Robustness Analysis of Linear Parameter Varying Systems Using Integral Quadratic Constraints

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Abstract

A general approach is presented to analyze the worst case input/output gain for an interconnection of a linear parameter varying (LPV) system and an uncertain or nonlinear element. The LPV system is described by state matrices that have an arbitrary, i.e. not necessarily rational, dependence on the parameters. The input/output behavior of the nonlinear/uncertain block is described by an integral quadratic constraint (IQC). A dissipation inequality is proposed to compute an upper bound for this gain. This worst-case gain condition can be formulated as a semidefinite program and efficiently solved using available optimization software. Moreover, it is shown that this new condition is a generalization of the well-known Bounded Real Lemma type result for LPV systems. The results contained in this paper complement known results that apply IQCs for analysis of LPV systems whose state matrices have a rational dependence on the parameters. The effectiveness of the proposed method is demonstrated on simple numerical examples.

I. INTRODUCTION

This paper presents a method to analyze the robustness of a linear parameter varying (LPV) system with respect to nonlinearities and/or uncertainties. LPV systems are a class of linear systems where the state matrices depend on (measurable) time-varying parameters. The existing analysis and synthesis results for LPV systems provide a rigorous framework for design of gain-scheduled controllers. There are several special classes of LPV systems that are categorized based on how the state matrices depend on the scheduling parameters. One special class assumes the

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state matrices of the LPV system have a rational dependence on the parameters. In this case finite dimensional semidefinite programs (SDPs) can be formulated to synthesize LPV controllers [1], [2], [3]. Another class of LPV systems assumes the state matrices have an arbitrary dependence on the parameters. The controller synthesis problem for this class of systems leads to an infinite collection of parameter-dependent linear matrix inequalities (LMIs) [4], [5]. A brief review of this technical result is provided in Section II. The computational solution of such parameter-dependent LMIs requires some finite-dimensional approximation and is typically more involved. The benefit is that arbitrary parameter dependence can be considered which appears in many applications, e.g. aeroelastic vehicles [6] and wind turbines [7], [8], by linearization of nonlinear models.

Integral quadratic constraints (IQCs) are used in this paper to model the uncertain and/or nonlinear components. IQCs, introduced in [9], provide a general framework for robustness analysis. In [9] the system is separated into a feedback connection of a known linear time-invariant (LTI) system and a perturbation whose input-output behavior is described by an IQC. An IQC stability theorem was formulated in [9] with frequency domain conditions and was proved using a homotopy method.

The contribution of this paper is to provide robust performance conditions for a feedback interconnection of an LPV system and a perturbation whose input-output behavior is described by an IQC. The frequency-domain stability condition in [9] does not apply for this case because the LPV system is time-varying. Instead, this paper uses a time-domain interpretation for IQCs, described in Section II, that builds upon the work in [10]. The time-domain IQC framework is more general than the original formulation in the frequency-domain [9]. Specifically, the time-domain approach enables generalizations to interconnections where the known system is nonlinear and/or time-varying. The time-domain viewpoint is used in Section III to derive dissipation-inequality conditions that bound the worst-case induced gain of the system. The conditions are derived for LPV systems whose state matrices have arbitrary dependence on the scheduling parameters. The robust performance analysis conditions can thus be viewed as generalizations of those given for nominal (not-uncertain) LPV systems in [4], [5].

It is also important to place the results of this paper in the context of existing results that use IQCs to analyze the performance of uncertain LPV systems. Specifically, there are several recent robust performance results obtained for LPV systems whose state matrices have rational dependence on the scheduling parameters [11], [12], [13]. In contrast to [11], [12], [13], the results in this paper are for the class of LPV systems whose state matrices have an arbitrary dependence on the parameters. As noted above, the results in this paper enable applications to systems, e.g. aeroelastic vehicles or wind turbines, for which arbitrary dependence on scheduling variables is a natural modeling framework. This paper builds on the initial work contained in [14].

II. BACKGROUND

A. Notation

$\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. $\mathbb{RL}_\infty$ denotes the set of rational functions with real coefficients that are proper and have no poles on the imaginary axis. $\mathbb{RH}_\infty$ is the subset of functions in $\mathbb{RL}_\infty$ that are analytic in the closed right half of the complex plane. $\mathbb{C}^{m\times n}$, $\mathbb{RL}_\infty^{m\times n}$ and $\mathbb{RH}_\infty^{m\times n}$ denote the sets of $m\times n$ matrices whose elements are in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{RL}_\infty$, $\mathbb{RH}_\infty$, respectively. A single superscript index is used for vectors, e.g. $\mathbb{R}^n$ denotes the set of $n\times 1$ vectors whose elements are in $\mathbb{R}$. Vertical concatenation of $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^m$ is denoted by $[x;y]\in\mathbb{R}^{n+m}$. $\mathbb{S}^n$ denotes the set of $n\times n$ symmetric matrices. $\mathbb{R}^+$ describes the set of nonnegative real numbers. For $z\in\mathbb{C}$, $\bar{z}$ denotes the complex conjugate of z. For a matrix $M\in\mathbb{C}^{m\times n}$, M^T denotes the transpose and M^* denotes the complex conjugate transpose. The para-Hermitian conjugate of $G\in\mathbb{RL}_\infty^{m\times n}$, denoted as $G^\sim$, is defined by $G^\sim(s) := G(-\bar{s})^*$. Note that on the imaginary axis, $G^\sim(j\omega) = G(j\omega)^*$. $L_2^n[0,\infty)$ is the space of functions $v:[0,\infty)\to\mathbb{R}^n$ satisfying $\|v\|_2<\infty$ where

$$\|v\|_2 := \left[\int_0^\infty v(t)^T v(t)\, dt\right]^{0.5} \tag{1}$$

Given $v\in L_2^n[0,\infty)$, v_T denotes the truncated function:

$$v_T(t) := \begin{cases} v(t) & \text{for } t\le T \\ 0 & \text{for } t>T \end{cases} \tag{2}$$

The extended space, denoted L_{2e}, is the set of functions v such that $v_T\in L_2$ for all $T\ge 0$. Similarly, $L_\infty^n[0,\infty)$ is the space of functions $v:[0,\infty)\to\mathbb{R}^n$ satisfying $\|v\|_\infty<\infty$ where

$$\|v\|_\infty := \sup_t \left[v(t)^T v(t)\right]^{0.5}. \tag{3}$$

B. Analysis of LPV Systems

Linear parameter varying (LPV) systems are a class of systems whose state space matrices depend on a time-varying parameter vector $\rho : \mathbb{R}^+ \to \mathbb{R}^{n_\rho}$. The parameter is assumed to be a continuously differentiable function of time and admissible trajectories are restricted, based on physical considerations, to a known compact subset $\mathcal{P} \subset \mathbb{R}^{n_\rho}$. In addition, the parameter rates of variation $\dot{\rho} : \mathbb{R}^+ \to \dot{\mathcal{P}}$ are assumed to lie within a hyperrectangle $\dot{\mathcal{P}}$ defined by

$$\dot{\mathcal{P}} := \{q \in \mathbb{R}^{n_\rho} | \, \underline{\nu}_i \leq q_i \leq \bar{\nu}_i, \; i = 1, \ldots, n_\rho\}. \tag{4}$$

The set of admissible trajectories is defined as $\mathcal{A} := \{\rho : \mathbb{R}^+ \to \mathbb{R}^{n_\rho} \; : \; \rho(t) \in \mathcal{P}, \; \dot{\rho}(t) \in \dot{\mathcal{P}} \; \forall t \geq 0\}$. The parameter trajectory is said to be rate unbounded if $\dot{\mathcal{P}} = \mathbb{R}^{n_\rho}$.

The state-space matrices of an LPV system are continuous functions of the parameter: $A : \mathcal{P} \to \mathbb{R}^{n_x \times n_x}$, $B : \mathcal{P} \to \mathbb{R}^{n_x \times n_d}$, $C : \mathcal{P} \to \mathbb{R}^{n_e \times n_x}$ and $D : \mathcal{P} \to \mathbb{R}^{n_e \times n_d}$. An n_x^{th} order LPV system, G_ρ, is defined by

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))d(t) \\ e(t) &= C(\rho(t))x(t) + D(\rho(t))d(t) \end{aligned} \tag{5}$$

The state matrices at time t depend on the parameter vector at time t. Hence, LPV systems represent a special class of time-varying systems. Throughout the remainder of the paper the explicit dependence on t is occasionally suppressed to shorten the notation. Moreover, it is important to emphasize that the state matrices are allowed to have an arbitrary dependence on the parameters.

The performance of an LPV system G_ρ can be specified in terms of its induced L_2 gain from input d to output e. The induced L_2-norm is defined by

$$\|G_\rho\| := \sup_{d \neq 0, d \in L_2, \rho \in \mathcal{A}, x(0)=0} \frac{\|e\|_2}{\|d\|_2}. \tag{6}$$

The notation $\rho \in \mathcal{A}$ refers to the entire (admissible) trajectory as a function of time. The analysis below leads to conditions that involve the parameter and rate at a single point in time, i.e. $(\rho(t), \dot{\rho}(t))$. The parametric description $(p, q) \in \mathcal{P} \times \dot{\mathcal{P}}$ is introduced to emphasize that such conditions only depend on the (finite-dimensional) sets $\mathcal{P}$ and $\dot{\mathcal{P}}$.

In [5] a generalization of the LTI Bounded Real Lemma is stated, which provides a sufficient condition to upper bound the induced L_2 gain of an LPV system. The sufficient condition uses a quadratic storage function that is defined using a parameter-dependent matrix $P : \mathcal{P} \to \mathbb{S}^{n_x}$. It is assumed that P is a continuously differentiable function of the parameter ρ. In order to shorten the notation, a differential operator $\partial P : \mathcal{P} \times \dot{\mathcal{P}} \to \mathbb{S}^{n_x}$ is introduced as in [15]. ∂P is defined as:

$$\partial P(p, q) := \sum_{i=1}^{n_\rho} \frac{\partial P(p)}{\partial p_i} q_i \tag{7}$$

The next theorem states the condition provided in [5] to bound the L_2 gain of an LPV system.

