Robust H_∞ Filter Design Using Frequency Gridding

Peter Seiler, Balint Vanek, József Bokor, and Gary J. Balas

***Abstract*— This paper considers the design of robust H_∞ filters for continuous-time linear systems with uncertainties described by integral quadratic constraints (IQCs). The synthesis problem can be converted into an infinite-dimensional optimization with frequency dependent linear matrix inequality constraints on the filter and IQC multipliers. This optimization is approximated by a finite dimensional semidefinite program by restricting the filter to be a linear combination of basis functions and enforcing the constraints on a finite, but dense, grid of frequencies. A heuristic algorithm is described to quickly solve the resulting finite dimensional optimization. A small example is provided to demonstrate the proposed algorithm.**

I. Introduction

Estimation is important for both signal processing and feedback control. The well-known Kalman Filter [14], [15], [13] provides an optimal minimum-variance estimator for linear systems subject to Gaussian noise. The rise of robust control techniques in the 1980s led to an interest in alternative filters, e.g. the H_2 filter (a generalization of the Kalman filter) and the H_∞ filter ([28], [11]). These methods assume the signals are generated by a known dynamic model and robustness with respect to model uncertainty is an important consideration. Numerous papers on robust filter design have appeared [1], [20], [17], [33], [16], [7], [21], [4], [23], [8], [9], [27], [30], [31], [29], [32], [25], [26].

This paper considers the robust H_∞ filtering problem for uncertain, continuous-time systems with the uncertainties described by Integral Quadratic Constraints (IQCs). IQCs, introduced in [18], provide a general framework for robustness analysis of linear systems with respect to nonlinearities and uncertainties. Robust filter design has been considered with static IQC multipliers in [16], [23], [27] and with dynamic multipliers in [27], [25], [26]. The current paper also considers dynamic IQCs multipliers. The problem formulation is equivalent to that in [25], [26] but the solution procedure is distinct.

The design problem requires a search for the filter and the IQC multiplier. For the case of LTI uncertainties, this can be recast as a μ-synthesis problem and the coordinate-wise D-K iteration has been applied to solve for the filter and uncertainty multipliers [1]. The D-K iteration yields sub-optimal solutions but is a standard method to handle the nonconvexity that arises in robust control synthesis. In robust filter design problem, the filter enters the design interconnection in an open loop (rather than a feedback) configuration and this structure can be exploited. In [25], [26], the filter synthesis problem is converted into a semi-definite program (SDP) [3] using a special IQC factorization to enforce nominal stability. The set of allowable IQC multipliers is, in general, infinite dimensional. The approach in [25], [26] obtains a finite dimensional optimization by restricting the multipliers to be combinations of chosen basis functions.

A frequency-gridding approach is taken in this paper. First, it is shown that IQC performance condition can be turned into a frequency-dependent linear matrix inequality (LMI) in the filter and multipliers. Next, a finite dimensional optimization is obtained by enforcing the frequency-dependent LMI on a dense frequency grid and restricting the filter to be a linear combination of chosen basis functions. The frequency-dependent IQC multipliers are allowed to be arbitrary functions on the frequency grid. One drawback of this approach is that some optimization variables are frequency independent and this couples together all frequency-dependent constraints. The resulting finite-dimensional optimization is convex but with a large number of constraints and variables. This paper proposes a heuristic method to obtain a reasonably fast algorithm to solve this problem. The proposed algorithm has similarities to frequency-gridding approaches applied for robust feedforward design [10], [5], [6] and for solving LMIs derived from the KYP lemma [19].

Finally, it is worth noting that the current paper minimizes an upper bound on the worst-case H_∞ filter performance. Specifically, the IQC performance condition used in this paper is only a sufficient condition for the filter to achieve a given level of performance over the set of allowable uncertainties. The use of upper bounds on worst-case performance is common in the literature but one notable exception is [31], [29]. In [31], [29] it is observed that directly minimizing the worst-case performance over the model uncertainty, rather than an upper bound, is an infinite-dimensional convex optimization in the filter. This is a useful insight but the algorithms in [31], [29] are more computationally demanding than the one proposed here. Moreover, the algorithms in [31], [29] are developed for repeated real parameter uncertainties and it does not seem possible to extend this to the classes of uncertainties/nonlinearities that can be handled within the IQC framework.

