Exponential Decay Rate Conditions for Uncertain Linear Systems Using Integral Quadratic Constraints

Bin Hu and Peter Seiler

Abstract—**This paper develops linear matrix inequality (LMI) conditions to test whether an uncertain linear system is exponentially stable with a given decay rate α. These new α-exponential stability tests are derived for an uncertain system described by an interconnection of a nominal linear time-invariant system and a "troublesome" perturbation. The perturbation can contain uncertain parameters, time delays, or nonlinearities. This paper presents two key contributions. First, α-exponential stability of the uncertain LTI system is shown to be equivalent to (internal) linear stability of a related scaled system. This enables derivation of α-exponential stability tests from linear stability tests using integral quadratic constraints (IQCs). This connection requires IQCs to be constructed for a scaled perturbation operator. The second contribution is a list of IQCs derived for the scaled perturbation using the detailed structure of the original perturbation. Finally, connections between the proposed approach and related work are discussed.**

I. INTRODUCTION

This paper presents a unified framework to test whether an uncertain linear system is exponentially stable with a specifically given decay rate α. The uncertain system is described as an interconnection of a nominal linear time-invariant (LTI) system and a "troublesome" perturbation. The perturbation considered in this paper can be uncertain parameters, delays, or nonlinearities. The exponential convergence rate is an important metric quantifying the performance of a controller designed to regulate an uncertain system [4]. Hence, the results in this paper can be used to assess controller performance. Moreover, many optimization algorithms can be viewed as uncertain linear systems [11]. The results in this paper can be tailored for the convergence rate analysis of these optimization algorithms.

Integral quadratic constraints (IQCs) provide a general framework for robust analysis of uncertain systems [13]. The IQC theory developed in [13] addresses input-output stability of uncertain LTI systems based on frequency domain inequalities. Related stability theorems have also been formulated using time domain dissipation inequality techniques [18], [17]. Input-output stability implies exponential stability for LTI systems [12]. Hence the existing IQC theory can be used to prove exponential stability of an uncertain LTI system. Similarly, Popov IQCs have been used to show exponential stability of nonlinear systems [7]. These type of results prove existence of exponential convergence but do not provide an accurate estimate/bound for the convergence rate.

There are several related exponential decay rate conditions in the literature. Most existing α-exponential stability tests lead to computable conditions in the form of (convex) generalized eigenvalue problems (GEVPs) [2]. An early result of this nature is obtained for sector-bounded nonlinearities [4]. Recently, the IQC framework has been modified to formulate GEVP-type α-exponential stability tests for discrete-time systems [11], [1]. In [11], an α-exponential stability test formulated using a time domain dissipation inequality has been used to analyze a class of first-order optimization algorithms. In [1], the standard frequency domain stability theorem in [13] has been extended to a GEVP for estimating exponential rates of uncertain LTI systems. The framework in [1] connects the standard IQC setup in [13] to α-exponential stability tests. The resultant conditions may also be numerically solved by frequency domain gridding. The work in [11], [1] relies on the constructions of α-IQCs[1] (defined in Section V) for the perturbation operator. An α-IQC Zames-Falb multiplier has been successfully constructed for nonlinear perturbations.

Two key contributions are made in this paper. First, α-exponential stability of the uncertain LTI system is shown to be equivalent to (internal) linear stability of a related scaled system (Section III). This leads to the α-exponential stability tests presented in Theorem 3. The proposed α-exponential stability tests require (standard) IQCs[2] to be constructed for a related scaled perturbation operator. The second contribution is that a library of IQCs for this scaled perturbation operator is derived in Section IV. Section V discusses the connections between the proposed framework and the α-IQC approach. This paper focuses on uncertain LTI systems. However, similar to the IQC extension for linear parameter varying (LPV) systems [15], [16], the derivation procedures in this paper rely on time domain arguments and can be easily extended to other uncertain linear systems.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. Notation

The projection operator P_T maps any function u as follows: $(P_T u)(t) = u(t)$ for $t \le T$ and $(P_T u)(t) = 0$ otherwise. The extended space, denoted L_{2e}, is the set of functions v such that $P_T v \in L_2$ for all $T \ge 0$. An operator $F : L_{2e} \to L_{2e}$ is causal if $P_T F = P_T F P_T$. An operator is bounded if it has a finite L_2 gain. The condition number of matrix P is denoted as $\text{cond}(P)$.

