A Dissipation Inequality Formulation for Stability Analysis with Integral Quadratic Constraints

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***Abstract*— Integral quadratic constraints (IQCs) provide a general framework for robustness analysis of feedback interconnections. The main IQC stability theorem by Megretski and Rantzer was formulated with frequency domain conditions that depend on the IQC multiplier. Their proof of this theorem uses a homotopy method and operator theory. An interesting aspect of this theory is that input/output stability (defined as uniformly bounded gain over all finite horizons) is established using integral constraints that only hold, in general, on infinite time horizons. The use of IQCs that only hold over infinite time horizons is related to the use of noncausal multipliers in absolute stability theory. This paper shows that if the conditions of the IQC stability theorem are satisfied by any rational IQC multiplier then a dissipation inequality is satisfied by a quadratic storage function. This provides a new interpretation for IQC analysis in terms of quadratic storage functions and a causal, finite-horizon dissipation inequality.**

I. Introduction

Integral quadratic constraints (IQCs), introduced in [10], [11], [12], provide a general framework for robustness analysis of linear systems with respect to nonlinearities and uncertainties. In this framework the system is separated into a feedback connection of a known, linear time-invariant system and a perturbation whose input-output behavior is described by an integral quadratic constraint. The IQC stability theorem in [10], [12] was formulated with frequency domain conditions and was proved using a homotopy method and operator theory. In [4] it was shown that the use of IQC multipliers that satisfy a certain convexity condition is equivalent to the use of multipliers in the absolute stability theory [3], [25].

The integral constraint used to describe the perturbation can be expressed in either the frequency domain or the time domain. For the time domain interpretation, [12] draws a distinction between hard IQCs for which the integral constraint is valid over all finite time intervals and soft IQCs for which the integral constraint need not hold over finite time intervals. The use of soft IQCs amounts to proving stability (defined as having uniformly bounded input/output gain on all finite horizons) using integral constraints that only hold, in general, on infinite time horizons. Moreover, in [12] a connection is drawn between soft IQCs and the use of non-causal multipliers in the theory of absolute stability.

This paper proves IQC stability theorems using dissipativity theory [22], [23]. Previous work [2], [9], [17], [6] has related IQC analysis to dissipativity theory only for the special case of hard IQCs. This paper shows that if the conditions of the IQC stability theorem are satisfied by any rational IQC multiplier then stability can be established by a quadratic storage function. The dissipation inequality satisfied by the storage function uses a Lagrange multiplier to capture information provided by the IQC. This result is important because it provides a connection between dissipativity theory and IQC stability theory with rational IQC multipliers. The dissipation inequality is a time-domain condition and hence this might lead to certain generalizations that would be difficult, if not impossible, to obtain with the frequency domain IQC conditions. This result also has pedagogical importance because the dissipation inequality leads to a finite horizon interpretation for the use of soft IQCs. As a corollary, it also gives a causal interpretation for the use of non-causal multipliers in absolute stability theory.

The dissipation inequality interpretation relies on an IQC factorization theorem that is of independent interest. In most cases IQC multipliers are specified in the frequency domain and the time-domain interpretation arises by performing a factorization of the multiplier [12]. This factorization is not unique and structured factorizations have been proposed for robust filter design [18], [19]. One fact that seems unnoticed in the literature is that an IQC multiplier can be classified as soft or hard depending on the factorization. In other words, the notions of soft and hard are not inherent to the IQC multiplier but are factorization dependent. An example is provided to demonstrate this fact. It is then shown that hard IQC factorizations exist for a large class of known multipliers. This hard IQC factorization theorem forms the basis for connecting IQC stability theory to dissipativity theory. Due to space constraints, most results in this paper are presented without proofs.

II. Integral Quadratic Constraints

A. Background

Let $\Pi : j\mathbb{R} \rightarrow \mathbb{C}^{(l+m)\times(l+m)}$ be a measurable Hermitian-valued function. Two signals $z \in L_2^m[0,\infty)$ and $v \in L_2^l[0,\infty)$ satisfy the integral quadratic constraint (IQC) defined by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{z}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{z}(j\omega) \end{bmatrix} d\omega \geq 0 \qquad (1)$$

where $\hat{v}(j\omega)$ and $\hat{z}(j\omega)$ are Fourier transforms of v and z. IQCs were introduced in [12] for stability analysis of feedback interconnections. Π is called an "IQC multiplier" or simply a "multiplier". The term "multiplier" is used in a

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different context in absolute stability theory [3], [25]. The connections between IQC multipliers and absolute stability multipliers is clarified in Section III-D. IQCs can be used to describe the relationship between input-output signals of system components. A bounded operator $\Delta : L_{2e}^l[0,\infty) \rightarrow L_{2e}^m[0,\infty)$ satisfies the IQC defined by Π if Equation 1 holds for all (v, z) where $v \in L_2^l[0,\infty)$ and $z = \Delta(v)$.