II. Notation

$\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. $\mathbb{RH}_\infty$ denotes the set of proper, rational func-

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tions with real coefficients that are analytic in the closed right half of the complex plane. $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$, and $\mathbb{RH}_\infty^{m\times n}$ denote the sets of $m \times n$ matrices whose elements are in $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{RH}_\infty$, respectively. A single superscript index is used to denote vectors, e.g. $\mathbb{R}^l$ denotes the set of $l \times 1$ vectors whose elements are in $\mathbb{R}$. For a matrix M in $\mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$, M^T denotes the transpose and M^* denotes the complex conjugate transpose.

$L_2^l[0,\infty)$ is the space of functions $f : [0,\infty) \to \mathbb{R}^l$ satisfying $\|f\| < \infty$ where

$$\|f\| := \left[\int_0^\infty f(t)^T f(t) dt\right]^{0.5} \tag{1}$$

f_T denotes the truncated function:

$$f_T(t) := \begin{cases} f(t) & \text{for } t \le T \\ 0 & \text{for } t > T \end{cases} \tag{2}$$

The extended space, denoted L_{2e}, is the set of functions f such that $f_T \in L_2$ for all $T \ge 0$.

III. Integral Quadratic Constraints

This section briefly reviews the IQC framework introduced in [18]. Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(l+m)\times(l+m)}$ be a measurable Hermitian-valued function. Two signals $w \in L_2^m[0,\infty)$ and $v \in L_2^l[0,\infty)$ satisfy the IQC defined by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} \ge 0 \tag{3}$$

where $\hat{v}(j\omega)$ and $\hat{w}(j\omega)$ are Fourier transforms of v and w, respectively. Π is called an “IQC multiplier” or simply a “multiplier”. IQCs can be used to describe the relationship between input-output signals of system components. A bounded operator $\Delta : L_{2e}^l[0,\infty) \to L_{2e}^m[0,\infty)$ satisfies the IQC defined by Π if Equation 3 holds for all (v, w) where $v \in L_2^l[0,\infty)$ and $w = \Delta(v)$.

Consider the feedback interconnection specified by the following equations:

$$v_1 = Gv_2 + f_1 \tag{4}$$
$$v_2 = \Delta(v_1) + f_2 \tag{5}$$

where $f_1 \in L_{2e}^l[0,\infty)$ and $f_2 \in L_{2e}^m[0,\infty)$ are exogenous inputs. G is a causal, linear time-invariant operator on $L_{2e}^m[0,\infty)$ with transfer function $G(s) \in \mathbb{RH}_\infty^{l\times m}$. Δ is a causal operator on $L_{2e}^l[0,\infty)$ with bounded gain.

Definition 1: The feedback interconnection of G and Δ is <u>well-posed</u> if the map $(v_1, v_2) \to (f_1, f_2)$ defined by Equations 4 and 5 has a causal inverse on $L_{2e}^{m+l}[0,\infty)$.

Definition 2: The feedback interconnection of G and Δ is <u>stable</u> if the interconnection is well-posed and if the map $(v_1, v_2) \to (f_1, f_2)$ has a bounded inverse, i.e. there exists a constant $\gamma > 0$ such that

$$\int_0^T (v_1^T v_1 + v_2^T v_2) dt \le \gamma \int_0^T (f_1^T f_1 + f_2^T f_2) dt \tag{6}$$

for any $T \ge 0$ and for any solution of the feedback interconnection.

The following theorem, from [18], formulates a stability condition for the feedback interconnection in terms of IQCs and a frequency-domain matrix inequality.