Theorem 1. *([5]): An LPV system G_ρ is exponentially stable and $\|G_\rho\| < \gamma$ if there exists a continuously differentiable $P : \mathcal{P} \to \mathbb{S}^{n_x}$, such that $\forall (p, q) \in \mathcal{P} \times \dot{\mathcal{P}}$*

$$P(p) > 0, \tag{8}$$

$$\begin{bmatrix} P(p)A(p) + A(p)^T P(p) + \partial P(p,q) & P(p)B(p) \\ B^T(p)P(p) & -I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C(p)^T \\ D(p)^T \end{bmatrix} \begin{bmatrix} C(p) & D(p) \end{bmatrix} < 0. \tag{9}$$

Proof: The proof is based on the existence of a quadratic storage function that satisfies a dissipation inequality. Let $d \in L_2$ be an arbitrary input and $\rho \in \mathcal{A}$ be any admissible parameter trajectory. Let x and e denote the state and output responses of the LPV system G_ρ for the input d and trajectory ρ assuming zero initial conditions.

Equation (9) is a strict matrix inequality and hence there exists some $\epsilon > 0$ such that the perturbed matrix inequality holds:

$$\begin{bmatrix} P(p)A(p) + A(p)^T P(p) + \partial P(p,q) & P(p)B(p) \\ B^T(p)P(p) & -(1-\epsilon)I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C(p)^T \\ D(p)^T \end{bmatrix} \begin{bmatrix} C(p) & D(p) \end{bmatrix} \leq 0. \tag{10}$$

Evaluating (10) at $(p, q) = (\rho, \dot{\rho})$ and multiplying on the left/right by $[x^T, d^T]$ and $[x^T, d^T]^T$ respectively gives

$$\dot{x}^T P(\rho)x + x^T P(\rho)\dot{x} + x^T \partial P(\rho, \dot{\rho})x + \frac{1}{\gamma^2} e^T e - (1-\epsilon) d^T d \leq 0. \tag{11}$$

A storage function $V : R^{n_x} \times \mathcal{P} \to \mathbb{R}^+$ for G_ρ can be defined by $V(x, \rho) = x^T P(\rho) x$. Condition (8) implies that this storage function is globally non-negative, i.e. $V(x, \rho) \geq 0$. This storage function can be evaluated along the state/parameter trajectory to obtain the function of time $V(x(t), \rho(t))$ which is simply denoted as $V(t)$. It follows from (11) that the derivative of V with respect to time satisfies

$$\dot{V} + \frac{1}{\gamma^2} e^T e - (1-\epsilon) d^T d \leq 0. \tag{12}$$

Integrating this equation over the interval $[0,T]$ and recalling that $x(0)=0$ results in

$$V(T) + \frac{1}{\gamma^2}\int_0^T e(t)^T e(t)dt - (1-\epsilon)\int_0^T d(t)^T d(t)dt \leq 0. \quad (13)$$

$V(T) \geq 0$ implies that $\int_0^T e(t)^T e(t)dt \leq (1-\epsilon)\gamma^2 \int_0^T d(t)^T d(t)dt$ holds for all $T \geq 0$. Let $T \to \infty$ to obtain the bound $\|e\|_2 < \gamma \|d\|_2$. ■

Two details must be considered in order to turn this theoretical result into a useful numerical algorithm. First, the conditions (8) and (9) are parameter-dependent LMIs that must be satisfied for all possible $(p,q) \in \mathcal{P}\times\dot{\mathcal{P}}$. Thus (8) and (9) represent an infinite collection of LMI constraints. Since q enters only affinely into the LMI and the set $\dot{\mathcal{P}}$ is a polytope, it is sufficient to check the LMI on the vertices of $\dot{\mathcal{P}}$. On the other hand, p can enter (9) nonlinearly through the state matrices and the set $\mathcal{P}$ need not be convex. A remedy to this problem, which works in many practical examples, is to approximate the set $\mathcal{P}$ by a finite set $\mathcal{P}_{grid} \subset \mathcal{P}$. Specifically, the finite set $\mathcal{P}_{grid} := \{\rho^{(k)}\}_{k=1}^N$ represents a grid of points over the set $\mathcal{P}$. The conditions (8) and (9) are enforced only on these grid points leading to a finite dimensional SDP. Note that the gridding approach is only an approximation for the parameter-dependent LMI conditions in Theorem 1. Hence, no rigorous performance guarantees are provided by this gridding approach and special care must be taken when drawing conclusions. A pragmatic implementation of this approach is as follows: Enforce the LMI conditions on a ”coarse” grid consisting of a small number of points in order to reduce computation time. The resulting solution can then be checked on a ”dense” grid of many points to ensure the accuracy of the solution. The SDP can be re-solved on a less coarse grid if required.

The second detail is that the main decision variable in conditions (8) and (9) is the function $P(p)$. $P(p)$ must be restricted to a finite dimensional subspace in order to avoid an infinite dimensional space for the decision variable. A common practice [5], [16] is to restrict the storage function variable $P(p)$ to be a linear combination of basis functions,

$$P(p) = \sum_{i=1}^{N_b} g_i(p) P_i \quad (14)$$

where $g_i : \mathbb{R}^{n_\rho} \to \mathbb{R}$ are user-specified basis functions ($i = 1, \ldots, N_b$) and the matrix coefficients $P_i \in \mathcal{S}^{n_x}$ are the decision variables in the optimization.

C. Integral Quadratic Constraints

An IQC is defined by a symmetric matrix $M \in \mathbb{S}^{n_z}$ and a stable linear system $\Psi \in \mathbb{RH}_\infty^{n_z \times (m_1+m_2)}$. Ψ is denoted as

$$\Psi(j\omega) := C_\psi (j\omega I - A_\psi)^{-1} [B_{\psi 1} \ B_{\psi 2}] + [D_{\psi 1} \ D_{\psi 2}] \tag{15}$$

A bounded, causal operator $\Delta : L_{2e}^{m_1} \to L_{2e}^{m_2}$ satisfies an IQC defined by (Ψ, M) if the following inequality holds for all $v \in L_2^{m_1}[0, \infty)$, $w = \Delta(v)$ and $T \geq 0$:

$$\int_0^T z(t)^T M z(t) \, dt \geq 0 \tag{16}$$

where z is the output of the linear system Ψ:

$$\dot{x}_\psi(t) = A_\psi x_\psi(t) + B_{\psi 1} v(t) + B_{\psi 2} w(t), \ x_\psi(0) = 0 \tag{17}$$

$$z(t) = C_\psi x_\psi(t) + D_{\psi 1} v(t) + D_{\psi 2} w(t) \tag{18}$$

The notation $\Delta \in IQC(\Psi, M)$ is used if Δ satisfies the IQC defined by (Ψ, M). Fig. 1 provides a graphical interpretation of the IQC. The input and output signals of Δ are filtered through Ψ. If $\Delta \in IQC(\Psi, M)$ then the output signal z satisfies the (time-domain) constraint in (16) for any finite-horizon $T \geq 0$. The validity of the constraint over finite-horizons (rather than infinite-horizons) is significant for technical reasons as it enables the constraint to be used within the dissipation inequality framework. Two simple examples are provided below to connect this terminology to standard results used in robust control.

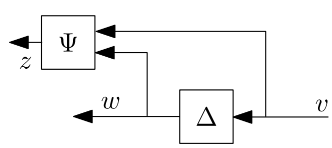


Fig. 1. Graphical interpretation of the IQC

Example 1. *Consider a causal (SISO) nonlinear operator Δ that satisfies the bound $\|\Delta\| \leq b$. The norm bound on Δ implies that $\|w\|_2 \leq b \|v\|_2$ for any input/output pair $v \in L_2$ and $w = \Delta(v)$. This constraint on (v, w) can be expressed as the following infinite-horizon inequality:*

$$\int_0^\infty \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0 \tag{19}$$

Next, the causality of Δ *is used to demonstrate that, in fact, the inequality involving* (v, w) *holds over all finite horizons. Given any* $T \geq 0$*, define a new input* $\tilde{v}$ *by* $\tilde{v}(t) = v(t)$ *for* $t \leq T$ *and* $\tilde{v}(t) = 0$ *otherwise. The truncated signal* $\tilde{v}$ *generates an output* $\tilde{w} = \Delta(\tilde{v})$ *and this new input/output pair* $(\tilde{v}, \tilde{w})$ *also satisfies* $\|\tilde{w}\|_2 \leq b\,\|\tilde{v}\|_2$*. This implies*

$$
\begin{aligned}
0 &\leq \int_0^\infty \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix}^T \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix} dt \\
&\overset{(a)}{\leq} \int_0^T \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix}^T \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix} dt \\
&\overset{(b)}{\leq} \int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt
\end{aligned}
$$

Inequality (a) follows because $\tilde{v}(t) = 0$ *for* $t \geq T$*. Inequality (b) follows from the causality of* Δ*. Specifically,* $\tilde{v}(t) = v(t)$ *for* $t \leq T$ *and hence* $\tilde{w}(t) = w(t)$ *for* $t \leq T$*. The final conclusion is that the finite horizon inequality holds for any* $T \geq 0$ *and any input/output pair* (v, w) *of* Δ*. Thus* Δ *satisfies the IQC defined by* (Ψ, M) *with* $\Psi = I_2$ *and* $M = \left[\begin{smallmatrix} b^2 & 0 \\ 0 & -1 \end{smallmatrix}\right]$*. In this example* Ψ *contains no dynamics and hence* $z = [v; w]$*.*