The IQC in Equation 1 can be equivalently expressed in the time domain. Assume that Π is a rational and uniformly bounded function. Then $\Pi(j\omega)$ can be factored as $\Pi(j\omega) = \Psi(j\omega)^* M \Psi(j\omega)$ where M is a constant matrix and $\Psi(j\omega) \in \mathbb{RH}_\infty^{(l+m)\times(l+m)}$ [24], [17], [9]. $\Psi(j\omega)$ is denoted as:

$$\Psi(j\omega) := C_\psi(j\omega I - A_\psi)^{-1}[B_{\psi 1}\ B_{\psi 2}] + [D_{\psi 1}\ D_{\psi 2}] \quad (2)$$

The IQC (Equation 1) can be expressed as:

$$\int_0^\infty y_\psi(t)^T M y_\psi(t)\ dt \geq 0 \quad (3)$$

where y_ψ is the output of the following linear system:

$$\dot{x}_\psi(t) = A_\psi x_\psi(t) + B_{\psi 1} v(t) + B_{\psi 2} z(t) \quad (4)$$

$$y_\psi(t) = C_\psi x_\psi(t) + D_{\psi 1} v(t) + D_{\psi 2} z(t) \quad (5)$$

$$x_\psi(0) = 0 \quad (6)$$

To shorten notation, the factorization $\Psi(j\omega)^* M \Psi(j\omega)$ will occasionally be denoted (Ψ, M).

The time domain constraint (Equation 3) holds, in general, over infinite time intervals. A hard IQC satisfies the following more restrictive property: If Δ is any causal, bounded operator that satisfies the IQC (Equation 1) then $\int_0^T y_\psi(t)^T M y_\psi(t)\ dt \geq 0$ holds for all $T \geq 0$. By contrast, IQCs for which the time domain constraint need not hold over all finite time intervals are called soft IQCs. This distinction is important because the main technical and pedagogical difficulties in the IQC framework arise from the use of soft IQCs (e.g., see the comments in Section I.B of [12]). Unfortunately there is ambiguity surrounding the terms soft IQC and hard IQC. Specifically, the factorization of $\Pi(j\omega)$ as $\Psi(j\omega)^* M \Psi(j\omega)$ is not unique.

Simple examples can be constructed to show that a multiplier can be interpreted as being either soft or hard depending on the factorization. This demonstrates that the characterizations of soft and hard are not inherent to the IQC multiplier but instead depend on the factorization. This dependence of the notions soft and hard on the factorization seems unnoticed in the literature. For clarity, the following definition will be used in the remainder of the paper.

Definition 1: A rational function $\Pi : j\mathbb{R} \rightarrow \mathbb{C}^{(l+m)\times(l+m)}$ admits a hard IQC factorization if there exists $\Psi \in \mathbb{RH}_\infty^{(l+m)\times(l+m)}$ and $M \in \mathbb{C}^{(l+m)\times(l+m)}$, such that $\Pi = \Psi^\sim M \Psi$, and any bounded, causal operator Δ which satisfies the IQC defined by Π also satisfies

$$\int_0^T y_\psi(t)^T M y_\psi(t)\ dt \geq 0 \quad (7)$$

for all $T \geq 0$ and for all $v \in L_2^l[0,\infty)$, $z = \Delta(v)$. (Ψ, M) is a hard IQC factorization of Π.

B. Hard IQC Factorizations

Two closely related classes of multipliers are considered in this section. It is shown that hard IQC factorizations exist for one of these classes of multipliers. This result, which is of independent interest, forms the basis for connecting IQC theory to dissipativity theory.