Theorem 1: Let $G(s) \in \mathbb{RH}_\infty^{l\times m}$ and let Δ be a bounded causal operator. Assume that:

i) for every $\tau \in [0,1]$, the interconnection of G and $\tau\Delta$ is well-posed.
ii) for every $\tau \in [0,1]$, the IQC defined by Π is satisfied by $\tau\Delta$.
iii) $\exists \epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le -\epsilon I \quad \forall \omega \in \mathbb{R} \tag{7}$$

then the feedback interconnection of G and Δ is stable.

The IQC framework can be extended to robust performance analysis. Consider the feedback interconnection shown in Figure 1 and partition $G := \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ conformably with the pairs of input/output signals. Define:

$$\begin{aligned} M(G,\Pi,\gamma) := &\begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix}^* \Pi \begin{bmatrix} G_{11} & G_{12} \\ I & 0 \end{bmatrix} \\ &+ \begin{bmatrix} G_{21} & G_{22} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} G_{21} & G_{22} \\ 0 & I \end{bmatrix} \end{aligned} \tag{8}$$

For any Δ satisfying the IQC defined by Π, the feedback interconnection shown in Figure 1 has L_2-gain from d to e less than γ if there exists $\epsilon > 0$ such that:

$$M(G(j\omega), \Pi(j\omega), \gamma) \le -\epsilon I \;\forall \omega \in \mathbb{R} \tag{9}$$

This result requires minor modifications to the definitions of stability and well-posedness adapted to the LFT interconnection in Figure 1. The proof is straightforward and details can be found in Section 6.5 of [24]. More general quadratic performance indices can also be considered. In this paper, the focus will remain on the L_2-gain.

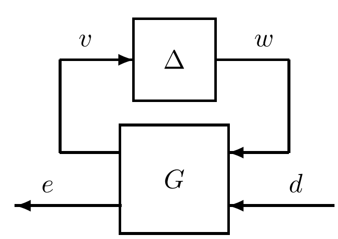

Fig. 1. Interconnection for robust performance assessment

The frequency domain inequality in Equation 9 is, in general, a sufficient but not necessary condition for the worst-case L_2 gain to be less than γ. The bound on the L_2 gain can be improved by searching over any set of IQC multipliers satisfied by Δ. Specifically, if Δ satisfies the IQCs defined by $\{\Pi_i\}_{i=1}^N$ then it satisfies the IQC defined by any Π in the set

$$\mathbf{\Pi} := \left\{ \sum_{i=1}^{N} \alpha_i \Pi_i \ : \ \alpha_i \geq 0, \ i = 1, \ldots, N \right\} \tag{10}$$

An improved bound on the L_2 gain can be computed by solving:

$$\begin{aligned} &\inf_{\Pi \in \mathbf{\Pi}} \gamma \\ &\text{s.t. } M(G(j\omega), \Pi(j\omega), \gamma) \leq -\epsilon I \ \forall \omega \in \mathbb{R} \end{aligned} \tag{11}$$

The KYP lemma [22] can be used to convert the frequency domain inequality constraint into a LMI in the decision variables. Thus this optimization can be recast as a finite-dimensional SDP [3].

The form of $\mathbf{\Pi}$ in Equation 10 arises naturally in many instances. For example, let Δ be the saturation nonlinearity. Then Δ satisfies the IQC defined by multiplier for the [0,1] sector: $\Pi_1 := \left[\begin{smallmatrix} 0 & 1 \\ 1 & -2 \end{smallmatrix}\right]$. If $H := \frac{1}{s+1}$ then Δ also satisfies the IQC defined by $\Pi_2 := \left[\begin{smallmatrix} 0 & 1+H \\ 1+H & -2-(H+H^*) \end{smallmatrix}\right]$. Π_2 corresponds to a Zames/Falb multiplier for monotonic, odd, static nonlinearities [34]. Δ also satisfies the IQC defined by any $\Pi \in \mathbf{\Pi} := \{\alpha_1 \Pi_1 + \alpha_2 \Pi_2 \ : \ \alpha_1, \ \alpha_2 \geq 0\}$.

