0.005$, the open loop aircraft model becomes unstable. In this particular example, a sign in the state matrix has flipped. The magnitude of λ_α is large enough to completely alter the open loop flight dynamics of the X-15. A positive deflection in angle of attack now induces a positive pitch acceleration, which causes instability in the longitudinal axis. However, the closed loop dynamics remain stable. The current SOS algorithm is unable to reach a feasible result when the open loop plant is unstable. Hence, gain bounds are only computed for uncertainties for which the open loop plant P_λ is stable.

B. Time Delay Margin Analysis

In this section, time delay margins for the MRAC system are calculated. SOS methods are used to satisfy the conditions in Lemma 3, constructing a polynomial Lyapunov function sufficient to prove stability in the presence of time delay. Figure 3 illustrates the system architecture considered in this analysis. The reference signal is neglected and the remaining components form a system equivalent to a regulator control problem with time delay. The time delay occurs between the controller and the plant.

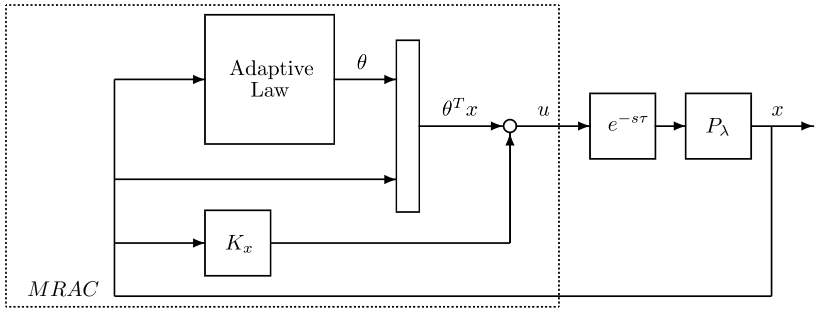


Figure 3: Architecture for MRAC time delay margin analysis

It should be noted that the construction of Lyapunov functions that prove local stability in the presence of time delays is not at odds with known results about MRAC with a sigma-modification. Recall that the simple Lyapunov function V in Equation 11 is used to design the MRAC. In the absence of time-delay, $\dot{V}$ is given by Equation 13. As previously mentioned, $\dot{V}$ is not necessarily negative semidefinite for small values of e and hence this Lyapunov function can only be used to prove that e converges to a neighborhood of the origin if $\sigma > 0$. When the reference signal is omitted it is possible to construct quadratic Lyapunov functions that prove converge of both x and θ to the origin in the absence of time delays. The main point is that it is possible to construct Lyapunov functions that prove asymptotic converge of the state and parameters when the reference signal is omitted.

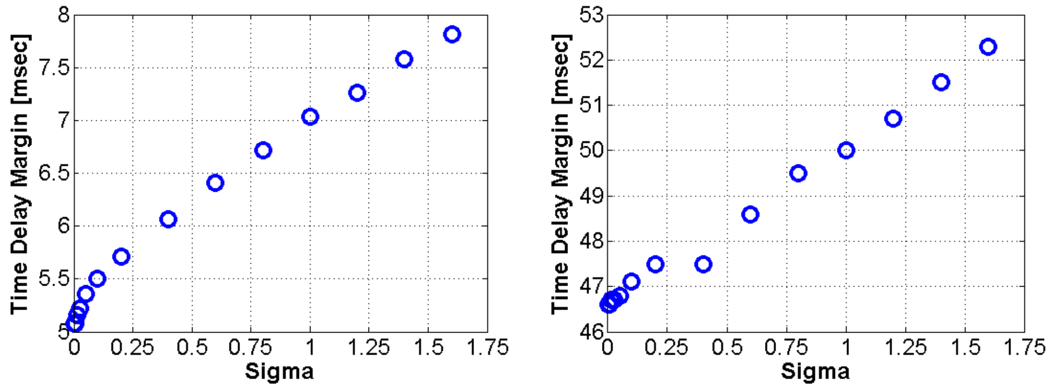
A region in the closed loop aircraft model state space around the origin is designated for local stability analysis. The MRAC system is nonlinear, hence local stability is a relevant and sufficient result. The local region in the plant state space is selected as $\pm$ 2 deg on α and $\pm$ 5 deg/sec on q. This region is large enough to be relevant for engineering analysis. The definition of a local region in the adaptive law parameter θ state space is also required. Gridding and simulating over allowable initial conditions in the plant, the response of the full system state is observed. Based on these simulations, trajectories of the two adaptive law parameter states satisfy a local region constrained by $\pm$ 0.8 and $\pm$ 1.4 in each direction, respectively. Together, these 4 state space constraints form a box that contains all possible trajectories originating from admissible plant initial conditions. This local analysis constraint is added to the SOS conditions as described in Section IV. A limitation of the SOS conditions used in this analysis is the inability to obtain feasible results when the local region in the state space is arbitrarily large.