B. Problem Statement

This paper considers the exponential convergence rate analysis for uncertain continuous-time LTI systems. As shown in Figure 1, the uncertain system $F_u(G, \Delta)$ is described by the interconnection of an LTI system G and an uncertain perturbation Δ. G is described by the following state-space model:

$$\begin{aligned} \dot{x}_G(t) &= A_G\, x_G(t) + B_G w(t) \\ v(t) &= C_G\, x_G(t) + D_G w(t) \end{aligned} \tag{1}$$

where $x_G \in \mathbb{R}^{n_G}$, $w \in \mathbb{R}^{n_w}$, and $v \in \mathbb{R}^{n_v}$. The perturbation $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ is a causal operator. More specifically, the perturbation considered in this paper can be a block diagonal concatenation of uncertain parameters, time delays, and/or nonlinearities. LTI norm bounded uncertainty is not considered since it may lead to an arbitrarily slow convergence rate of $F_u(G, \Delta)$.

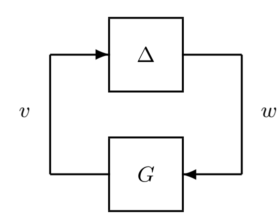

Fig. 1. Interconnection for an Uncertain LTI System

[1]Originally the terminology "ρ-IQC" is used since discrete-time systems are considered. A continuous-time formulation is adopted in this paper, and "α-IQC" will be used.

[2]The framework in this paper relies on the construction of "IQCs" for a scaled perturbation operator, while the original work in [11], [1] makes use of the "α-IQCs" for the original perturbation operator.

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Definition 1. *The interconnection $F_u(G,\Delta)$ is well-posed if for each $x_G(0) \in \mathbb{R}^{n_G}$ there exists a unique solution $x_G \in L_{2e}^{n_G}$, $v \in L_{2e}^{n_v}$ and $w \in L_{2e}^{n_w}$ satisfying Equation (1) and $w = \Delta(v)$.*

Definition 2. *$F_u(G,\Delta)$ is exponentially stable with rate α (≥ 0) if it is well-posed and if $\exists\, c \geq 0$ such that $\|x_G(t)\| \leq c e^{-\alpha t}\|x_G(0)\|$, $\forall t \geq 0$.*

The objective of this paper is to derive linear matrix inequality (LMI) conditions to test whether $F_u(G,\Delta)$ is exponentially stable with a given rate α. These conditions are referred to as α-exponential stability tests. These α-exponential stability tests are useful since a bisection algorithm can then be used to find the best (i.e. smallest) exponential rate bound for $F_u(G,\Delta)$.

C. Integral Quadratic Constraints

This section briefly reviews the stability analysis framework provided by integral quadratic constraints (IQCs) [13], [17]. The key idea is to replace the troublesome block Δ with quadratic constraints on its inputs and outputs. IQCs can be specified either in the frequency or time domain. The definitions of IQCs are given as follows.

Definition 3. *Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(n_v+n_w)\times(n_v+n_w)}$ be a measurable Hermitian-valued function. A bounded, causal operator $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ satisfies the frequency domain IQC defined by the multiplier Π, if the following inequality holds for all $v \in L_2^{n_v}[0,\infty)$ and $w = \Delta(v)$*

$$\int_{-\infty}^{\infty} \begin{bmatrix} V(j\omega) \\ W(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} V(j\omega) \\ W(j\omega) \end{bmatrix} d\omega \geq 0 \tag{2}$$

where V and W are Fourier transforms of v and w.