However, the form of $\mathbf{\Pi}$ in Equation 10 is not sufficiently general to handle the class of multipliers for LTI uncertainties. For example, let $\mathbf{\Delta}$ denote the set of unit norm-bounded, LTI uncertainties. Then $\Delta \in \mathbf{\Delta}$ satisfies the IQC defined by any multiplier Π in the set:

$$\mathbf{\Pi} := \left\{ \begin{bmatrix} \beta(j\omega) I & 0 \\ 0 & -\beta(j\omega) I \end{bmatrix} \ : \ \beta(j\omega) \geq 0, \ \forall \omega \right\} \tag{12}$$

This set involves an arbitrary function of frequency, $\beta(j\omega)$, subject to the frequency domain inequality $\beta(j\omega) \geq 0, \ \forall \omega$. For this set of multipliers Equation 11 is an infinite dimensional optimization. The standard approach in IQC analysis is to use basis functions to represent such arbitrary functions of frequency [12]. In other words, $\beta(j\omega) = \sum_{i=1}^{M} \beta_i \phi_i(j\omega)$ where $\{\phi_i(j\omega)\}_{i=1}^{M}$ are chosen basis functions. This is the Ritz approximation method for solving infinite dimensional optimizations [2]. With this approximation, the optimization in Equation 11 can again be recast as a finite-dimensional SDP.

The current paper will not approximate the multipliers using basis functions. The set of uncertainties $\mathbf{\Delta}$ is allowed to be block structured and Π will also have block structure that depends on this set of allowable uncertainties. Some blocks of Π will depend on frequency-independent variables and other blocks will depend on frequency dependent variables that are subject to frequency dependent LMI constraints. A general form for the set of multipliers is:

$$\begin{aligned} \mathbf{\Pi} := \Bigg\{ & \Pi(j\omega) := \sum_{i=1}^{N} \alpha_i \Pi_i(j\omega) + \sum_{i=1}^{M} \beta_i(j\omega) \Pi_{N+i}(j\omega) \\ & : \ LMI(\omega, \alpha, \beta(j\omega)) \leq -\epsilon I \ \forall \omega \in \mathbb{R} \Bigg\} \end{aligned} \tag{13}$$

$\alpha \in \mathbb{R}^N$ is a vector of frequency independent variables and $\beta(j\omega) : j\mathbb{R} \rightarrow \mathbb{C}^M$ is a vector of frequency dependent variables. Any of the $\{\Pi_i\}_{i=1}^{N+M}$ may either be functions of frequency or constant. More explicit details on multipliers for block structured uncertainty can be found in [12], [24].

IV. Robust Filter Design

A. Problem Formulation

Figure 2 shows the interconnection structure for the robust filter design problem considered in this paper. The generalized plant P has two inputs and three outputs. $d \in L_2^{n_d}$ denotes the input disturbances. $y \in L_{2e}^{n_y}$ and $z \in L_{2e}^{n_z}$ denote the measurements and signals to be estimated, respectively. Any noises in the measurements are included in d. $v \in L_{2e}^{n_v}$ and $w \in L_{2e}^{n_w}$ are interconnection signals associated with the plant uncertainty. The blocks of P partitioned according to these input/output signals are denoted as:

$$P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \\ P_{31} & P_{32} \end{bmatrix} \tag{14}$$

The filter $F \in \mathbb{RH}_\infty^{n_z \times n_y}$ uses the measurements to construct an estimate $\hat{z}$.