The first set of results explore the change in time delay margin as a function of sigma modification. The calculated time delay margins from the SOS algorithm serves as a lower bound on the true value. Simulations are used to find upper bounds. For these simulations, the plant initial condition is $\alpha = 0$ deg and $q = 5$ deg/sec. The time delay is increased until the signals exhibit growing oscillations. Figure 4 summarizes the effect of sigma modification on the lower and upper bounds of time delay margin.

For both the lower and upper bounds, an increase in sigma modification results in a larger time delay margin. This was expected, as the purpose of sigma modification is increased robustness. The lower bound is conservative compared to the upper bound, with an order of magnitude difference. However, this result is a dramatic improvement over recent work on time delay margins for MRAC systems. A similar problem was studied in Ref. 28 and the gap between bounds was of 3 orders of magnitude. Although the result of this analysis is still conservative, the gap between lower and upper bounds is greatly reduced. Exploring more advanced lower bound methods is a next step in reducing the gap, and have been shown to obtain less conservative results in simple examples.[23]

Interpreting the results in Figure 4a, as the sigma modification approaches zero, the time delay margin reaches a value of 5.1 msec. The SOS algorithm has numerical problems when sigma modification equals zero, i.e. a feasible solution cannot be found for any value of time delay. The time delay margin increases as sigma modification increases, but this analysis considers σ values up to 2. Higher sigma modification masks the intended MRAC dynamics by significantly altering the adaptive parameters dynamics and causing worse performance in the tracking error.

The next set of results calculate time delay margin in the presence of plant uncertainty. Sigma modifcation with a value of 1 is used in the control design. As noted previously from the results in Figure 2, the plant is not sensitive to fluctuations in λ_q. A less refined grid is used in this direction in the interest of reducing

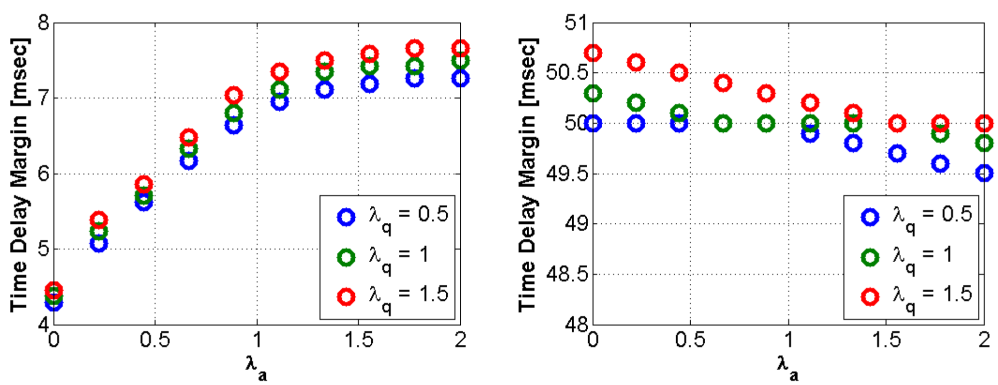


(a) Lower Bound (b) Upper Bound

Figure 4: Time delay margin versus sigma modification

computation time. This is particularly convenient since the computation time exceeds 20 minutes per data point.Therefore, the grid on λ_q is coarse, and fine on λ_α.