Example 2. *Next consider an LTI (SISO) system* Δ *that satisfies the bound* $\|\Delta\| \leq b$*. Since* Δ *is LTI it commutes with any stable, minimum phase system* $D(s)$*, i.e.* $\Delta D = D\Delta$*. Thus the frequency-scaled system* $\bar{\Delta} := D\Delta D^{-1}$ *is also norm-bounded by* b*. Let* $(\bar{v}, \bar{w})$ *be any input-output*

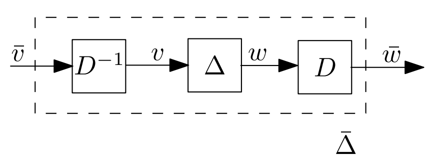


Fig. 2. Scaling of the LTI system Δ

pair for the scaled system $\bar{\Delta}$*, see Fig. 2. The first example implies that*

$$
\int_0^T \begin{bmatrix} \bar{v}(t) \\ \bar{w}(t) \end{bmatrix}^T \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{v}(t) \\ \bar{w}(t) \end{bmatrix} dt \geq 0 \tag{20}
$$

The associated input/output pair for the original system $w = \Delta(v)$ is related to the input/output pair for the scaled system by $\bar{w} = Dw$ and $\bar{v} = Dv$. Thus the inequality in (20) *can be equivalently written as*

$$\int_0^T z(t)^T M z(t) \geq 0 \tag{21}$$

where $M = \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix}$ and z is the output of $\Psi := \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$ generated by the signals $\begin{bmatrix} v \\ w \end{bmatrix}$, see Fig. 1. Hence Δ satisfies the IQC defined by (Ψ, M). Note that the use of $D(s)$ directly corresponds to the multipliers used in classical robustness analysis, e.g. the structured singular value μ [17], [18], [19], [20].

The two examples above are simple instances of IQCs. The proposed approach applies to the more general IQC framework introduced in [9] but with some technical restrictions. In particular, [9] provides a library of IQC multipliers that are satisfied by many important system components, e.g. saturation, time delay, and norm bounded uncertainty. The IQCs in [9] are expressed in the frequency domain as an integral constraint defined using a multiplier Π. The multiplier Π can be factorized as $\Pi = \Psi^{\sim} M \Psi$ and this connects the frequency domain formulation to the time-domain formulation used in this paper. One technical point is that, in general, the time domain IQC constraint only holds over infinite horizons ($T = \infty$). The work in [9], [10] draws a distinction between hard/complete IQCs for which the integral constraint is valid over all finite time intervals and soft/conditional IQCs for which the integral constraint need not hold over finite time intervals. The formulation of an IQC in this paper as a finite-horizon (time-domain) inequality is thus valid for any frequency-domain IQC that admits a hard/complete factorization (Ψ, M). While this is somewhat restrictive, it has recently been shown in [10] that a wide class of IQCs have a hard factorization. The remainder of the paper will simply treat, without further comment, (Ψ, M) as the starting point for the definition of the finite-horizon IQC. A more detailed treatment of the connection of time-domain IQCs to frequency domain ones is given in Section A.

III. LPV Robustness Analysis

An uncertain LPV system is described by the interconnection of an LPV system G_ρ and an uncertainty Δ, as depicted in Fig. 3. This interconnection represents an upper linear fractional transformation (LFT), which is denoted $\mathcal{F}_u(G_\rho, \Delta)$. The uncertainty Δ is assumed to satisfy an

IQC described by (Ψ, M). Note that Δ can include hard nonlinearities (e.g. saturations) and infinite dimensional operators (e.g. time delays) in addition to true system uncertainties. The term "uncertainty" is used for simplicity when referring to the perturbation Δ.

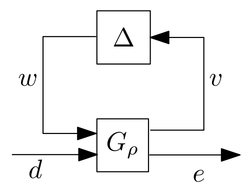


Fig. 3. Uncertain LPV System

The robust performance of $\mathcal{F}_u(G_\rho, \Delta)$ is measured in terms of the worst case induced L_2 gain from the input d to the output e. The worst-case gain is defined as

$$\sup_{\Delta \in IQC(\Psi, M)} \|\mathcal{F}_u(G_\rho, \Delta)\|. \tag{22}$$

This gain is worst-case over all uncertainties Δ that satisfy the IQC defined by (Ψ, M) and admissible trajectories ρ.

A. Bounded Real Lemma including IQCs

The analysis is based on the basic LFT interconnection in Fig. 3 is where Δ satisfies the IQC defined by (Ψ, M). In this basic interconnection the filter Ψ is included as shown in Fig. 4.

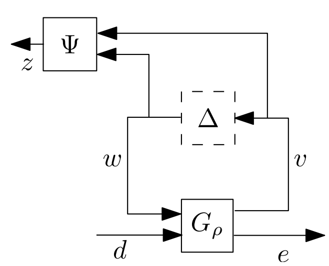


Fig. 4. Analysis Interconnection Structure

The dynamics of the interconnection in Fig. 4 are described by $w = \Delta(v)$ and

$$\begin{aligned} \dot{x} &= A(\rho)x + B_1(\rho)w + B_2(\rho)d \\ z &= C_1(\rho)x + D_{11}(\rho)w + D_{12}(\rho)d \\ e &= C_2(\rho)x + D_{21}(\rho)w + D_{22}(\rho)d, \end{aligned} \tag{23}$$

where the state vector is $x = [x_G; x_\psi]$ with x_G and x_ψ being the state vectors of the LPV system G_ρ and the filter Ψ respectively. The uncertainty Δ is shown in the dashed box in Fig. 4 to signify that it is removed for the analysis. The signal w is treated as an external signal subject to the constraint

$$\int_0^T z(t)^T M z(t)\, dt \geq 0. \tag{24}$$

This effectively replaces the precise relation $w = \Delta(v)$ with the quadratic constraint on z.

A dissipation inequality can be formulated to upper bound the worst-case L_2 gain of $\mathcal{F}_u(G_\rho, \Delta)$ using the system (23) and the time domain IQC (24).

Theorem 2. *Assume $\mathcal{F}_u(G_\rho, \Delta)$ is well posed for all $\Delta \in$ IQC(Ψ, M). Then the worst-case gain is $< \gamma$ if there exists a continuously differentiable $P : \mathcal{P} \to \mathbb{S}^{n_x}$ and a scalar $\lambda > 0$ such that $\forall (p, q) \in \mathcal{P} \times \dot{\mathcal{P}}$*

$$P(p) > 0, \tag{25}$$

$$\begin{bmatrix} P(p)A(p) + A(p)^T P(p) + \partial P(p,q) & P(p)B_1(p) & P(p)B_2(p) \\ B_1(p)^T P(p) & 0 & 0 \\ B_2(p)^T P(p) & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} C_1(p)^T \\ D_{11}(p)^T \\ D_{12}(p)^T \end{bmatrix} M \begin{bmatrix} C_1(p) & D_{11}(p) & D_{12}(p) \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C_2(p)^T \\ D_{21}(p)^T \\ D_{22}(p)^T \end{bmatrix} \begin{bmatrix} C_2(p) & D_{21}(p) & D_{22}(p) \end{bmatrix} < 0 \tag{26}$$

Proof: Let (x, e, w, v, z) be the signals generated for the interconnected system Fig. 4 for the input $d \in L_2$ and parameter trajectory $\rho \in \mathcal{A}$ assuming zero initial conditions. Well-posedness of the interconnection implies that these signals are well-defined. By assumption, the uncertainty Δ satisfies the IQC defined by (Ψ, M) and hence the signal z must satisfy the time domain IQC (24) for any $T > 0$. Define a storage function $V : R^{n_x} \times \mathcal{P} \to \mathbb{R}^+$ for G_ρ by $V(x, \rho) = x^T P(\rho) x$. Equation (26) is a strict inequality, so there exists a $\epsilon > 0$ that the following perturbed matrix inequality holds:

$$
\begin{bmatrix} P(p)A(p) + A(p)^T P(p) + \partial P(p,q) & P(p)B_1(p) & P(p)B_2(p) \\ B_1(p)^T P(p) & 0 & 0 \\ B_2(p)^T P(p) & 0 & -(1-\epsilon)I \end{bmatrix}
$$

$$
+\lambda \begin{bmatrix} C_1(p)^T \\ D_{11}(p)^T \\ D_{12}(p)^T \end{bmatrix} M \begin{bmatrix} C_1(p) & D_{11}(p) & D_{12}(p) \end{bmatrix} \tag{27}
$$

$$
+\frac{1}{\gamma^2} \begin{bmatrix} C_2(p)^T \\ D_{21}(p)^T \\ D_{22}(p)^T \end{bmatrix} \begin{bmatrix} C_2(p) & D_{21}(p) & D_{22}(p) \end{bmatrix} \leq 0
$$

Left and right multiplying (27) by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ respectively and evaluating (27) at $(p, q) = (\rho(t), \dot{\rho}(t))$ shows that (27) is equivalent to the following dissipation inequality:

$$
\nabla_x V \dot{x} + \nabla_\rho V \dot{\rho} \leq (1-\epsilon) d^T d - \frac{1}{\gamma^2} e^T e - \lambda z^T M z. \tag{28}
$$

The dissipation inequality (28) can be integrated along the state/parameter trajectory from $t = 0$ to $t = T$. Recalling that $x(0) = 0$, the integration yields:

$$
\frac{1}{\gamma^2} \int_0^T e(t)^T e(t)\, dt + V(x(T), \rho(T)) \leq (1-\epsilon) \int_0^T d(t)^T d(t)\, dt - \lambda \int_0^T z(t)^T M z(t)\, dt \tag{29}
$$

It follows from $P(\rho) > 0$ and the IQC condition (24) that

$$
\frac{1}{\gamma^2} \int_0^T e(t)^T e(t) \leq (1-\epsilon) \int_0^T d(t)^T d(t)\, dt. \tag{30}
$$

Let $T \to \infty$ to conclude that $\|e\|_2 < \gamma \|d\|_2$. This holds for any input $d \in L_2$, admissible trajectory $\rho \in \mathcal{A}$, and uncertainty $\Delta \in \text{IQC}(\Psi, M)$. Thus the worst-case gain is $< \gamma$. ■