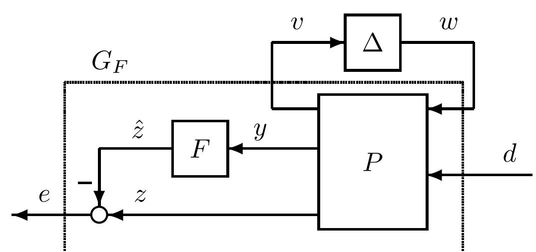

Fig. 2. Interconnection for robust filter design

Let $\mathbf{\Delta}$ denote a set of uncertainties / nonlinearities. Let $\mathbf{\Pi}$ denote a set of multipliers such that for any $\Delta \in \mathbf{\Delta}$ and any $\Pi \in \mathbf{\Pi}$, Δ satisfies the IQC defined by Π. It is assumed that $\mathbf{\Pi}$ is in the form of Equation 13. The problem considered in this paper is to design a filter that minimizes the IQC upper bound on the worst-case performance. In other words, the objective is to solve the optimization:

$$\begin{aligned} \gamma^* := & \inf_{F \in \mathbb{RH}_\infty^{n_z \times n_y}, \ \Pi \in \mathbf{\Pi}} \gamma \\ & \text{s.t. } M(G_F(j\omega), \Pi(j\omega), \gamma) \leq -\epsilon I \ \forall \omega \in \mathbb{R} \end{aligned} \tag{15}$$

where:

$$G_F := \begin{bmatrix} P_{11} & P_{12} \\ -FP_{21} + P_{31} & -FP_{22} + P_{32} \end{bmatrix} \tag{16}$$

G_F is the system contained in the dashed box of Figure 2. G_F maps the inputs (w, d) to the outputs (v, e). The optimization in Equation 15 involves solving for the filter and the IQC multipliers.

B. Filter Synthesis

The constraint in the robust filter design problem (Equation 15) contains one term that involves a product of $F(j\omega)$ with itself. By Schur complements [3], the constraint is equivalent to (suppressing the functional dependence on $j\omega$):

$$\left[\begin{array}{c|c} \begin{bmatrix} P_{11} & P_{12} \\ I & 0 \end{bmatrix}^* \Pi \begin{bmatrix} P_{11} & P_{12} \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\gamma^2 I \end{bmatrix} & (\cdot) \\ \hline \begin{bmatrix} -FP_{21} + P_{31} & -FP_{22} + P_{32} \end{bmatrix} & -I \end{array}\right] \leq -\epsilon I \quad \forall \omega \in \mathbb{R} \tag{17}$$

The $(.)$ term in the (1,2) block can be inferred from symmetry. At each frequency this matrix inequality is jointly affine in $F(j\omega)$ and $\Pi(j\omega)$. Thus the robust filter design problem can be expressed as an infinite-dimensional optimization with LMI constraints in the multiplier variables and filter:

$$\begin{aligned} \gamma^* := & \inf_{F \in \mathbb{RH}_\infty^{n_z \times n_y},\ \alpha \in \mathbb{R}^M,\ \beta : j\mathbb{R} \to \mathbb{C}^N} \gamma \\ & \text{s.t. } LMI(\omega, F(j\omega), \alpha, \beta(j\omega), \gamma) \leq -\epsilon I \ \forall \omega \in \mathbb{R} \end{aligned} \tag{18}$$

The LMI in Equation 18 includes the constraint in Equation 17 and any LMI constraints that are required to specify the IQC multipliers in Equation 13. The remainder of this section develops finite dimensional optimizations that can be used to compute upper and lower bounds on γ^*.

A lower bound is obtained by enforcing the constraint at only one frequency, ω_0:

$$\begin{aligned} \underline{\gamma}(j\omega_0) := & \inf_{F \in \mathbb{C}^{n_z \times n_y},\ \alpha \in \mathbb{R}^M,\ \beta \in \mathbb{C}^N} \gamma \\ & \text{s.t. } LMI(\omega_0, F, \alpha, \beta, \gamma) \leq -\epsilon I \end{aligned} \tag{19}$$

This optimization can be performed on a grid of frequencies $\{\omega_k\}_{k=1}^{n_\omega}$ and then $\underline{\gamma}^* := \max_k \underline{\gamma}(j\omega_k)$ is a lower bound for γ^*. This is a finite-dimensional SDP at each frequency and these optimizations can be solved quickly since the problems are decoupled across frequency. There will generally be a gap between $\underline{\gamma}^*$ and γ^*. One could attempt to improve the lower bound by enforcing the constraint on more than one frequency. This increases the computational cost for the lower bound and is not pursued in this paper.