The SOS algorithm is able to prove stability by establishing a lower bound on the time delay margin. Simulations are used to find suitable upper bounds. Figure 5 shows the calculated lower and upper bounds on time delay margin in the presence of plant uncertainty.



(a) Lower Bound (b) Upper Bound

Figure 5: Time delay margins in the presence of plant uncertainty

The lower bound on time delay margin found with the SOS algorithm is again conservative. In the lower bound, the time delay margin grows as the uncertainty parameter λ_α increases. Simulations find that the upper bound on the time delay margin is relatively constant at around 50 msec. As predicted, the time delay margin is not very sensitive to variations in the uncertainty parameter λ_q. Future work includes improving the SOS conditions with algorithms known to provide less conservative results.[23] Monte carlo simulations can also be used to find a smaller upper bounds on the time delay margin.

VI. Conclusion

Adaptive control algorithms have the potential to improve performance and robustness in aerospace systems. However, there is a lack of tools available to rigorously analyze these systems. This paper uses polynomial optimization tools to show the suitability of such analysis in the verification and validation of adaptive control systems. The performance and robustness of a model-reference adaptive controller for a short-period aircraft model is examined. Input-output gains and time delay margins are calculated. The input-output gain analysis details the sensitivity of the system to plant uncertainty. The time delay margin analysis provides insight to the effects of sigma modification as well as plant uncertainty on stability in the presence of time delay. The findings are less conservative than previous results. Future work includes refining the SOS conditions with algorithms known to provide less conservative results. More refined simulations will be performed to seek lesser upper bounds on time delay margin.

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References

[1]Parrilo, P., *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*, Ph.D. thesis, California Institute of Technology, 2000.

[2]Lasserre, J., "Global Optimization with Polynomials and the Problem of Moments," *SIAM Journal on Optimization*, Vol. 11, No. 3, 2001, pp. 796–817.

[3]Parrilo, P., "Semidefinite programming relaxations for semialgebraic problems," *Mathematical Programming Ser. B*, 2003.

[4]Tibken, B., "Estimation of the domain of attraction for polynomial systems via LMIs," *Proceedings of the IEEE Conference on Decision and Control*, 2000, pp. 3860–3864.

[5]Hachicho, O. and Tibken, B., "Estimating domains of attraction of a class of nonlinear dynamical systems with LMI methods based on the theory of moments," *Proceedings of the IEEE Conference on Decision and Control*, 2002, pp. 3150–3155.

[6]Jarvis-Wloszek, Z., *Lyapunov Based Analysis and Controller Synthesis for Polynomial Systems using Sum-of-Squares Optimization*, Ph.D. thesis, University of California, Berkeley, 2003.

[7]Jarvis-Wloszek, Z., Feeley, R., Tan, W., Sun, K., and Packard, A., "Some Controls Applications of Sum of Squares Programming," *Proceedings of the 42nd IEEE Conference on Decision and Control*, Vol. 5, 2003, pp. 4676–4681.

[8]Tan, W. and Packard, A., "Searching for control Lyapunov functions using sums of squares programming," *42nd Annual Allerton Conference on Communications, Control and Computing*, 2004, pp. 210–219.

[9]Chesi, G., "On the estimation of the domain of attraction for uncertain polynomial systems via LMIs," *Proceedings of the IEEE Conference on Decision and Control*, 2004, pp. 881–886.

[10]Papachristodoulou, A., *Scalable analysis of nonlinear systems using convex optimization*, Ph.D. thesis, California Institute of Technology, 2005.

[11]Jarvis-Wloszek, Z., Feeley, R., Tan, W., Sun, K., and Packard, A., *Positive Polynomials in Control*, Vol. 312 of *Lecture Notes in Control and Information Sciences*, chap. Controls Applications of Sum of Squares Programming, Springer-Verlag, 2005, pp. 3–22.

[12]Chesi, G., Garulli, A., Tesi, A., and Vicino, A., "LMI-based computation of optimal quadratic Lyapunov functions for odd polynomial systems," *International Journal of Robust and Nonlinear Control*, Vol. 15, 2005, pp. 35–49.