Let $\Pi : j\mathbb{R} \rightarrow \mathbb{C}^{(l+m)\times(l+m)}$ be a Hermitian-valued function partitioned as $\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$ where Π_{11} is $l \times l$ and Π_{22} is $m \times m$. Π is called a PN IQC multiplier if it satisfies the following conditions:

a) $\Pi \in \mathbb{RL}_\infty^{(l+m)\times(l+m)}$
b) $\Pi_{11}(j\omega) \geq 0\ \forall \omega \in \mathbb{R}$
c) $\Pi_{22}(j\omega) \leq 0\ \forall \omega \in \mathbb{R}$

The PN terminology refers to the Positive semidefinite and Negative semidefinite properties specified by conditions b) and c). The first condition is required so Π and (Ψ, M) have state-space representations. The second condition is necessary and sufficient for the zero operator to satisfy the IQC defined by Π. The third condition implies that if the operator Δ satisfies the IQC defined by Π then Δ maps zero input to zero output. The third condition also implies that the set of all operators that satisfy the IQC defined by Π is a convex set [5], [4]. The PN class of multipliers is quite general and covers the most typical multipliers used in IQC analysis. In fact, all of the IQCs listed in [12] satisfy conditions b) and c) except for the IQCs for certain sector bounded nonlinearities and polytopic uncertainties.

Π is called a Strict-PN IQC multiplier if it satisfies condition a) and if there exists an $\epsilon > 0$ such that:

b') $\Pi_{11}(j\omega) \geq \epsilon I_l\ \forall \omega \in \mathbb{R}$
c') $\Pi_{22}(j\omega) \leq -\epsilon I_m\ \forall \omega \in \mathbb{R}$

The Strict-PN terminology refers to the Strict Positive definite and Negative definite properties specified by conditions b') and c'). The next theorem proves that Strict-PN IQC multipliers admit a hard IQC factorization. The proof is based on the Lemmas provided in the appendix.

Theorem 1: Let $\Pi : j\mathbb{R} \rightarrow \mathbb{C}^{(l+m)\times(l+m)}$ be a Hermitian-valued function. If Π is a Strict-PN IQC multiplier then there exists a hard IQC factorization (Ψ, M) such that $M := \begin{bmatrix} I_l & 0 \\ 0 & -I_m \end{bmatrix}$ and $\Psi := \begin{bmatrix} \Psi_{11} & 0 \\ \Psi_{21} & \Psi_{22} \end{bmatrix}$ where Ψ_{11}, $\Psi_{11}^{-1} \in \mathbb{RH}_\infty^{l\times l}$, $\Psi_{22} \in \mathbb{RH}_\infty^{m\times m}$, and $\Psi_{21} \in \mathbb{RH}_\infty^{m\times l}$.

Proof: By Lemma 1, if Π is a Strict-PN IQC multiplier then it can be factored as $\Psi^\sim M \Psi$ with $M := \begin{bmatrix} I_l & 0 \\ 0 & -I_m \end{bmatrix}$ and $\Psi := \begin{bmatrix} \Psi_{11} & 0 \\ \Psi_{21} & \Psi_{22} \end{bmatrix}$ where Ψ_{11}, $\Psi_{11}^{-1} \in \mathbb{RH}_\infty^{l\times l}$, $\Psi_{22} \in \mathbb{RH}_\infty^{m\times m}$, and $\Psi_{21} \in \mathbb{RH}_\infty^{m\times l}$.

Let Δ be any casual, bounded operator that satisfies the IQC defined by Π. Given any $v \in L_2[0,\infty)$ and $T \geq 0$ define $w = \Delta(v)$ and $y_\psi = \Psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$. By Lemma 2 there exists $z(t) \in L_2[T,\infty)$ such that for the input

$$\tilde{v}(t) = \begin{cases} v(t) & t < T \\ z(t) & t \geq T \end{cases} \quad (8)$$

the signal $\tilde{y}_1 = \Psi_{11}\tilde{v}$ satisfies $\tilde{y}_1(t) = 0$ for all $t \geq T$. Define $\tilde{w} := \Delta(\tilde{v})$ and $\tilde{y}_\psi := \Psi \left[\begin{smallmatrix} \tilde{v} \\ \tilde{w} \end{smallmatrix}\right]$. $\tilde{y}_\psi$ can be partitioned as $\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}$ where $\tilde{y}_2 := \Psi_{21}\tilde{v} + \Psi_{22}\tilde{w}$. Since Δ satisfies the IQC defined by Π, the pair $(\tilde{v}, \tilde{w})$ satisfies

$$0 \leq \int_0^\infty \tilde{y}_\psi(t)^T M \tilde{y}_\psi(t)\ dt \qquad (9)$$

It now follows that:

$$0 \leq \int_0^T \tilde{y}_\psi(t)^T M \tilde{y}_\psi(t)\ dt = \int_0^T y_\psi(t)^T M y_\psi(t)\ dt$$

The first inequality follows from Equation 9 because $\tilde{y}_1(t) = 0$ for all $t \geq T$ by construction and the lower right block of M is $-I_m$. The equality follows by causality of Δ. ■

The previous theorem shows that a hard IQC factorization exists for all Strict-PN IQC multipliers. The factorization in Lemma 1 can be numerically computed with state space operations involving Lyapunov and Riccati equations.