The upper bound is computed by restricting F to lie in the space of chosen basis functions: $F(j\omega) := \sum_{i=1}^{P} \tau_i F_i(j\omega)$ where $\{F_i(j\omega)\}_{i=1}^{P}$ are the chosen (stable) basis functions and $\tau \in \mathbb{R}^P$. With this approximation, an upper bound for the robust filter design problem can be expressed as an SDP with an infinite number of constraints:

$$\begin{aligned} \bar{\gamma} := & \inf_{\alpha \in \mathbb{R}^M,\ \beta : j\mathbb{R} \to \mathbb{C}^N,\ \tau \in \mathbb{R}^P} \gamma \\ & \text{s.t. } LMI\left(\omega, \sum_{i=1}^{M} \tau_i F_i(j\omega), \alpha, \beta(j\omega), \gamma\right) \leq -\epsilon I \ \forall \omega \in \mathbb{R} \end{aligned} \tag{20}$$

$\bar{\gamma}$ is an upper bound on γ^* due to the approximation of F by basis functions. To obtain a finite dimensional optimization, the constraints are only enforced on a finite, but dense grid of frequencies $\{\omega_k\}_{k=1}^{n_\omega}$:

$$\begin{aligned} \bar{\gamma} \approx & \inf_{\alpha \in \mathbb{R}^M,\ \beta_k \in \mathbb{C}^N,\ \tau \in \mathbb{R}^P} \gamma \\ & \text{s.t. } LMI\left(\omega_k, \sum_{i=1}^{M} \tau_i F_i(j\omega_k), \alpha, \beta_k, \gamma\right) \leq -\epsilon I \\ & \text{for } k = 1, \ldots, n_\omega \end{aligned} \tag{21}$$

The β_k are the frequency-dependent variables in the IQC multiplier defined at ω_k. The proposed approach uses basis functions for the filter but allows the multipliers to be arbitrary functions on the frequency grid. This can roughly be viewed as dual to the approach taken in [25], [26] where basis functions are chosen for the multipliers but the filter is allowed to be an arbitrary, linear system. It will be assumed that the frequency grid is sufficiently dense that the differences between the optimizations in Equation 20 and 21 are negligible. The optimal selection of the frequencies to include in this grid is an important research problem for the development of software for many problems in control. Alternatively, the frequency sweeping method in [5], [6] can, at the expense of additional computation, be used to compute a true upper bound on the optimal performance.

The optimization in Equation 21 is a finite-dimensional SDP. However it involves n_ω LMI constraints and $(M + P + Nn_\omega + 1)$ variables. The LMI constraints are coupled due to the frequency independent variables α and τ. The computation time to solve this problem with current SDP algorithms would be significant for even small to moderate sized frequency grids, e.g. $n_\omega \approx 50$.

A heuristic algorithm is used to quickly compute the optimal solution. The basic idea is to solve the optimization on a coarse frequency grid, check the solution on the dense frequency grid and then add new frequency points, as needed, to the coarse grid. The steps of the heuristic algorithm are:

0) Let $\{\omega_k\}_{k=1}^{n_\omega}$ be a given dense frequency grid. Set $i = 0$ and $S_i := \{\omega_1, \omega_{n_\omega}\}$.
1) Solve Equation 21 enforcing the constraints on the coarse grid S_i. Stop if not feasible otherwise go to Step 2.
2) Compute the frequency dependent IQC variables on the dense grid $\{\omega_k\}_{k=1}^{n_\omega}$ by linearly interpolating between the solution computed on the coarse grid.
3) Evaluate the LMI constraint on the dense grid using the optimal α, τ, γ computed in Step 1 and the linearly interpolated β_k computed in Step 2. Stop if feasible otherwise let ω^* denote the frequency of maximal violation of the LMI constraint.
4) Set $i = i + 1$ and $S_{i+1} = S_i \cup \{\omega^*\}$. Return to Step 1.