The same considerations in regard to gridding and basis functions for $P(p)$ as in Section II-B have to be taken into account, in order to turn Theorem 2 into a computational tractable optimization problem. It should also be noted that the multiple uncertainties can be incorporated in the analysis. Specifically, assume

$$
\Delta = \begin{bmatrix} \Delta_1 & & \\ & \ddots & \\ & & \Delta_N \end{bmatrix} \tag{31}
$$

and each Δ_k satisfies the IQC defined by (Ψ_k, M_k) for $k = 1, \cdots, N$. Each Ψ_k can be appended to the inputs/outputs of the corresponding Δ_k to yield a filtered output z_k. Theorem 2 remains valid if the LMI condition (26) is modified to include the term

$$\sum_{k=1}^{N} \lambda_k \begin{bmatrix} C_{1k}(p)^T \\ D_{11k}(p)^T \\ D_{12k}(p)^T \end{bmatrix} M_k \begin{bmatrix} C_{1k}(p) & D_{11k}(p) & D_{12k}(p) \end{bmatrix} \tag{32}$$

for any constants $\lambda_k \geq 0$. In this case the extended system includes the dynamics of G_ρ as well as the dynamics of each Ψ_k $(k = 1, \cdots, N)$. In addition, $(C_{11k}, D_{11k}, D_{12k})$ denote the output state matrices of the extended system associated with output z_k. The stability analysis consists of a search for the matrix function $P : \mathcal{P} \to \mathbb{S}^{n_x}$, gain bound γ, and the constants λ_k that lead to feasibility of the conditions in Theorem 2. This approach also enables many IQCs for a single Δ to be incorporated into the analysis.

B. Parameter-Varying IQCs

The dissipation inequality framework opens up the possibility for new classes of IQCs. Unlike the classical frequency domain approach to IQCs, the time domain interpretation allows for Ψ and/or M to be time varying and/or nonlinear. The IQC theory given in [9] cannot deal with this generalized class of IQCs. Specifically in the context of linear parameter varying systems it is straightforward to extend Theorem 2 to consider parameter-varying IQCs where Ψ and/or M could depend on ρ. Note that results for parameter-dependent multipliers exist in the literature. For example, the results in [21], [22] consider a known LTI system G interconnected with real parameter uncertainty Δ specified as the LFT $F_u(G, \Delta)$. Analysis conditions are developed based on a parameter-dependent multiplier. The multiplier for Δ depends on Δ itself. This section considers a related but slightly different parameter-dependent multiplier. Specifically, the problem formulation considers the interconnection $F_u(G_\rho, \Delta)$ of an LPV system G_ρ and an uncertainty Δ. The objective is to develop a multiplier for Δ that depends on the parameter ρ that appears in the known part of the model. This idea appears implicitly, for example, in the literature on time-delayed LPV systems. However, the generality of this approach for gridded LPV systems does not seem to be fully exploited.

One example of this extension is given for the analysis of time-delayed LPV systems. A constant delay $w = \mathcal{D}_\tau(v)$ is defined by $w(t) = v(t - \tau)$ where τ specifies the constant delay.

To be precise the constant delay is defined as $\mathcal{D}_\tau : L_2^n[0,\infty) \to L_2^n[0,\infty)$ such that $w = \mathcal{D}_\tau(v)$ satisfies $w(t) = 0$ for $t \in [0,\tau)$ and $w(t) = v(t-\tau)$ for $t \geq \tau$. The next lemma provides a parameter-varying IQC for $\mathcal{D}_\tau$. Note that this IQC is delay independent i.e. it does not depend on the amount of delay. The basic idea for the proposed IQC is taken from [23] which develops stability conditions for a delayed LPV system using the Lyapunov-Krasovskii framework.

Lemma 1. *Let $Q : \mathcal{P} \to \mathbb{R}^{n\times n}$ satisfy $Q(\rho) > 0$ for all $\rho \in \mathcal{P}$. Then $\mathcal{D}_\tau \in IQC(\Psi, M)$ where $\Psi := I_{2n}$ and*

$$M := \begin{bmatrix} Q(\rho(t)) & 0 \\ 0 & -Q(\rho(t-\tau)) \end{bmatrix} \tag{33}$$

Proof: Let $v \in L_2^n[0,\infty)$ and define $w = \mathcal{D}_\tau(v)$. The assumption $Q(\rho) > 0$ implies that the following inequality holds for all $v \in L_2^n[0,\infty)$ and for all $T \geq 0$:

$$\int_{T-\tau}^{T} v^T(t) Q(\rho(t)) v(t) \ dt \geq 0 \tag{34}$$

This integral term appears as one term of the Lyapunov-Krasovskii function of Theorem 4.1 in [23]. With some algebra this expression can be re-written in the following IQC form:

$$\int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} Q(\rho(t)) & 0 \\ 0 & -Q(\rho(t-\tau)) \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \ dt \geq 0 \tag{35}$$

Thus $\mathcal{D}_\tau \in IQC(\Psi, M)$. ■

This IQC can be also be obtained via a scaling argument similar to that used in Example 2. Specifically, the norm bound $\|\mathcal{D}_\tau\| \leq 1$ leads to the simple IQC $\mathcal{D}_\tau \in IQC(\Psi, M)$ where $\Psi = I_{2n}$ and $M = \left[\begin{smallmatrix} I_n & 0 \\ 0 & -I_n \end{smallmatrix}\right]$. The delay $\mathcal{D}_\tau$ also satisfies the following swapping relation: $\mathcal{D}_\tau D(\rho(t)) = D(\rho(t-\tau))\mathcal{D}_\tau$ Thus the matrix-scaled system $\bar{\Delta} = D(\rho(t-\tau))\mathcal{D}_\tau D(\rho(t))^{-1}$ is also norm bounded by 1. A similar argument to that given in Example 2 leads to the conclusion of Lemma 1: $D_\tau \in IQC(\Psi, M)$ where $Q(\rho) = D^T(\rho)D(\rho) > 0$.

This parameter-varying IQC can be used within the dissipation inequality framework to develop a less conservative analysis condition for delayed LPV systems. Let $\mathcal{F}_u(G_\rho, \mathcal{D}_\tau)$ denote a delayed LPV system and consider the IQC $\mathcal{D}_\tau \in IQC(\Psi, M)$. The analysis interconnection (Fig. 4 and (23)) simplifies in this case because $\Psi = I_{2n}$ means that $z = \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$. This leads to the following analysis result which, for simplicity, is written for the rate-unbounded case.

Theorem 3. *Assume $\mathcal{F}_u(G_\rho, \mathcal{D}_\tau)$ is well posed for the constant delay $\tau > 0$. Then the worst-case gain is $< \gamma$ if there exists a matrix $P = P^T > 0$ and a function $Q : \mathcal{P} \to \mathbb{R}^{n\times n}$ such that $\forall p_1, p_2 \in \mathcal{P}$*

$$Q(p_1) > 0 \tag{36}$$

$$\begin{bmatrix} PA(p_1) + A(p_1)^T P & PB_1(p_1) & PB_2(p_1) \\ B_1(p_1)^T P & 0 & 0 \\ B_2(p_1)^T P & 0 & -I \end{bmatrix} + \begin{bmatrix} C_1(p_1)^T \\ D_{11}(p_1)^T \\ D_{12}(p_1)^T \end{bmatrix} M(p_1, p_2) \begin{bmatrix} C_1(p_1) & D_{11}(p_1) & D_{12}(p_1) \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C_2(p_1)^T \\ D_{21}(p_1)^T \\ D_{22}(p_1)^T \end{bmatrix} \begin{bmatrix} C_2(p_1) & D_{21}(p_1) & D_{22}(p_1) \end{bmatrix} < 0 \tag{37}$$

where

$$M(p_1, p_2) := \begin{bmatrix} Q(p_1) & 0 \\ 0 & -Q(p_2) \end{bmatrix} \tag{38}$$

Proof: The proof uses the IQC for $\mathcal{D}_\tau$ defined in Lemma 1 and is similar to that given for Theorem 2. Details are omitted. ■

Theorem 3 gives additional flexibility relative to the use of the constant IQC(Ψ, M) with $\Psi = I_{2n}$ and $M = \left[\begin{smallmatrix} Q & 0 \\ 0 & -Q \end{smallmatrix}\right]$. This added flexibility comes at the cost of introducing additional unknowns in the problem formulation. Note that the concept of a parameter-dependent IQC can be potentially extended to a broader class of operators by applying a swapping lemma (see [24]). So far only constant Ψ and parameter dependent $M(\rho)$ have been considered, although in principal there is no reason to restrict Ψ to be constant. This idea will be pursued in future work.

C. Connection to Existing Bounded Real Lemma for LPV Systems

The main result of this section shows that Theorem 2 is a generalization of the Bounded Real condition for LPV systems (Theorem 1). Additionally, it can be shown that an inherent robustness is included in Theorem 1. Specifically consider the uncertain feedback system $\mathcal{F}_u(G_\rho, \Delta)$ where Δ is a causal operator that satisfies the norm bound $\|\Delta\| \le b$. For each value of $b > 0$ Theorem 2 can be used to compute a bound on the worst-case gain, denoted γ_b. The value $b = 0$ corresponds to a (nominal) LPV system and a bound on the L_2 gain of this system, denoted γ_0, can be computed using the standard Bounded Real type condition in Theorem 1. It is shown below that the bounds computed using the condition in Theorem 2 have the property that $\gamma_b \to \gamma_0$ as $b \to 0$. In other words, the worst-case gain bounds computed via Theorem 2 converge to those computed by Theorem 1 when the uncertainty is "small". In this sense, the nominal performance condition in Theorem 1 is robust to small uncertainty perturbations. The remainder of this section provides a precise statement and proof of this connection.