[13]Prajna, S., *Optimization-Based Methods for Nonlinear and Hybrid Systems Verification*, Ph.D. thesis, California Institute of Technology, 2005.

[14]Tan, W., Packard, A., and Wheeler, T., "Local gain analysis of nonlinear systems," *Proceedings of the American Control Conference*, pp. 92–96.

[15]Tan, W., *Nonlinear Control Analysis and Synthesis using Sum-of-Squares Programming*, Ph.D. thesis, University of California, Berkeley, 2006.

[16]Tibken, B. and Fan, Y., "Computing the domain of attraction for polynomial systems via BMI optimization methods," *Proceedings of the American Control Conference*, 2006, pp. 117–122.

[17]Topcu, U., Packard, A., Seiler, P., and Wheeler, T., "Stability region analysis using simulations and sum-of-squares programming," *Proceedings of the American Control Conference*, 2007, pp. 6009–6014.

[18]Tan, W., Topcu, U., Seiler, P., Balas, G., and Packard, A., "Simulation-aided Reachability and Local Gain Analysis for Nonlinear Dynamical Systems," *Proceedings of the IEEE Conference on Decision and Control*, 2008, pp. 4097–4102.

[19]Topcu, U., Packard, A., and Seiler, P., "Local stability analysis using simulations and sum-of-squares programming," *Automatica*, Vol. 44, No. 10, 2008, pp. 2669–2675.

[20]Topcu, U., *Quantitative Local Analysis of Nonlinear Systems*, Ph.D. thesis, University of California, Berkeley, 2008.

[21]Topcu, U. and Packard, A., "Linearized analysis versus optimization-based nonlinear analysis for nonlinear systems," *submitted to the 2009 American Control Conference*, 2009.

[22]Papachristodoulou, A., "Analysis of Nonlinear Time-Delay Systems using Sum of Squares Decomposition," *Proceedings of the American Control Conference*, 2004, pp. 4153–4158.

[23]Papachristodoulou, A., Peet, M., and Lall, S., "Analysis of Polynomial Systems With Time Delays via Sum of Squares Decompostion," *IEEE Transactions on Automatic Control*, Vol. 54, No. 5, 2009.

[24]Prajna, S., Papachristodoulou, A., Seiler, P., and Parrilo, P. A., *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*, 2004.

[25]Lofberg, J., "YALMIP : A Toolbox for Modeling and Optimization in MATLAB," *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.

[26]Balas, G., Packard, A., Seiler, P., and Topcu, U., "Robustness Analysis of Nonlinear Systems," http://www.cds.caltech.edu/ utopcu/LangleyWorkshop.html.

[27]Seiler, P., Balas, G., Packard, A., and Topcu, U., "Analytical Validation Tools for Safety Critical Systems," *AIAA Infotech@Aerospace Conference*, No. AIAA 2009-1991, 2009.

[28]Annaswamy, A., Jang, J., and Lavretsky, E., "Stability margins for adaptive controllers in the presence of time delay," *AIAA Guidance, Navigation, and Control Conference*, No. AIAA 2008-6659, 2008.

[29]Narendra, K. and Annaswamy, A., *Stable Adaptive Systems*, Prentice Hall, 1989.

[30]Ioannou, P. and Sun, J., *Robust Adaptive Control*, Prentice Hall, 1995.

[31]Sturm, J., "SeDuMi version 1.05," http://fewcal.kub.nl/sturm/software/sedumi.html, 2001.

[32]Sturm, J., "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, 1999, pp. 625–653.

[33]Gatermann, K. and Parrilo, P., "Symmetry groups, semidefinite programs, and sums of squares," *Journal of Pure and Applied Algebra*, Vol. 192, 2004, pp. 95–128.

[34]Vidyasagar, M., *Nonlinear Systems Analysis*, Prentice Hall, 2nd ed., 1993.

[35]Khalil, H., *Nonlinear Systems*, Prentice Hall, 3rd ed., 2002.