If the algorithm terminates to due infeasibility in Step 1 then there is no filter for which robust stability can be proven via IQCs. This simply means that robust stability of the open loop system P can not be proven with respect to the uncertainties. This termination condition will not occur if it is assumed that the open loop system (without the filter) can be proven to be robustly stable using the given IQC multipliers.

If the algorithm terminates due to feasibility in Step 3 then optimal IQC multipliers and filter have been computed on the dense grid. The algorithm must terminate in a finite number of iterations. In particular, a new frequency point is added to the coarse grid at each iteration and the algorithm must terminate if S_i is equal to $\{\omega_k\}_{k=1}^{n_\omega}$. The algorithm typically terminates with the coarse grid containing many fewer points than the dense grid.

This algorithm is similar to that proposed in [5], [6] for robust feedforward design with respect to LTI uncertainties. The key distinction lies in Step 3. In particular, [5], [6] compute the optimal multipliers at each frequency in the dense grid using a μ upper bound calculation. This step alone can be costly to perform on a dense grid for each iteration of the algorithm. The linear interpolation used in the algorithm proposed in the current paper is sub-optimal but fast. The linear interpolation is typically less than 1 to 2 percent of the total time. The sub-optimality of the linear interpolants has not been a significant issue in test examples. The worst-case performance is achieved on a small number of frequencies in the test examples and sub-optimal multipliers are acceptable away from these frequencies.

Finally, it is noted that the optimal filter returned by the algorithm is stable. Stability is assured simply by selecting stable basis functions because stable transfer functions form a subspace of all rational, proper transfer functions. In fact, the search for filters can be performed over any subspace simply by choosing the basis functions to lie within the desired subspace. Thus it would be quite easy to design robust filters with a certain structure, e.g. decentralized filters. A similar observation was made in [7] with regards to a different robust filter design algorithm for systems with polytopic uncertainty. It is also worth noting that a good basis functions can be computed by solving the nominal filter design problem, i.e. the optimal filter for $\Delta = 0$. If Δ contains only LTI uncertainties, then another good basis function can be computed using D-K iteration to find a sub-optimal solution to the related μ-synthesis problem.

V. EXAMPLE

This section demonstrates the proposed algorithm on the two-mass example considered in [25], [26]. The generalized plant P is:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -2 & 2 \\ 2 & -2 & 4 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -1.5 \\ 3.0 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} d \tag{22}$$

$$v = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix} x \tag{23}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \end{bmatrix} d \tag{24}$$

$$z = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x \tag{25}$$

$$w = \delta v \tag{26}$$

The first entry of d is a plant input disturbance and the second entry represents sensor noise. The uncertainty is a single norm-bounded real parameter, $\delta \in \mathbb{R}$ with $|\delta| \leq 1$. This represents uncertainty in the damping between the two masses. There is only one copy of this uncertainty since it enters the model in [25], [26] via a rank one matrix. The norm-bounded uncertainty δ satisfies the IQC defined by any multiplier Π in the set:

$$\mathbf{\Pi} := \left\{ \begin{bmatrix} \beta_1(j\omega) & j\beta_2(j\omega) \\ -j\beta_2(j\omega) & -\beta_1(j\omega) \end{bmatrix} \ : \ \beta_1(j\omega) \geq 0, \ \forall \omega \right\} \tag{27}$$

where $\beta_1, \beta_2 : j\omega \to \mathbb{R}$ are arbitrary functions of frequency. The robust filter computed in [26] achieves a worst-case H_∞ performance of 2.64.

A lower bound on the optimal performance was computed using frequency-gridding method described in Section IV-B. The frequency grid consisted of 250 logarithmically spaced points between 0.1 and 10 rad/sec. Figure 3 shows the lower bounds versus frequency. The total time to compute the bounds at all 250 frequency points was 7.8 sec. The largest value across frequency is $\underline{\gamma} = 2.64$ and hence the method of [25], [26] achieves the optimal filter within the reported accuracy.