Definition 4. *Let Ψ be an $n_z \times (n_v + n_w)$ LTI system, and $M = M^T \in \mathbb{R}^{n_z \times n_z}$. A causal operator $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ satisfies the time domain IQC defined by (Ψ, M), if the following inequality holds for all $v \in L_{2e}^{n_v}[0,\infty)$, $w = \Delta(v)$ and $T \geq 0$*

$$\int_0^T z^T(t) M z(t)\, dt \geq 0 \tag{3}$$

where z is the output of Ψ driven by inputs (v, w) with zero initial conditions.

Definition 4 does not require Δ to be bounded, although in many cases the IQCs specified on Δ will imply its boundedness. Input-output stability theorems can be formulated using either frequency domain IQCs [13] or time domain IQCs [18], [17]. A library of frequency domain IQCs for different bounded perturbation operators was summarized in [13]. Additional frequency domain IQCs have been developed for time delay [5], [9], [15] and nonlinearities [6].

Time domain IQCs have been applied to various types of systems, e.g. LPV systems [16], nonlinear systems [15], and stochastic systems [14]. However, constructing time domain IQCs is a nontrivial issue. A systematic approach is the J-spectral factorization method [17]. This approach can be used to factorize frequency domain IQCs into time domain IQCs for bounded Δ under mild technical conditions. There is also some work on directly deriving time domain IQCs without involving frequency domain IQCs, e.g. discrete-time Zames-Falb IQCs for gradients of strongly convex functions [11].

One related result is reviewed here to demonstrate the application of IQCs. In Section III, this result will be extended to an LMI condition for α-exponential stability. The following concept is required.

Definition 5. *The interconnection $F_u(G,\Delta)$ is linearly stable[3] if it is well-posed and if $\exists\, c \geq 0$ such that $\|x_G(t)\| \leq c\|x_G(0)\|$, $\forall t \geq 0$.*

[3] Linear stability is a special case of the so-called global uniform stability [10, Lemma 4.5] when the required class $\mathcal{K}$ function is a linear function.

In the traditional IQC setup [13], the problem formulation and the concept of well-posedness are related to two exogenous inputs for the purpose of input-output stability analysis. This paper relies on internal linear stability analysis. The exogenous inputs are dropped from the problem formulation and the definition of well-posedness. An LMI condition for linear stability of $F_u(G,\Delta)$ can be formulated using time domain IQCs as follows. The uncertainty Δ is assumed to satisfy multiple time domain IQCs defined by $\{(\Psi_k, M_k)\}_{k=1}^N$. All $\{\Psi_k\}_{k=1}^N$ are first aggregated into the following single filter Ψ:

$$\begin{bmatrix} \dot\psi(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A_\Psi & B_{\Psi 1} & B_{\Psi 2} \\ C_\Psi & D_{\Psi 1} & D_{\Psi 2} \end{bmatrix} \begin{bmatrix} \psi(t) \\ v(t) \\ w(t) \end{bmatrix} \tag{4}$$

where $z := [z_1^T \dots z_N^T]^T$ and z_k is the output of Ψ_k. The linear stability analysis is based on the extended system shown in Figure 2.

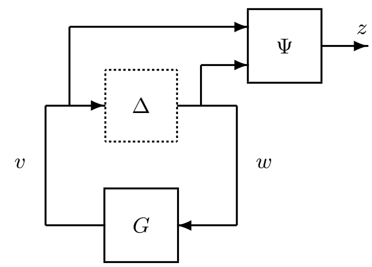

Fig. 2. Uncertain LTI system extended to include filter Ψ

Define a map $H(G,\Psi)$ which maps G and Ψ to the extended system governed by the following state space model:

$$\begin{bmatrix} \dot x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \tag{5}$$

The extended state vector is $x := \left[\begin{smallmatrix} x_G \\ \psi \end{smallmatrix}\right] \in \mathbb{R}^{n_G+n_\psi}$. The state matrices for the extended system $H(G,\Psi)$ are determined by:

$$\mathcal{A} := \begin{bmatrix} A_G & 0 \\ B_{\Psi 1} C_G & A_\Psi \end{bmatrix}, \qquad \mathcal{B} := \begin{bmatrix} B_G \\ B_{\Psi 1} D_G + B_{\Psi 2} \end{bmatrix} \tag{6}$$

$$\mathcal{C} := \begin{bmatrix} D_{\Psi 1} C_G & C_\Psi \end{bmatrix}, \qquad \mathcal{D} := D_{\Psi 1} D_G + D_{\Psi 2} \tag{7}$$