There are many PN IQC multipliers that are not not Strict-PN multipliers. Such multipliers do not satisfy the conditions of Theorem 1 and hence a hard factorization may not exist. However, PN IQC multipliers can be perturbed so that the Strict-PN conditions are satisfied. For example, the Popov IQC can be perturbed so that it satisfies the Strict-PN conditions. This is of interest because it means the Popov IQC, given as the canonical soft IQC in [12], is arbitrarily close to an IQC which admits a hard IQC factorization.

III. IQC Stability Theorems

The formulation in this section is taken from [10], [12]. Consider the feedback interconnection:

$$v = Gw + f \qquad (10)$$

$$w = \Delta(v) + e \qquad (11)$$

where $f \in L_{2e}^l[0,\infty)$ and $e \in L_{2e}^m[0,\infty)$ are exogenous inputs. G is a causal, linear time-invariant operator on $L_{2e}^m[0,\infty)$ with transfer function $G(s) \in \mathbb{RH}_\infty^{l\times m}$. Δ is a causal operator on $L_{2e}^l[0,\infty)$ with bounded gain.

Definition 2: The feedback interconnection of G and Δ is well-posed if the map $(v,w) \to (e,f)$ defined by Equations 10 and 11 has a causal inverse on $L_{2e}^{m+l}[0,\infty)$.

Definition 3: The feedback interconnection of G and Δ is stable if the interconnection is well-posed and if the map $(v,w) \to (e,f)$ has a bounded inverse, i.e. there exists a constant $\gamma > 0$ such that

$$\int_0^T (v^T v + w^T w)dt \leq \gamma \int_0^T (f^T f + e^T e)dt \qquad (12)$$

for any $T \geq 0$ and for any solution of the feedback interconnection.

A. Frequency Domain IQC Condition ([10], [12])

IQCs can be used to formulate a frequency-domain condition proving stability of the feedback interconnection. This section reviews the IQC stability condition formulated by Megretski and Rantzer.

Theorem 2: [10], [12] Let $G(s) \in \mathbb{RH}_\infty^{l\times m}$ and let Δ be a bounded causal operator. Assume that:

i) for every $\tau \in [0,1]$, the interconnection of G and $\tau\Delta$ is well-posed.
ii) for every $\tau \in [0,1]$, the IQC defined by Π is satisfied by $\tau\Delta$.
iii) $\exists\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \quad \forall\omega \in \mathbb{R} \qquad (13)$$

then the feedback interconnection of G and Δ is stable.

This theorem was proved in [10], [12] using a homotopy method. The technical conditions involving τ are a result of the homotopy method used in the proof. Condition ii) implies that the IQC defined by Π must be satisfied by the zero operator $\Delta = 0$. As mentioned previously, the IQC is satisfied by the zero operator if and only if $\Pi_{11}(j\omega) \geq 0$ $\forall\omega \in \mathbb{R}$. Thus $\Pi_{11}(j\omega) \geq 0$ $\forall\omega \in \mathbb{R}$ is necessary for any IQC multiplier that satisfies the conditions of Theorem 2. Condition ii) can also be simplified for PN IQC multipliers. In particular, it follows from the properties of PN IQC multipliers (Section II-B) that $\tau\Delta$ satisfies the IQC for all $\tau \in [0,1]$ if and only if Δ satisfies the IQC.

For rational IQC multipliers, Condition iii) in Theorem 2 is equivalent to a linear matrix inequality. Assume $\Pi \in \mathbb{RL}_\infty^{(l+m)\times(l+m)}$ and factorize Π as $\Pi := \Psi^\sim M \Psi$ where $\Psi \in \mathbb{RH}_\infty^{(l+m)\times(l+m)}$. Assume Ψ has the state matrices as specified in Equation 2. In addition, let $G(s) := C(sI - A)^{-1}B + D$. The left side of Equation 13 can be expressed in terms of the state matrices of G and Ψ:

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} = \begin{bmatrix} (j\omega I - \hat{A})^{-1}\hat{B} \\ I \end{bmatrix}^* \left(\begin{bmatrix} \hat{C}^T \\ \hat{D}^T \end{bmatrix} M \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix} \right) \begin{bmatrix} (j\omega I - \hat{A})^{-1}\hat{B} \\ I \end{bmatrix} \qquad (14)$$

where $\hat{A} := \begin{bmatrix} A & 0 \\ B_{\psi 1}C & A_\psi \end{bmatrix}$, $\hat{B} := \begin{bmatrix} B \\ B_{\psi 2} + B_{\psi 1}D \end{bmatrix}$, $\hat{C} := \begin{bmatrix} D_{\psi 1}C & C_\psi \end{bmatrix}$, and $\hat{D} := D_{\psi 2} + D_{\psi 1}D$.