The norm bound $\|\Delta\| \le b$ implies that the operator Δ satisfies the IQC defined by (Ψ, M) with $\Psi = I$ and $M = \left[\begin{smallmatrix} b^2 & 0 \\ 0 & -I \end{smallmatrix}\right]$. Using this factorization, the signal $z = C_1 x + D_{11} w + D_{12} d$ in the interconnection Fig. 4, see (23), is given by

$$z = \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} C_{11} & D_{111} & D_{121} \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ d \end{bmatrix}. \tag{39}$$

Inserting the IQC for a norm bounded operator into (26) yields

$$\begin{bmatrix} P(p)A(p) + A(p)^T P(p) + \partial P(p,q) & P(p)B_1(p) & P(p)B_2(p) \\ B_1(p)^T P(p) & -\lambda I & 0 \\ B_2(p)^T P(p) & 0 & -I \end{bmatrix}$$
$$+ b^2 \lambda \begin{bmatrix} C_{11}(p)^T \\ D_{111}(p)^T \\ D_{121}(p)^T \end{bmatrix} \begin{bmatrix} C_{11}(p) & D_{111}(p) & D_{121}(p) \end{bmatrix} \tag{40}$$
$$+ \frac{1}{\gamma^2} \begin{bmatrix} C_2(p)^T \\ D_{21}(p)^T \\ D_{22}(p)^T \end{bmatrix} \begin{bmatrix} C_2(p) & D_{21}(p) & D_{22}(p) \end{bmatrix} < 0.$$

The following theorem proves that feasibility of the nominal conditions in (9) is equivalent to feasibility of the worst-case gain conditions in (40) for some interval $b \in [0, \bar{b}]$.

Theorem 4. *There exists a $P(\rho) > 0$, $\gamma > 0$ and $\lambda > 0$ such that the nominal LMI condition from Theorem 1*

$$\begin{bmatrix} P(p)A(p) + A(p)^T P(p) + \partial P(p,q) & P(p)B_2(p) \\ B_2(p)^T P(p) & -I \end{bmatrix} + \frac{1}{\gamma^2}\begin{bmatrix} C_2(p)^T \\ D_{22}(p)^T \end{bmatrix}\begin{bmatrix} C_2(p) & D_{22}(p) \end{bmatrix} < 0. \tag{41}$$

is feasible if and only if there exists a $\bar{b} > 0$ such that $P(\rho)$, γ and λ are a feasible solution of (40) *for all $b \in [0, \bar{b}]$.*

Proof: It is first shown that feasibility of (41) is equivalent to feasibility of (40) for the zero operator, i.e. $b = 0$. Applying the Schur complement, (40) holds if and only if

$$\begin{aligned} V - S(R - \lambda I)^{-1} S^T < 0 \\ (R - \lambda I) < 0 \end{aligned} \tag{42}$$

with V is defined to be the matrix on the left side of (41),

$$S = \begin{bmatrix} P(p)B_1(p) \\ 0 \end{bmatrix} + \frac{1}{\gamma^2}\begin{bmatrix} C_2(p)^T D_{21}(p) \\ D_{22}(p)^T D_{21}(p) \end{bmatrix} \tag{43}$$

and

$$R = \frac{1}{\gamma^2} D_{21}(p)^T D_{21}(p). \tag{44}$$

If conditions (42) are satisfied then $V < 0$. Conversely, feasibility of $V < 0$ implies that (42) can be satisfied by a sufficiently large λ. Choosing a sufficiently large λ enforces $(R - \lambda I) < 0$. Thus feasibility of (41) is equivalent to feasibility of (40) with $b = 0$. Next note that strict matrix inequalities are robust to small perturbations, i.e. a matrix $M < 0$ implies that $M + \Delta M < 0$ if $\bar{\sigma}(\Delta M)$ is sufficiently small. Thus (40) is feasible with $b = 0$ if and only if it is feasible for all $b \in [0, \bar{b}]$ for some sufficiently small $\bar{b}$.

■

D. Worst-Case Energy-to-Peak Gain Computation

The LPV robust performance analysis can be easily extended to include other performance measurements than the L_2 gain. One example of a different input/output gain is the L_2-L_∞ gain.

The L_2-L_∞ gain from input d to output e of an LPV system G_ρ is defined by

$$\|G_\rho\|_{L_2\to L_\infty} := \sup_{d\neq 0, d\in\mathcal{L}_2, \rho\in\mathcal{A}, x(0)=0} \frac{\|e\|_\infty}{\|d\|_2}. \tag{45}$$

Note that the norm is only finite if there exists no direct feedthrough term from the input to the output. Similarly the worst-case L_2-L_∞ gain of an interconnection $\mathcal{F}_u(G_\rho, \Delta)$, see Fig. 3, is defined by

$$\sup_{\Delta\in IQC(\Psi,M)} \|\mathcal{F}_u(G_\rho,\Delta)\|_{L_2\to L_\infty}. \tag{46}$$

Using these definitions, the following theorem states conditions to bound the worst-case energy-to-peak gain of $\mathcal{F}_u(G_\rho, \Delta)$.

Theorem 5. *Assume $\mathcal{F}_u(G_\rho, \Delta)$ is well posed for all $\Delta \in IQC(\Psi, M)$. Then the worst-case energy-to-peak gain is $< \gamma$ if there exists a continuously differentiable $P : \mathcal{P} \to \mathbb{S}^{n_x}$ and a scalar $\lambda > 0$ such that $\forall (p, q) \in \mathcal{P} \times \dot{\mathcal{P}}$*

$$P(p) > 0, \tag{47}$$

$$\begin{bmatrix} P(p)A(p) + A(p)^T P(p) + \partial P(p,q) & P(p)B_1(p) & P(p)B_2(p) \\ B_1(p)^T P(p) & 0 & 0 \\ B_2(p)^T P(p) & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} C_1(p)^T \\ D_{11}(p)^T \\ D_{12}(p)^T \end{bmatrix} M \begin{bmatrix} C_1(p) & D_{11}(p) & D_{12}(p) \end{bmatrix} < 0 \tag{48}$$

and

$$\begin{bmatrix} P(p) & C_2(p)^T \\ C_2(p) & \gamma^2 I \end{bmatrix} > 0. \tag{49}$$

Proof: The approach to the proof is similar to the proof of Theorem 2. Let (x, e, w, v, z) be the signals generated for the interconnected system Fig. 4 for the input $d \in L_2$ and parameter trajectory $\rho \in \mathcal{A}$ assuming zero initial conditions. Well-posedness of the interconnection implies that these signals are well-defined. By assumption, the uncertainty Δ satisfies the IQC defined by (Ψ, M) and hence the signal z must satisfy the time domain IQC (24) for any $T > 0$. Define a storage function $V : R^{n_x} \times \mathcal{P} \to \mathbb{R}^+$ for G_ρ by $V(x, \rho) = x^T P(\rho) x$. Use a similar perturbation argument as in the proof of Theorem 2. Then left and right multiplying the perturbed LMI by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ respectively and evaluating it at $(p, q) = (\rho(t), \dot{\rho}(t))$ shows that the perturbed LMI is equivalent to the following dissipation inequality:

$$\nabla_x V \dot{x} + \nabla_\rho V \dot{\rho} \leq (1-\epsilon) d^T d - \lambda z^T M z. \tag{50}$$

The dissipation inequality (50) can be integrated along the state/parameter trajectory from $t = 0$ to $t = T$. Recalling that $x(0) = 0$, the integration yields:

$$V(x(T), \rho(T)) \leq (1-\epsilon) \int_0^T d(t)^T d(t)\, dt - \lambda \int_0^T z(t)^T M z(t)\, dt \tag{51}$$

It follows from the IQC condition (24) that $\int_0^T z(t)^T M z(t)\, dt \geq 0$, so that

$$x(T)^T P(\rho(T)) x(T) \leq (1-\epsilon) \int_0^T d(t)^T d(t)\, dt. \tag{52}$$

Using Schur complement on (49) and left and right multiplying it by x^T and x respectively gives

$$\frac{1}{\gamma^2} e(t)^T e(t) \leq x(t)^T P(\rho(t)) x(t). \tag{53}$$

Evaluating (53) at $t = T$, and applying (52) yields:

$$\frac{1}{\gamma^2} e(T)^T e(T) \leq (1-\epsilon) \int_0^T d(t)^T d(t)\, dt. \tag{54}$$

Take the supremum over T to conclude that $\|e\|_\infty < \gamma \|d\|_2$. This holds for any input $d \in L_2$, admissible trajectory $\rho \in \mathcal{A}$, and uncertainty $\Delta \in$ IQC(Ψ, M).

■

IV. NUMERICAL EXAMPLES

This section provides three numerical examples to demonstrate the proposed robust analysis conditions. The first two examples consider a feedback system with a perturbation that is assumed to be norm bounded (Section IV-A) or a memoryless nonlinearity (Section IV-B). The third example uses a parameter-varying IQC to model a time-delay perturbation (Section IV-C).

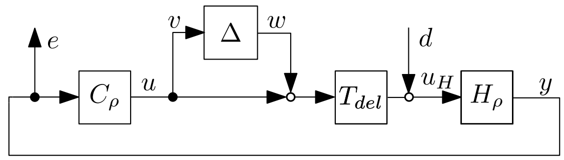


Fig. 5. Closed Loop Interconnection with Dynamic Uncertainty

A. First-order LPV System

The first example is a feedback interconnection of a first-order LPV system with a gain-scheduled proportional-integral controller as shown in Fig. 5.