$H(G,\Psi)$ is a specific state-space realization for the system $\Psi \left[\begin{smallmatrix} G \\ I \end{smallmatrix}\right]$. This specific realization is used below to prove linear stability with respect to the states of G. Define $M_\lambda := diag(\lambda_1 M_1, \dots, \lambda_N M_N)$, where the "$diag$" notation means block diagonal concatenation. The next theorem presents an LMI condition for linear stability of $F_u(G,\Delta)$ using time domain IQCs and a dissipation inequality. This theorem uses an LMI defined by G and $\{(\Psi_k, M_k)\}_{k=1}^N$:

$$LMI_{(G,\Psi,M)}(P,\lambda) := \begin{bmatrix} \mathcal{A}^T P + P\mathcal{A} & P\mathcal{B} \\ \mathcal{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{C}^T \\ \mathcal{D}^T \end{bmatrix} M_\lambda \begin{bmatrix} \mathcal{C} & \mathcal{D} \end{bmatrix}$$

Theorem 1. *Let G be a LTI system defined by (1) and $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ be a causal operator such that $F_u(G,\Delta)$ is well-posed. Assume Δ satisfies the time domain IQCs defined by $\{(\Psi_i, M_i)\}_{i=1}^N$. If one of the following conditions holds*

(a) $\exists$ a matrix $P = P^T > 0$ and scalars $\lambda_i \geq 0$ such that $LMI_{(G,\Psi,M)}(P,\lambda) \leq 0$.
(b) $\exists$ a matrix $P = P^T \geq 0$ and scalars $\lambda_i \geq 0$ such that $LMI_{(G,\Psi,M)}(P,\lambda) < 0$.

Then $F_u(G,\Delta)$ is linearly stable.

Proof: Assume Condition (a) holds. Define a storage function by $V(x) = x^T P x$. Left and right multiply $LMI_{(G,\Psi,M)}(P,\lambda) \leq 0$ by $[x^T, w^T]$ and $[x^T, w^T]^T$ to show that V satisfies:

$$\dot V(x(t)) + \sum_{i=1}^N \lambda_i z_i(t)^T M_i z_i(t) \leq 0 \tag{8}$$

This dissipation inequality can be integrated from $t = 0$ to $t = T$ with initial condition $x(0) = \left[\begin{smallmatrix} x_G(0) \\ 0 \end{smallmatrix}\right]$ to yield:

$$V\left(x(T)\right) + \sum_{i=1}^{N} \lambda_i \int_0^T z_i(t)^T M_i z_i(t)\, dt \le V(x(0)) \qquad (9)$$

Applying the time domain IQC conditions with the fact $\lambda_i \geq 0$, one can get $V\left(x(T)\right) \leq V(x(0))$. Therefore, $\|x_G(T)\|^2 \leq \|x(T)\|^2 \leq \text{cond}(P)\|x(0)\|^2 = \text{cond}(P)\|x_G(0)\|^2$. Thus $\|x_G(T)\| \leq \sqrt{\text{cond}(P)}\|x_G(0)\|$ and $F_u(G, \Delta)$ is linearly stable.

Now assume Condition (b) holds. Since $LMI_{(G,\Psi,M)}(P, \lambda) < 0$, $\exists\ \epsilon > 0$ such that $LMI_{(G,\Psi,M)}(P + \epsilon I, \lambda) \leq 0$. Linear stability follows from Condition (a) due to the fact $P + \epsilon I > 0$. ■

The dissipation inequality approach presented above relies on the fact that the constraint in (3) holds for any finite-horizon $T \geq 0$. It does not require either G or Ψ to be stable. It only requires that the states of $H(G, \Psi)$ have no finite escape time. Hence, Definition 4 does not enforce the stability of Ψ. In principle, one can use time domain IQCs with unstable Ψ, although the J-spectral factorization of any frequency domain IQC always leads to stable Ψ.