Condition iii) in Theorem 2 is equivalent to a linear matrix inequality condition by the KYP Lemma [16], [21].

Theorem 3: $\exists\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \quad \forall\omega \in \mathbb{R} \qquad (15)$$

if and only if there exists a matrix $P = P^T$ such that

$$\begin{bmatrix} \hat{A}^T P + P\hat{A} & P\hat{B} \\ \hat{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \hat{C}^T \\ \hat{D}^T \end{bmatrix} M \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix} < 0 \qquad (16)$$

Moreover, if the upper left corner of M (conformable with the row dimension of $\hat{C}$) is positive semidefinite then any solution of P of Equation 16 satisfies $P \geq 0$.

B. Dissipation Inequality for IQCs with Hard Factorizations

IQCs can be used to formulate an alternative stability condition in terms of a dissipation inequality. The dissipation inequality formulation only applies to IQCs for which a hard factorization exists. By Theorem 1, this encompasses all Strict-PN IQC multipliers. The restriction to Strict-PN IQC multipliers will be relaxed in subsequent sections.

Consider again the basic feedback interconnection and assume Δ satisfies the IQC defined by Π. Assume that $\Pi \in \mathbb{RL}_\infty^{(l+m)\times(l+m)}$ and let (Ψ, M) be a hard IQC factorization for Π. The basic feedback interconnection including Ψ is described by $z = \Delta(v)$ and the following linear dynamics:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_\psi \end{bmatrix} = \hat{A}\begin{bmatrix} x \\ x_\psi \end{bmatrix} + \hat{B}z + \hat{B}_2\begin{bmatrix} f \\ e \end{bmatrix} := F(x, x_\psi, z, f, e) \quad (17)$$

$$\begin{bmatrix} v \\ w \end{bmatrix} = \hat{C}_1\begin{bmatrix} x \\ x_\psi \end{bmatrix} + \hat{D}_{11}z + \hat{D}_{12}\begin{bmatrix} f \\ e \end{bmatrix} \quad (18)$$

$$y_\psi = \hat{C}\begin{bmatrix} x \\ x_\psi \end{bmatrix} + \hat{D}z + \hat{D}_{22}\begin{bmatrix} f \\ e \end{bmatrix} \quad (19)$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ are defined beneath Equation 14. The other state matrices are defined as $\hat{B}_2 := \begin{bmatrix} 0 & B \\ B_{\psi 1} & B_{\psi 1}D \end{bmatrix}$, $\hat{C}_1 := \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}$, $\hat{D}_{11} := \begin{bmatrix} D \\ I \end{bmatrix}$, $\hat{D}_{12} := \begin{bmatrix} I & D \\ 0 & I \end{bmatrix}$, and $\hat{D}_{22} := \begin{bmatrix} D_{\psi 1} & D_{\psi 1}D \end{bmatrix}$.

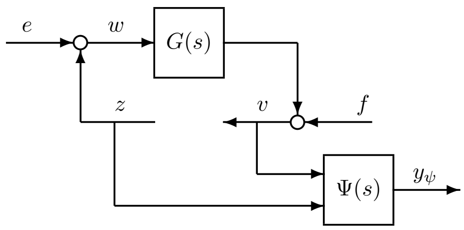

Fig. 1. Analysis Interconnection Structure

The IQC encapsulates the knowledge about the input-output behavior of Δ: given any input v, the output $z = \Delta(v)$ must be such that y_ψ satisfies the time-domain IQC. Analysis with hard IQC factorizations can be viewed as replacing the precise relation $z = \Delta(v)$ with the constraint:

$$\int_0^T y_\psi(t)^T M y_\psi(t)\ dt \geq 0 \quad (20)$$

Equation 20 implicitly constrains the possible values of z consistent with the behavior of Δ. The analysis proceeds using the interconnection structure in Figure 1. The uncertainty Δ is removed and z is viewed as an external signal subject only to the the quadratic constraint on y_ψ (Equation 20). The following theorem uses the hard IQC factorization to provide a dissipation inequality stability condition for the feedback interconnection of G and Δ.