The system H_ρ, taken from [25], is a first order system with dependence on a single parameter ρ. It can be written as

$$\begin{aligned} \dot{x}_H &= -\frac{1}{\tau(\rho)} x_H + \frac{1}{\tau(\rho)} u_H \\ y &= K(\rho) x_H \end{aligned} \tag{55}$$

with the time constant $\tau(\rho)$ and output gain $K(\rho)$ depending on the scheduling parameter as follows:

$$\tau(\rho) = \sqrt{133.6 - 16.8\rho}, \qquad K(\rho) = \sqrt{4.8\rho - 8.6}. \tag{56}$$

The scheduling parameter ρ is restricted to the interval $[2, 7]$. For all the following analysis scenarios a grid of six points is used that span the parameter space equidistantly. A time-delay of 0.5 seconds is included at the control input. The time delay in the system is represented by a second order Pade approximation:

$$T_{del}(s) = \frac{\frac{(T_d s)^2}{12} - \frac{T_d s}{2} + 1}{\frac{(T_d s)^2}{12} + \frac{T_d s}{2} + 1}, \tag{57}$$

where $T_d = 0.5$.

A gain-scheduled PI-controller C_ρ is designed that guarantees a closed loop damping $\zeta_{cl} = 0.7$ and a closed loop frequency $\omega_{cl} = 0.25$ at each frozen value of ρ. The controller gains that satisfy these requirements are given by

$$K_p(\rho) = -\frac{2\zeta_{cl}\omega_{cl}\tau(\rho) - 1}{K(\rho)}, \qquad K_i(\rho) = -\frac{\omega_{cl}^2 \tau(\rho)}{K(\rho)}. \tag{58}$$

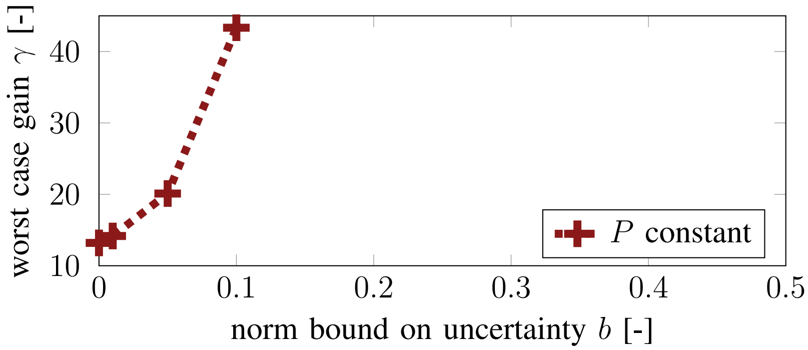
The controller is realized in the following state space form:

$$\begin{aligned} \dot{x}_c &= K_i(\rho)e \\ u &= x_c + K_p(\rho)e \end{aligned} \tag{59}$$

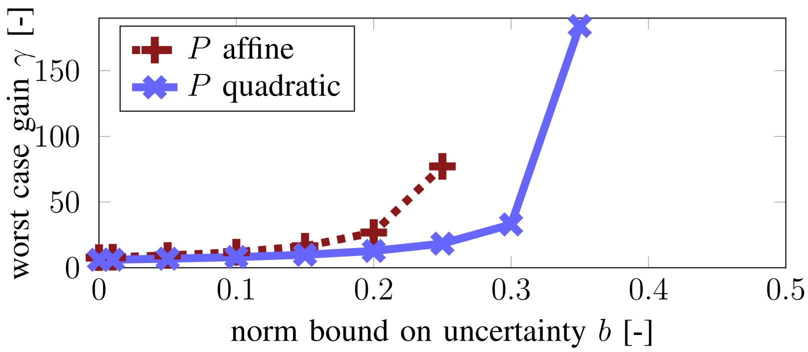
As an extension to the original example in [25], a multiplicative norm bounded LTI uncertainty Δ is inserted at the input of the plant. The norm of the uncertainty is assumed to be less than a positive scalar b, i.e. $\|\Delta\| \leq b$. The uncertainty is described by the integral quadratic constraints (Ψ_1, M_1) with $\Psi_1 = I$ and $M_1 = \left[\begin{smallmatrix} b^2 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and (Ψ_2, M_2) with $\Psi_2 = \left[\begin{smallmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{smallmatrix}\right]$ and $M_2 = \left[\begin{smallmatrix} b^2 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. The gain from a disturbance at the plant input d to the control error e is used as a performance measurement and γ denotes an upper bound on the worst case gain as defined by (22).

The initial analysis is performed assuming unbounded parameter variation rates. This corresponds to the use of a constant matrix P in the analysis conditions. First, the standard analysis conditions (Theorem 1) yield a bound on the L_2 gain of the nominal system ($b = 0$) of 13.2. Next, the analysis conditions in Theorem 2 are used to bound the worst-case L_2 gain for different uncertainty sizes. The results are shown in the top plot of Fig. 6. The convex optimization could not find feasible solutions for $b > 0.1$, meaning that for uncertainties larger than ten percent no finite gain can be guaranteed. For $b = 0.05$, the analysis yields a bound on the worst case L_2 gain of 20.1.

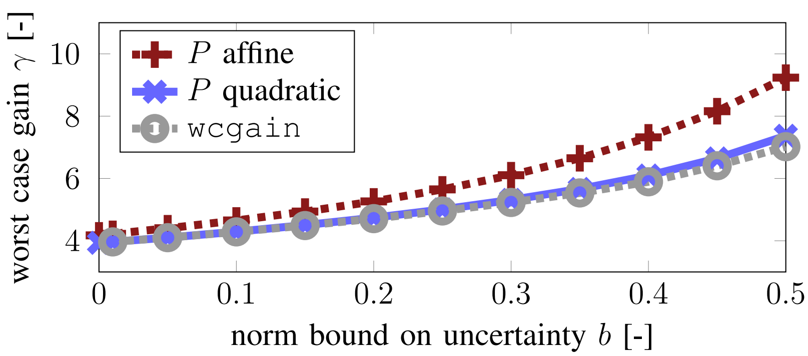
The following approach is used to estimate the conservativeness of the upper bounds shown in the top plot of Fig. 6. The worst-case L_2 gain is computed at each frozen value of ρ using the Matlab command `wcgain` [26]. The `wcgain` command computes upper and lower bounds on the largest L_2 gain of a known LTI system interconnected with LTI uncertainty. The computations are variations of those used for the structured singular value (μ) and details are provided in [27]. This frozen LTI analysis is performed for $b = 0.05$ to obtain the worst values of ρ and Δ. The worst value of Δ_0 returned by `wcgain` is then used to construct a (not-uncertain) LPV system $F_u(G_\rho, \Delta_0)$. The induced L_2 gain bound for this (not-uncertain) system is computed via the (nominal) conditions in Theorem 1. The result of this analysis is a gain of 19.4 which is close to the computed worst-case gain of 20.1. The approach using `wcgain` does not provide a true lower bound on the worst-case gain due to the conservatism in the Bounded Real Lemma for LPV systems. However, it does provide an indication of the conservatism in the proposed robustness bound.



(a) Unbounded Parameter Variation Rate



(b) Parameter Variation bounded by $|\dot{\rho}| \leq 2$



(c) Parameter Variation bounded by $|\dot{\rho}| \leq 0.1$

Fig. 6. Worst Case Gain vs Norm Bound on Uncertainty for the interconnection in Fig. 5

In the next analysis, the parameter variation rate is assumed to be bounded by $|\dot{\rho}| \leq 2$. Affine and quadratic parameter dependences are considered for P, i.e. $P(\rho) = P_0 + \rho P_1$ and $P(\rho) = P_0 + \rho P_1 + \rho^2 P_2$, respectively. The results are given in the middle plot of Fig. 6. Allowing a higher order $P(\rho)$ reduces the worst case gains and also allows finding finite gain bounds for uncertainties up to $b = 0.35$. In comparison, the affine storage function could only guarantee finite gains for $b \leq 0.25$.

Finally, bounds on the worst-case gains are computed when the parameter rate is bounded by $|\dot{\rho}| \leq 0.1$. The results for both affine and quadratic dependences of $P(\rho)$ are shown in the lower plot of Fig. 6. For comparison, this figure also shows a lower bound computed using the Matlab function `wcgain` described above. Specifically, `wcgain` was used to compute the worst-case gain at each frozen value of ρ. The curve labeled `wcgain` shows the largest gain across the parameter domain computed at each value of b. This frozen parameter analysis provides a lower bound on the worst-case gain of the uncertain LPV system. Note that in this example the lower bounds computed via `wcgain` are close to the upper bounds on the worst-case L_2 gain for the uncertain LPV system. In general, the gap between both methods can be large. For example, a similar analysis in [14] yields a much larger gap. The following fact mostly contributes to the difference between the results from `wcgain` and the proposed method with IQCs: `wcgain` is only an LTI analysis at each (frozen) grid point. The LPV analysis maximizes the gain over all allowable parameter trajectories while the `wcgain` analysis can be viewed as restricting the trajectory to be frozen at a single parameter value.

B. Sector bounded nonlinearity

The first example in Section IV-A uses only a simple static IQC to describe a norm bounded LTI operator. The second example described in this section considers more general IQCs. In particular, the same feedback interconnection (see Fig. 4) is considered. However, Δ is now assumed to be a memoryless nonlinearity in a sector $[0, \beta]$. Note that Δ belongs to a sector $[0, \beta]$ if

$$\Delta(v,t)\left(\beta v - \Delta(v,t)\right) \geq 0 \tag{60}$$

holds for all $v, t \in \mathbb{R}$. The sector condition in (60) can be expressed as a time-domain IQC using $\Psi = I$ and $M = \left[\begin{smallmatrix} 0 & \beta \\ \beta & -2 \end{smallmatrix}\right]$.