Many other linear stability conditions can be derived. For example, given stable G and bounded Δ, one can drop the constraint $P \geq 0$ in Condition (b) of the above theorem [17]. In addition, the conic combination can be extended to more general IQC parameterizations where M_λ is an affine function of λ [19]. This leads to less conservative factorization conditions for the combined multiplier [16]. Some alternative procedures (ν-gap metric theory in [3], dissipation inequality in [18], etc) are also available for deriving stability tests.

III. α-Exponential Stability Tests

This section establishes the connections between linear stability and α-exponential stability. The connections are built upon a specific loop transformation, as shown in Figure 3. For any fixed α, define the scaling operator $\mathcal{S}_{\alpha-} : L_{2e}^{n_v}[0, \infty) \rightarrow L_{2e}^{n_v}[0, \infty)$ that maps v_α to $v = \mathcal{S}_{\alpha-} v_\alpha$ as follows: $v(t) := e^{-\alpha t} v_\alpha(t)$. Similarly, define another scaling operator $\mathcal{S}_{\alpha+} : L_{2e}^{n_w}[0, \infty) \rightarrow L_{2e}^{n_w}[0, \infty)$ that maps w to $w_\alpha = \mathcal{S}_{\alpha+} w$ by setting $w_\alpha(t) := e^{\alpha t} w(t)$. $\mathcal{S}_{\alpha-}$ and $\mathcal{S}_{\alpha+}$ have well-defined inverse operators denoted by $\mathcal{S}_{\alpha-}^{-1}$ and $\mathcal{S}_{\alpha+}^{-1}$, respectively. Notice $\mathcal{S}_{\alpha-}^{-1} = \mathcal{S}_{\alpha+}$ and $\mathcal{S}_{\alpha+}^{-1} = \mathcal{S}_{\alpha-}$ if and only if $n_v = n_w$. The connections between $F_u(G, \Delta)$ and $F_u(\mathcal{S}_{\alpha-}^{-1} G \mathcal{S}_{\alpha+}^{-1}, \mathcal{S}_{\alpha+} \Delta \mathcal{S}_{\alpha-})$ are important for the results in this paper. A similar loop transformation has been used in [1], which defines the scaled plant $\mathcal{S}_{\alpha-}^{-1} G \mathcal{S}_{\alpha+}^{-1}$ in the frequency domain and relates the α-exponential stability of $F_u(G, \Delta)$ to the input-output stability of $F_u(\mathcal{S}_{\alpha-}^{-1} G \mathcal{S}_{\alpha+}^{-1}, \mathcal{S}_{\alpha+} \Delta \mathcal{S}_{\alpha-})$. The frequency domain scaling of G only focuses on the input-output relationship. This paper relates the α-exponential stability of $F_u(G, \Delta)$ to the (internal) linear stability of $F_u(\mathcal{S}_{\alpha-}^{-1} G \mathcal{S}_{\alpha+}^{-1}, \mathcal{S}_{\alpha+} \Delta \mathcal{S}_{\alpha-})$. This requires a specific time domain state space definition for $\mathcal{S}_{\alpha-}^{-1} G \mathcal{S}_{\alpha+}^{-1}$, which leads to useful relationships between the states of the original and transformed interconnections.

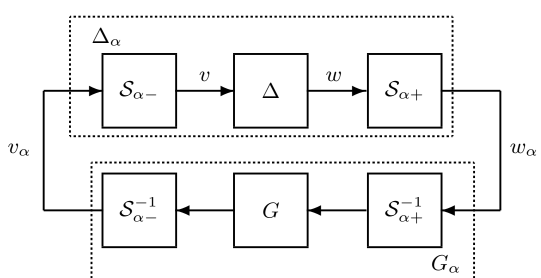

Fig. 3. Transformed Interconnection

Define the scaled systems $G_\alpha := \mathcal{S}_{\alpha-}^{-1} G \mathcal{S}_{\alpha+}^{-1}$ and $\Delta_\alpha := \mathcal{S}_{\alpha+} \Delta \mathcal{S}_{\alpha-}$. These are input/ouput definitions for the scaled systems. A specific, state-space realization for G_α can be obtained from the realization for G in Equation (1). Define $x_{G,\alpha}(t) := e^{\alpha t} x_G(t)$. A state-space realization for G_α is then given by:

$$\begin{aligned} \dot{x}_{G,\alpha}(t) &= (A_G + \alpha I)\, x_{G,\alpha}(t) + B_G w_\alpha(t) \\ v_\alpha(t) &= C_G\, x_{G,\alpha}(t) + D_G w_\alpha(t) \end{aligned} \qquad (10)$$

As a slight abuse of notation, the scaled system G_α will always refer to this specific linear time-invariant realization. The main loop transformation result is now stated.

Theorem 2. *$F_u(G, \Delta)$ is well-posed if and only if $F_u(G_\alpha, \Delta_\alpha)$ is well-posed. Moreover, $F_u(G, \Delta)$ is exponentially stable with rate α if and only if $F_u(G_\alpha, \Delta_\alpha)$ is linearly stable.*

Proof: It is straightforward to prove that $x_G \in L_{2e}^{n_G}$, $v \in L_{2e}^{n_v}$, and $w \in L_{2e}^{n_w}$ is a solution for Equation (1) and $w = \Delta(v)$ with initial condition $x_G(0) \in \mathbb{R}^{n_G}$ if and only if ($x_G(t) e^{\alpha t}$, $v(t) e^{\alpha t}$, $w(t) e^{\alpha t}$) provides an L_{2e} solution for Equation (10) and $w_\alpha = \Delta_\alpha(v_\alpha)$ with initial condition $x_{G,\alpha}(0) = x_G(0)$. Therefore, $F_u(G, \Delta)$ is well-posed if and only if $F_u(G_\alpha, \Delta_\alpha)$ is well-posed. Next suppose $F_u(G, \Delta)$ and $F_u(G_\alpha, \Delta_\alpha)$ are well-posed and have the same initial condition $x_G(0) = x_{G,\alpha}(0)$. The following holds

$$\begin{aligned} x_{G,\alpha}(t) &= e^{\alpha t} x_G(t) \\ w_\alpha(t) &= e^{\alpha t} w(t) \\ v_\alpha(t) &= e^{\alpha t} v(t) \end{aligned} \qquad (11)$$

where (x_G, v, w) and ($x_{G,\alpha}$, v_α, w_α) are the resultant L_{2e} solutions for $F_u(G, \Delta)$ and $F_u(G_\alpha, \Delta_\alpha)$, respectively. Moreover, $\|x_G(t)\| \leq c\|x_G(0)\| e^{-\alpha t} \Leftrightarrow \|x_{G,\alpha}(t)\| \leq c\|x_{G,\alpha}(0)\|$. Therefore, $F_u(G, \Delta)$ is exponentially stable with rate α if and only if $F_u(G_\alpha, \Delta_\alpha)$ is linearly stable. ■

Remark 1. *Proposition 5 in [1] states that input-output stability of the transformed loop is a sufficient condition for α -exponential stability of the original loop. Theorem 2 here states that linear stability of the transformed loop is a necessary and sufficient condition for α-exponential stability of the original loop.*