Theorem 4: Let $G(s) \in \mathbb{RL}_\infty^{l\times m}$ and let Δ be a bounded causal operator. Assume that:

i) the interconnection of G and Δ is well-posed.
ii) Δ satisfies the IQC defined by Π
iii) there exists a matrix $P \geq 0$ and scalars $\lambda, \gamma > 0$ such that $V(x, x_\psi) := \begin{bmatrix} x \\ x_\psi \end{bmatrix}^T P \begin{bmatrix} x \\ x_\psi \end{bmatrix}$ satisfies

$$\begin{aligned}\nabla V \cdot F(x, x_\psi, z, f, e) &\leq \gamma(f^T f + e^T e) \\ &- (v^T v + w^T w) - \lambda(y_\psi^T M y_\psi) \end{aligned} \quad (21)$$

$\forall x \in \mathbb{R}^{\cdot}, x_\psi \in \mathbb{R}^{\cdot}, z \in \mathbb{R}^m, f \in \mathbb{R}^l, e \in \mathbb{R}^m$ where v, w, y_ψ are defined by Equations 18 and 19.
iv) $\Pi \in \mathbb{RL}_\infty^{(l+m)\times(l+m)}$ and Π has a hard IQC factorization $\Pi = \Psi^\sim M \Psi$.

then the feedback interconnection of G and Δ is stable.

Note that G is not required to be stable in Theorem 4. It is easy to construct examples where G is unstable and the dissipation inequality is satisfied. However, if the IQC is satisfied by the zero operator, $\Delta = 0$, then stability of G is, of course, a necessary condition for the dissipation inequality to be satisfied. The remaining theorems in the paper add the assumption that $G(s) \in \mathbb{RH}_\infty^{l\times m}$.

Theorem 4 provides an alternative stability condition and proof for using IQCs. The dissipation inequality formulation is more restrictive than the frequency domain formulation (Theorem 2) because it only applies to IQCs for which a hard factorization exists. However, there are benefits to this formulation. One benefit of the dissipation inequality formulation is that it relaxes the technical conditions associated with the homotopy method in the original proof. Specifically, Theorem 4 only requires that Δ satisfy the IQC defined by Π and that the loop is well-posed for Δ. These conditions do not need to be verified for all $\tau\Delta$ with $\tau \in [0, 1]$ as in Theorem 2. Another benefit of the dissipation inequality formulation is that it involves a time-domain condition that is amenable to new generalizations. For example, one could easily extend the dissipation inequality condition to handle:

1) Integral polynomial constraints on the input/output behavior of Δ, i.e. replace $y_\psi^T M y_\psi$ in the time-domain IQC with a polynomial function of y_ψ.
2) G and/or Ψ described as nonlinear systems with polynomial vector fields.
3) Storage functions V of polynomial of degree greater than than two.
4) Feedback interconnections that are locally but not globally stable [20].

The connection to the frequency-domain condition (Equation 13) would be lost with these generalizations. However, Theorem 4 could be generalized and the dissipation inequality could be solved using sum-of-squares optimizations [13], [14], [7]. Freely available software [15], [8], [1] could then be used to solve for a storage function that satisfies the dissipation inequality. This would be computationally demanding but still within the limits of current desktop computers for small nonlinear problems.

The dissipation inequality (Equation 21) is an algebraic constraint on variables (x, x_ψ, z, f, e). This constraint involves the data of G and Ψ and does not depend on Δ. Specifically, the dissipation inequality can be equivalently expressed as a quadratic constraint using the extended linear system (Equations 17-19):

$$\begin{bmatrix} x \\ x_\psi \\ z \\ f \\ e \end{bmatrix}^T \left[\begin{array}{c|c} Q(P,\lambda) & S(P,\lambda) \\ \hline S(P,\lambda)^T & R(\lambda) - \gamma I \end{array}\right] \begin{bmatrix} x \\ x_\psi \\ z \\ f \\ e \end{bmatrix} \leq 0 \quad (22)$$

This shows a connection between the dissipation inequality and an LMI condition. The next theorem proves that feasibility of the dissipation inequality is equivalent to feasibility of the KYP LMI (Equation 16).