If the nonlinearity is also assumed to be static, i.e. $\Delta(v)$, then the Popov IQC can also be used. The Popov IQC is defined in the frequency domain (see Section A) as $\Pi = \pm \begin{bmatrix} 0 & -j\omega \\ j\omega & 0 \end{bmatrix}$. There are two technical issues associated with the use of the Popov IQC within the dissipation inequality framework. First the Popov IQC is "soft" in the sense that it describes a constraint in the time-domain that is satisfied over an infinite time horizon [9]. The dissipation inequality framework requires the use of a "hard" IQC for which the constraint holds over all finite times. Second, the Popov IQC as defined above is not a proper IQC multiplier because Π is not bounded on the imaginary axis. As a remedy for the second issue, a simple loop transformation $\bar{\Delta} := \Delta \circ \frac{1}{s+1}$ can be applied as shown in Fig. 7. The factor of $(s+1)$ introduced by this loop transformation can be absorbed into the known part of the system as along as the plant H_ρ is strictly proper. The first issue is resolved using the J-spectral factorization approach described in Section A. In particular, note that a nonlinearity Δ in the sector $[0, \beta]$ has a norm bound of β and hence satisfies the IQC defined by $\Pi = \begin{bmatrix} \beta^2 & 0 \\ 0 & -1 \end{bmatrix}$. Thus if Δ is a static, memoryless nonlinearity then it satisfies the slightly modified version of the Popov multiplier given by $\Pi = \begin{bmatrix} 0.01\beta^2 & -j\omega \\ j\omega & -0.01 \end{bmatrix}$. The transformed system $\bar{\Delta}$ thus satisfies the following version of the Popov IQC

$$\Pi := \begin{bmatrix} \frac{1}{j\omega+1} & 0 \\ 0 & 1 \end{bmatrix}^{\sim} \begin{bmatrix} 0.01\beta^2 & -j\omega \\ j\omega & -0.01 \end{bmatrix} \begin{bmatrix} \frac{1}{j\omega+1} & 0 \\ 0 & 1 \end{bmatrix} \tag{61}$$

This multiplier satisfies the conditions of Theorem 6 and hence a J-spectral factorization (Ψ, M) can be computed such that $\Pi = \Psi^{\sim} M \Psi$. This J-spectral factorization (Ψ, M) is a (hard) time-domain IQC that can be properly used in the dissipation inequality framework.

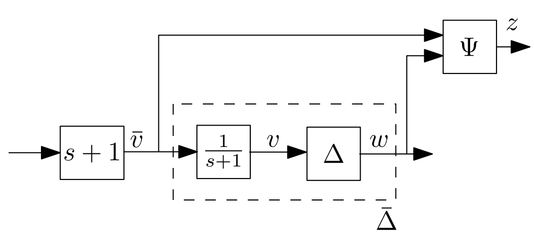


Fig. 7. Loop transformation for Popov IQC

Finally, if Δ is further restricted to be a monotonic, odd function then the Zames-Falb multiplier [28], [9] can be used. This IQC multiplier is defined by $\Pi = \begin{bmatrix} 0 & 1+H(j\omega) \\ 1+H(j\omega)^* & -2-(H(j\omega)+H(j\omega)^*) \end{bmatrix}$

where $H \in \mathbb{RL}_\infty$ is arbitrary except that the L_1 norm of its impulse response must be less than 1.

The Popov IQC has been adapted, as described above, to consider the modified $\bar{\Delta} = \Delta \circ \frac{1}{s+1}$. The sector-bound and Zames-Falb IQC multipliers must also be adapted for $\bar{\Delta}$. These can be expressed as:

$$\text{Sector bound IQC:} \qquad \Psi = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 1 \end{bmatrix}, \qquad M = \begin{bmatrix} 0 & \beta \\ \beta & -2 \end{bmatrix} \tag{62}$$

$$\text{Zames-Falb IQC:} \qquad \Pi = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 1 \end{bmatrix}^\sim \begin{bmatrix} 0 & 1 + H(s) \\ 1 + H(s)^\sim & -2 - (H(s) + H(s)^\sim) \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 1 \end{bmatrix} \tag{63}$$

Note that the Zames-Falb is expressed as a frequency domain multiplier Π. This must also be factorized in order to obtain a (hard) time domain IQC (Ψ, M). This was performed similar to the approach used to factorize the Popov multiplier. Specifically, the Zames-Falb multiplier was slightly modified using the norm-bound IQC. A J-spectral factorization was computed (Section A) to obtain the time domain (hard) IQC in the form (Ψ, M).

The parameter variation rate for this example is assumed to satisfy $|\dot{\rho}| \leq 2$. A quadratic parameter dependence is considered for P, i.e. $P(\rho) = P_0 + \rho P_1 + \rho^2 P_2$. Fig. 8 shows upper bounds on the worst case gain as a function of the sector bound β. The bounds are computed using various IQC multipliers. The first result (red line with $+$) only restricts Δ to belong to the sector $[0, \beta]$. The second curve (blue line with x) uses both the sector and modified Popov IQCs. The last curve (gray line with o) use the sector, modified Popov, and Zames-Falb IQCs. The Zames-Falb IQC was constructed with $H(j\omega) = \frac{1}{j\omega + 0.1}$. Each multiplier corresponds to additional assumptions and hence additional constraints on the nonlinearity Δ. Including more restrictions on the nonlinearity decreases the bound on the worst-case gain. A significant improvement can be seen by considering static nonlinearities (Popov and sector IQC) as compared to memoryless, time-varying nonlinearities (Sector IQC only). The additional assumption that Δ is a monotonic, odd function (Zames-Falb IQC) has minor impact on the results for this example. The results are problem dependent and the Zames-Falb IQC has a larger impact on other examples.

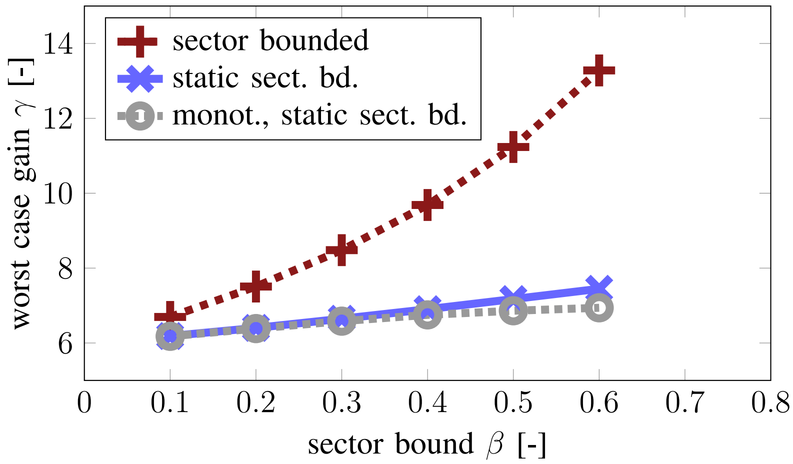


Fig. 8. Worst Case Gain vs Sector Bound

C. Parameter dependent IQCs

The third example is an LPV time-delayed system $\mathcal{F}_u(G_\rho, \Delta)$, where Δ describes $\Delta(v) = \mathcal{D}_\tau(v) - v$. In this description, $\Delta = 0$ corresponds to the (nominal) undelayed system, which is a common way of specifying robust analysis problems. A standard loop-shift argument can be used to obtain an IQC for Δ based on Lemma 1. Δ satisfies the $IQC(\Psi, M)$ where $\Psi := I_{2n}$ and

$$M := \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix}^T \begin{bmatrix} Q(\rho(t)) & 0 \\ 0 & -Q(\rho(t-\tau)) \end{bmatrix} \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix} \tag{64}$$

The LPV system G_ρ is given by

$$\begin{aligned} \dot{x} &= A(\rho)x + \begin{bmatrix} B_1(\rho) & B_2(\rho) \end{bmatrix} \begin{bmatrix} w \\ d \end{bmatrix} \\ \begin{bmatrix} v \\ e \end{bmatrix} &= \begin{bmatrix} C_1(\rho) \\ C_2(\rho) \end{bmatrix} x \end{aligned} \tag{65}$$

with

$$A(\rho) = \begin{bmatrix} -2.7 & -0.5 \\ 0 & -2 \end{bmatrix} + \rho \begin{bmatrix} -0.8 & 1 \\ 1 & -1 \end{bmatrix},$$

$$B_1(\rho) = \begin{bmatrix} 0.5 & 0.3 \\ 0 & 0.3 \end{bmatrix} + \rho \begin{bmatrix} -0.3 & -0.9 \\ -1.6 & 0.2 \end{bmatrix}, \qquad B_2(\rho) = \begin{bmatrix} 0 \\ -0.3 \end{bmatrix} + \rho \begin{bmatrix} 0 \\ -0.2 \end{bmatrix}, \tag{66}$$

$$C_1(\rho) = \begin{bmatrix} 0.5 & 0 \\ -2 & 1 \end{bmatrix} + \rho \begin{bmatrix} -3.5 & 0.5 \\ 2.7 & -0.5 \end{bmatrix}, \qquad C_2(\rho) = \begin{bmatrix} 1 & 0 \end{bmatrix} + \rho \begin{bmatrix} -1.4 & 0 \end{bmatrix}.$$

The scheduling parameter ρ can vary arbitrarily fast between $[0, 1]$. The system G_ρ depends affinely on ρ, so that only the two vertices $\rho = 0$ and $\rho = 1$ need to be considered.

The analysis conditions stated in Theorem 3 using a delay-independent IQC are used to study the robust performance of $\mathcal{F}_u(G_\rho, \Delta)$. The results of the study are summarized in Tab. I. First, an LTI analysis on the vertices of G_ρ is performed to gain some insight in the achievable worst case gain. For this analysis a "large" time delay of $\tau = 100$s is used for comparison against the delay-independent analysis results. The worst-case gain is estimated at the frozen grid points $\rho = 0$ and $\rho = 1$ using the frequency response of the system with delay $\tau = 100$s. This yields gains of 0.051 and 0.027 at parameter values of $\rho = 0$ and $\rho = 1$, respectively. Next, Theorem 3 is invoked using both a constant matrix Q and a parameter varying $Q(\rho) = Q_0 + \rho Q_1$. This example shows the benefit of parameter varying IQCs. Using a constant Q yields a significantly higher margin than $Q(\rho)$. Note that the LTI results only provide a crude lower bound for the achievable performance of the LPV system. Hence, the large gap between the LTI and robust LPV analysis results is not surprising.