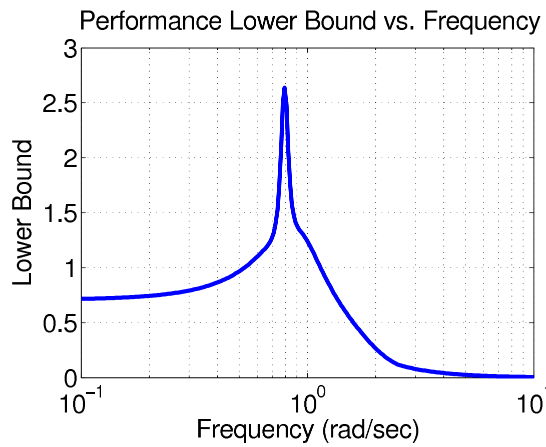

Fig. 3. Lower bound on optimal H_∞ filter performance, $\underline{\gamma}(j\omega)$

The method proposed in this paper uses the following basis functions for the filter:

$$\left\{ 1, \ \frac{1}{s+0.1}, \ \frac{1}{s+1}, \ \frac{1}{s+10}, \ \frac{1}{s^2+0.16s+0.64}, \right\} \tag{28}$$

The last basis function is a lightly damped second order system with natural frequency at 0.8 rad/sec. This particular basis function was chosen because the lower bound plot (Figure 3) has a sharp peak near this frequency. The optimal filter within the span of these bases functions was computed using the method described in Section IV-B. The frequency grid again consisted of 250 logarithmically spaced points between 0.1 and 10 rad/sec. The algorithm described in Section IV-B completed after 13 iterations. The total time to compute the optimal filter was 7.2 sec. The linear interpolation of the frequency-dependent IQC scalings took less than 0.1 sec of this total time. The optimal filter within the span of the basis function is

$$F(s) := \frac{1.042s^5 + 15.99s^4 + 14.22s^3 + 13.92s^2 + 7.764s + 1.032}{s^5 + 11.26s^4 + 13.52s^3 + 9.88s^2 + 7.264s + 0.64} \tag{29}$$

This filter achieves a worst-case H_∞ performance of 2.64 which is again the optimal performance. Figure 4 shows the Bode plot for this filter. The inclusion of the lightly-damped second order basis function is important. If this is removed from the list of basis function then the optimal filter can only achieve a worst-case gain of 3.26.

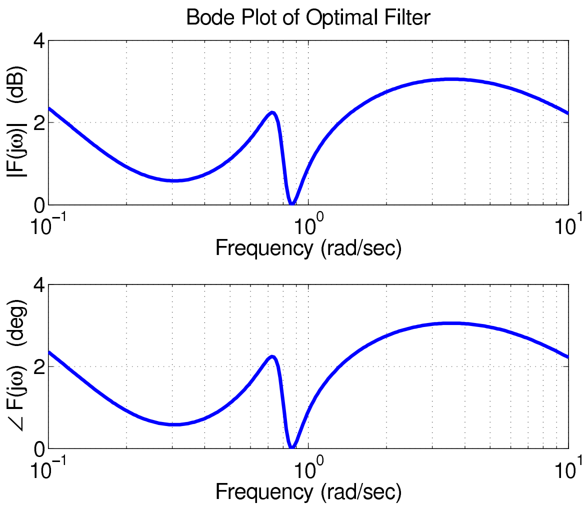

Fig. 4. Bode plot of optimal filter

VI. Conclusions

This paper considered the design of robust H_∞ filters for continuous-time linear systems with uncertainties described by IQCs. The synthesis problem was converted into an infinite-dimensional optimization with frequency dependent LMI constraints. A frequency-gridding approach was used to approximate this optimization by a large, finite-dimensional SDP. A heuristic algorithm was described to quickly solve the resulting optimization. A small example was provided to demonstrate the proposed algorithm. Future work will focus on applying this procedure to develop robust fault detection filters for aerospace applications.

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