Theorem 5: Assume $G(s) \in \mathbb{RH}_\infty^{l\times m}$. There exists a matrix $P \geq 0$ and scalars $\lambda, \gamma > 0$ such that $V(x, x_\psi) :=$

$\left[\begin{smallmatrix} x \\ x_\psi \end{smallmatrix}\right]^T P \left[\begin{smallmatrix} x \\ x_\psi \end{smallmatrix}\right]$ satisfies

$$\nabla V \cdot F(x, x_\psi, z, f, e) \leq \\ \gamma(f^T f + e^T e) - (v^T v + w^T w) - \lambda(y_\psi^T M y_\psi) \quad (23)$$

if and only if there exists a matrix $P \geq 0$ such that

$$\begin{bmatrix} \hat{A}^T P + P\hat{A} & P\hat{B} \\ \hat{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \hat{C}^T \\ \hat{D}^T \end{bmatrix} M \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix} < 0 \quad (24)$$

To summarize, for $G(s) \in \mathbb{RH}_\infty^{l \times m}$ the dissipation inequality (Equation 21) in Theorem 4 is equivalent to the existence of $P \geq 0$ that satisfies the KYP LMI (Equation 24). In contrast, the frequency domain stability condition (Equation 13) in Theorem 2 is equivalent to the existence of $P = P^T$ that satisfies the KYP LMI.

C. Dissipation Inequality for PN IQC Multipliers

The previous section showed that stability analysis with IQCs for which a hard factorizations exist can be interpreted with a dissipation inequality. By Theorem 1, this encompasses all Strict-PN multipliers. The objective of this section is to demonstrate that stability analysis with PN multipliers can also be interpreted with a dissipation inequality.

Theorem 6: Let $G(s) \in \mathbb{RH}_\infty^{l \times m}$ and let Δ be a bounded causal operator. Assume that:

i) the interconnection of G and Δ is well-posed.
ii) Δ satisfies the IQC defined by Π.
iii) $\exists \epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I \quad \forall \omega \in \mathbb{R} \quad (25)$$

iv) Π is a PN IQC multiplier

Then the feedback interconnection of G and Δ is stable.

Moreover, there exists a Strict-PN IQC $\tilde{\Pi}$ that satisfies conditions ii), iii). In addition, there is a $P \geq 0$ and scalars $\lambda, \gamma > 0$ such that $V(x, x_\psi) := \left[\begin{smallmatrix} x \\ x_\psi \end{smallmatrix}\right]^T P \left[\begin{smallmatrix} x \\ x_\psi \end{smallmatrix}\right]$ satisfies the dissipation inequality (Equation 21) with $\tilde{\Pi}$.

The main point of Theorem 6 is that if the IQC stability theorem holds with a PN IQC multiplier then it holds with a Strict-PN IQC multiplier. Hard factorizations exist for all Strict-PN IQC multipliers and a quadratic storage function can be constructed to satisfy the dissipation inequality. This provides provides an alternative proof for IQC stability analysis for PN multipliers based on connections to a dissipation inequality. Theorem 6 also provides a time-domain interpretation for IQC stability analysis for the most commonly used IQCs. Pedagogically, it is of interest that the dissipation inequality proof exploits finite-horizon information in the form of the hard constraint (Equation 20). This is in contrast to the use of, in general, infinite-horizon constraints in the original proof of the IQC stability theorem. A loop transformation argument can be used to construct storage functions for cases where the IQC multiplier does not satisfy the PN conditions.

D. Absolute Stability

This section provides a connection between absolute stability and the results contained in this paper. Section VI.9 of [3] provides a good summary of multiplier results in absolute stability theory. The term "multiplier" has a different meaning in absolute stability theory [3], [25]. In absolute stability theory, the term multiplier refers to a system $M(s)$ that is introduced into the feedback interconnection. The purpose is to prove stability of the original interconnection by showing that MG is strictly positive and ΔM^{-1} is positive. M is allowed to have poles in the right half-plane and M is interpreted as being a non-causal operator. The main absolute stability theorems require that M satisfy certain, restrictive factorization conditions. In particular, M must be factorizable as $M = M_- M_+$ where M_+, M_+^{-1}, M_-^*, and $(M_-^*)^{-1}$ are all stable (see, e.g., Theorem 20 on p.203 of [3] and Theorem 2 of [25]). The use of a non-causal absolute stability multiplier $M(s)$ can be interpreted in the IQC framework. The next result is a slight generalization of the Corollary on p.824 of [12].