V. Conclusions

In this paper the analysis frameworks for LPV systems and uncertainties described by IQCs have been combined. This leads to computationally efficient conditions to assess the robust performance of an LPV system interconnected with uncertainties and nonlinearities. The proposed robust LPV analysis framework is a generalization of the well known nominal LPV Bounded Real Lemma. Simple numerical examples were presented to show the potential of the proposed approach. All results were developed for the class of LPV system where the state matrices are

System	Method	Worst Case Gain
$G_\rho(\rho = 0)$	LTI, $\tau = 100$	0.051
$G_\rho(\rho = 1)$	LTI, $\tau = 100$	0.027
G_ρ	constant Q	2.177
G_ρ	parameter varying $Q(\rho) = Q_0 + \rho Q_1$	0.283

TABLE I

SUMMARY OF ROBUST PERFORMANCE OF THE TIME DELAYED LPV SYSTEM

arbitrary functions of the parameters. Thus the results complement existing similar results for the class of LPV systems whose state matrices are rational functions of the parameters. Future work will consider the synthesis of robust controllers for uncertain LPV systems.

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APPENDIX

APPENDIX

This appendix gives the standard definition of IQCs in the frequency domain and provides the connection back to the time-domain (finite-horizon) inequality used in this paper. The frequency domain interpretation is useful for constructing IQCs for certain components which are naturally given in the frequency domain, e.g. time delays, memoryless nonlinearities, etc. The material in this appendix clarifies how such frequency domain multipliers can be incorporated into the time-domain, dissipation inequality analysis described in this paper.

Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)}$ be a Hermitian-valued function. Two signals $v \in L_2^{m_1}[0,\infty)$ and $w \in L_2^{m_2}[0,\infty)$ satisfy the (frequency-domain) integral quadratic constraint (IQC) defined by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{67}$$

where $\hat{v}(j\omega)$ and $\hat{w}(j\omega)$ are Fourier transforms of v and w, respectively. A bounded, causal operator $\Delta : L_{2e}^{m_1}[0,\infty) \to L_{2e}^{m_2}[0,\infty)$ satisfies the IQC defined by Π, denoted $\Delta \in \text{IQC}(\Pi)$, if (67) holds for all $v \in L_2^{m_1}[0,\infty)$ and $w = \Delta(v)$.

Recall that the formulation of an IQC as a finite-horizon (time-domain) inequality (Equation (16)) is only valid for frequency-domain IQC that admit a hard/complete factorization

(Ψ, M). The next theorem provides a sufficient condition for the existence of a finite-horizon (time-domain) inequality for a given IQC defined by Π.

Theorem 6. *Let* $\Pi = \Pi^{\sim} \in \mathbb{RL}_{\infty}^{(m_1+m_2)\times(m_1+m_2)}$ *be partitioned as* $\begin{bmatrix} \Pi_{11} & \Pi_{21}^{\sim} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$ *where* $\Pi_{11} \in \mathbb{RL}_{\infty}^{m_1 \times m_1}$ *and* $\Pi_{22} \in \mathbb{RL}_{\infty}^{m_2 \times m_2}$*. Assume* $\Pi_{11}(j\omega) > 0$ *and* $\Pi_{22}(j\omega) < 0$ *for all* $\omega \in \mathbb{R} \cup \{\infty\}$*. Then the following statements hold:*

1) Π *can be factorized as* $\Psi^{\sim} M \Psi$ *where* Ψ *is square, stable, and stably invertible.*
2) *If* Δ *satisfies IQC*(Π) *(in the frequency domain) then* Δ *satisfies IQC*(Ψ, M) *(in the time-domain).*

Proof: The sign definite conditions on Π_{11} and Π_{22} ensure that Π has a factorization (Ψ, M) where Ψ is square and both Ψ, Ψ^{-1} are stable. This follows from Theorem 2.4 in [29] and Lemma 2 in Section A. Moreover, Section A provides a numerical algorithm to compute this special (J-spectral) factorization using state-space methods. The conclusion that (Ψ, M) defines a finite horizon IQC follows from Theorem 2.4 in [10]. ■

The IQC stability theorem in [9] uses multipliers specified in the frequency domain and involves a frequency domain inequality condition. This inequality is equivalent (by the KYP lemma [30], [31]) to the existence of a matrix $P = P^T$ satisfying a related linear matrix inequality (LMI). This paper develops stability theorems using dissipation inequalities and a time-domain IQC. The two approaches are related by a non-unique factorization of the frequency domain multiplier as $\Pi = \Psi^{\sim} M \Psi$. The dissipation theory requires the IQC to be "hard" in the sense that the integral constraint holds over all finite times. The J-spectral factorization satisfies this requirement by Theorem 6 and its proof. There is a second technical issue in connecting the frequency and time domain approaches. Specifically, the dissipation inequality that arises in the IQC stability theorem is equivalent to existence of a matrix $P \geq 0$ satisfying the KYP LMI. The constraint $P \geq 0$ ensures that P defines a valid storage function. Note that the frequency domain approach does not require $P \geq 0$. Thus the factorization $\Pi = \Psi^{\sim} M \Psi$ must ensure that a related KYP LMI has a positive definite solution. This requirement ensures that no additional conservatism is introduced in applying the time-domain conditions as compared to the frequency domain conditions. In [32] and [33] triangular factorizations were proposed which satisfy either the "hard" condition or the positive definiteness of the KYP LMI solution.

The J-spectral factorization given in these appendices satisfies both conditions [34]. Thus the J-spectral factorization of frequency domain IQC multipliers ensures that the dissipation inequality approach can be used with no additional conservatism in the analysis.

This appendix provides a specific numerical procedure to factorize $\Pi = \Pi^{\sim} \in \mathbb{RL}_{\infty}^{m\times m}$ as $\Psi^{\sim} M \Psi$. Such factorizations are not unique and this appendix only focuses on the J-spectral factorization. Additional details on various system factorizations can be found in [35], [36], [29], [37], [38].

The stability theorems in this paper require a special factorization such that Ψ is square, stable, and minimum phase. More precisely, given non-negative integers p and q, let $J_{p,q}$ denote the signature matrix $\begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$. Ψ is called a $J_{p,q}$-spectral factor of Π if $\Pi = \Psi^{\sim} J_{p,q} \Psi$ and $\Psi, \Psi^{-1} \in \mathbb{RH}_{\infty}^{m\times m}$. The term J-spectral factor will be used if the values of p and q are not important. J-spectral factorizations have been used to construct (sub-optimal) solutions to the H_∞ optimal control problem [39], [40], [36].

The numerical procedure to construct such a J-spectral factorization of $\Pi = \Pi^{\sim}$ can be summarized by the following steps. First, express Π with a minimal realization $(A_\pi, B_\pi, C_\pi, D_\pi)$. Second, compute a state space realization (A, B, C, D_π) for the stable part of Π. This step can be done using the Matlab command `stabsep`. The assumption that $\Pi = \Pi^{\sim}$ can be used to show that the poles of Π are symmetric about the imaginary axis and, moreover, the unstable part has a state space realization of the form $(-A^T, -C^T, B^T, 0)$ (Section 7.3 of [36]). Thus Π has a factorization in the form $\Pi = \Psi^{\sim} M \Psi$ where:

$$\Psi(s) := \begin{bmatrix} (sI - A)^{-1} B \\ I \end{bmatrix} \tag{68}$$

$$M := \begin{bmatrix} 0 & C^T \\ C & D_\pi \end{bmatrix} \tag{69}$$

This provides a factorization $\Pi = \Psi^{\sim} M \Psi$ where $M = M^T \in \mathbb{R}^{n_z \times n_z}$ and $\Psi \in \mathbb{RH}_{\infty}^{n_z \times m}$. For this factorization, Ψ is, in general, non-square and it may have right-half plane zeros. The remaining steps of the numerical procedure are used to construct a J-spectral factorization. Canonical factorization results in [38], [36] provide necessary and sufficient conditions for Π to have a J-spectral factorization. Specifically, Theorem 2.4 in [29] states that Π has a J-spectral factorization if and only if:

1) D_π is nonsingular and
2) there exists a solution $X = X^T$ to the following ARE:

$$A^T X + XA - (XB + C^T) D_\pi^{-1} (B^T X + C) = 0 \tag{70}$$

such that $A - BD^{-1}(BX + C)$ is Hurwitz.

The ARE in (70) can be solved using the Matlab command `care`. Note that no J-spectral factorization exists if the ARE fails to have a stabilizing solution. If the ARE has a unique stabilizing solution then the state-space realization for Ψ is given by $\left(A, B, J_{p,q} W^{-T} \left(B^T X + C\right), W\right)$ where W is a solution of $D_\pi = W^T J_{p,q} W$.

The last result of this appendix provides a simple frequency domain condition that is sufficient for the existence of a J-spectral factor of a multiplier $\Pi = \Pi^\sim$.

Lemma 2. *Let* $\Pi = \Pi^\sim \in \mathbb{RL}_\infty^{(m_1+m_2)\times(m_1+m_2)}$ *be partitioned as* $\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\sim & \Pi_{22} \end{bmatrix}$ *where* $\Pi_{11} \in \mathbb{RL}_\infty^{m_1 \times m_1}$ *and* $\Pi_{22} \in \mathbb{RL}_\infty^{m_2 \times m_2}$*. Assume* $\Pi_{11}(j\omega) > 0$ *and* $\Pi_{22}(j\omega) < 0$ *for all* $\omega \in \mathbb{R} \cup \{\infty\}$*. Then* Π *has a* J_{m_1,m_2}*-spectral factorization.*

Proof: The sign definite conditions on Π_{11} and Π_{22} can be used to show that Π has no equalizing vectors (as defined in [29]) and hence the corresponding ARE has a unique stabilizing solution (Theorem 2.4 in [29]). Details are given in [34]. ■