Corollary 1: Let $G(s) \in \mathbb{RH}_\infty^{l \times m}$ and let Δ be a bounded causal operator. Assume that:

i) the interconnection of G and Δ is well-posed.
ii) There exists $M \in \mathbb{RL}_\infty^{l \times m}$ such that

$$\int_{-\infty}^{\infty} Re(\hat{v}(j\omega)^* M(j\omega) \hat{z}(j\omega)) d\omega \geq 0 \quad (26)$$

for any $v \in L_2^l[0, \infty)$ and $z = \Delta(v)$.
iii) $\exists \epsilon > 0$ such that

$$M(j\omega)^* G(j\omega) + G(j\omega)^* M(j\omega) \leq -\epsilon I \quad \forall \omega \in \mathbb{R} \quad (27)$$

Then the feedback interconnection of G and Δ is stable.

Proof: The conditions of Theorem 6 are satisfied with the PN IQC multiplier $\Pi := \left[\begin{smallmatrix} 0 & M \\ M^* & 0 \end{smallmatrix}\right]$. ■

This result slightly generalizes the Corollary in [12]. In particular, the Corollary in [12] requires that the feedback interconnection is well-posed with $\tau\Delta$ for all $\tau \in [0, 1]$. Corollary 1 above only requires well-posedness for $\tau = 1$. It is stated in [12] that non-causal absolute stability multipliers are related to soft IQCs. It is of interest that Corollary 1 is a consequence of Theorem 6 the proof of which is based on constructing a hard IQC factorization.

This Corollary is important for two reasons. First, it provides a complete connection between absolute stability theory and dissipativity theory: if the absolute stability theory conditions are satisfied by any multiplier then a storage function can be constructed that satisfies a dissipation inequality. Second, it provides a causal interpretation for the use of non-causal absolute stability multiplier $M(s)$. Corollary 1 shows that the use of non-causal absolute stability multipliers is equivalent to the use of a PN IQC multiplier in the IQC framework. The proof of Theorem 6 demonstrates that the stability conditions can be satisfied with a Strict-PN IQC multiplier for which a hard IQC factorization exists. This factorization yields a stable filter Ψ that provides information about the behavior of Δ in terms of a causal, finite-horizon quadratic constraint. In addition, there is a quadratic storage function for the extended system that includes the states of Ψ. This storage function satisfies a dissipation inequality and hence proves stability (as defined in Section III-A) via integration over a finite-time horizon.

IV. Conclusions

This paper showed that if the conditions of the IQC stability theorem are satisfied by any rational IQC multiplier then a dissipation inequality is satisfied by a quadratic storage function. This result is important because it connects both IQC and absolute stability theories to dissipativity theory. In addition, this provides new interpretations for the use of infinite horizon (soft) constraints in IQC analysis and the use of non-causal multipliers in absolute stability theory. In both cases, stability can be proved using a quadratic storage functions and a causal, finite-horizon dissipation inequality. This result is also of practical importance because the dissipation inequality theorem can be generalized in several ways, e.g. using integral polynomial constraints. These generalizations would enable the analysis of new classes of nonlinear systems and will be the subject of future work. Finally, the dissipation inequality results were based on the construction of a hard IQC factorization for a class of IQC multipliers. It would be of theoretical interest to derive necessary and sufficient conditions for an IQC multiplier to have a hard IQC factorization.

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VI. Appendix

Lemma 1: Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(l+m)\times(l+m)}$ be a Hermitian-valued function. If Π is a Strict-PN IQC multiplier then Π can be factorized as $\Pi(j\omega) = \Psi(j\omega)^* M \Psi(j\omega)$ where $M := \left[\begin{smallmatrix} I_l & 0 \\ 0 & -I_m \end{smallmatrix}\right]$ and $\Psi := \left[\begin{smallmatrix} \Psi_{11} & 0 \\ \Psi_{21} & \Psi_{22} \end{smallmatrix}\right]$. In addition, Ψ_{11}, $\Psi_{11}^{-1} \in \mathbb{RH}_\infty^{l\times l}$, $\Psi_{22} \in \mathbb{RH}_\infty^{m\times m}$, and $\Psi_{21} \in \mathbb{RH}_\infty^{m\times l}$.

Lemma 2: Consider the system G with the realization:

$$\dot{x} = Ax + Bv \tag{28}$$

$$y = Cx + Dv \tag{29}$$

Assume $G \in \mathbb{RH}_\infty^{l\times l}$ and G is invertible with $G^{-1} \in \mathbb{RH}_\infty^{l\times l}$.

For any $T \geq 0$ and any $v \in L_2[0,T)$ there exists $z \in L_2[T,\infty)$ such that for the input

$$\tilde{v}(t) = \begin{cases} v(t) & t < T \\ z(t) & t \geq T \end{cases} \tag{30}$$

the system output satisfies $y(t) = 0$ for all $t \geq T$.