Theorem 2 states that a linear stability test for $F_u(G_\alpha, \Delta_\alpha)$ is equivalent to an α-exponential stability test for $F_u(G, \Delta)$. Thus LMI conditions formulated for linear stability of the scaled interconnection can be used to demonstrate α-exponential stability of the original loop. This approach requires IQCs to be specified for Δ_α. Most existing work on IQCs specifies multipliers for the unscaled operator Δ. A main contribution of this paper is that a library of IQC multipliers for Δ_α is derived in Section IV for a large class of perturbations. Note that application of Theorem 2 to the scaled system also requires the perturbation Δ_α to be causal. It is easily shown that causality of Δ_α is equivalent to causality of Δ. This follows because $\mathcal{S}_{\alpha-}$ and $\mathcal{S}_{\alpha+}$ are memoryless, pointwise-in-time multiplication operators. The frequency domain construction of IQC multipliers for Δ_α requires its boundedness, which is not as straightforward. Notice $\mathcal{S}_{\alpha+}$ is an unbounded operator. It is possible for a bounded operator Δ to yield an unbounded scaled operator Δ_α. The boundedness of Δ_α needs to be proven for each specific Δ. This issue is addressed in Section IV. Theorem 2 does not require G to be controllable or observable. The time domain scaling used in Theorem 2 can be extended to uncertain LPV systems or uncertain linear Markovian jump systems.

An LMI condition for α-exponential stability of $F_u(G, \Delta)$ is now formulated using the loop transformation result in Theorem 2. The scaled perturbation Δ_α is assumed to satisfy multiple time domain IQCs defined by $\{(\Phi_i, L_i)\}_{i=1}^N$. The construction of these IQCs will be discussed in Section IV. The analysis is based on the extended system $H(G_\alpha, \Phi)$ and the $LMI_{(G_\alpha,\Phi,L)}$ defined previously.

Theorem 3. *Let G be an $n_v \times n_w$ LTI system defined by (1) and $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ be a causal operator such that $F_u(G,\Delta)$ is well-posed. Assume Δ_α satisfies the time domain IQCs defined by $\{(\Phi_i, L_i)\}_{i=1}^N$. If one of the following conditions holds*

1) *$\exists$ a matrix $P = P^T > 0$ and scalars $\lambda_i \geq 0$ such that $LMI_{(G_\alpha,\Phi,L)}(P,\lambda) \leq 0$*
2) *$\exists$ a matrix $P = P^T \geq 0$ and scalars $\lambda_i \geq 0$ such that $LMI_{(G_\alpha,\Phi,L)}(P,\lambda) < 0$*

then $F_u(G,\Delta)$ is exponentially stable with rate α.

Proof: By Theorem 2, the well-posedness of $F_u(G,\Delta)$ implies that $F_u(G_\alpha,\Delta_\alpha)$ is well-posed. Moreover, causality of Δ implies causality of Δ_α. Clearly, Δ_α maps L_{2e} signals to L_{2e} signals. It follows from Theorem 1 that $F_u(G_\alpha,\Delta_\alpha)$ is linearly stable. Based on Theorem 2, $F_u(G,\Delta)$ is exponentially stable with rate α. ■

Theorem 3 demonstrates the utility of Theorem 2 using the linear stability test in Theorem 1. Similarly, the other linear stability tests mentioned after Theorem 1 can also be used to formulate α - exponential stability tests based on Theorem 2.

IV. Boundedness and IQCs for Scaled Perturbation

This section checks the boundedness and provides a list of IQCs for the scaled perturbation Δ_α. The results are developed for several important types of (unscaled) components Δ. A notable absence is the case where Δ is an LTI norm-bounded uncertainty. This uncertainty class is problematic for exponential convergence analysis, e.g. Δ may lead to an arbitrarily slow convergence rate of $F_u(G,\Delta)$. Finally, most IQCs developed in this section are specified as frequency domain multipliers. If the boundedness of Δ_α is checked then the J-spectral factorization results in [17], [16] can be used to construct corresponding time-domain IQCs. The J-spectral factorizations of some multipliers require the following perturbation argument. First, Δ_α will be proved to satisfy $\|\Delta_\alpha\| \leq \gamma$ for some $\gamma > 0$. Hence, Δ_α satisfies the multiplier $\Pi_0 = \begin{bmatrix} \gamma^2 & 0 \\ 0 & -1 \end{bmatrix}$. Any multiplier Π with a positive semidefinite upper left block and a negative semidefinite lower right block can be perturbed to $\Pi + \epsilon\Pi_0$ for sufficiently small $\epsilon > 0$. The perturbed multipliers satisfy the conditions required to construct J-spectral factorizations.

A. Multiplication with an Uncertain Parameter

A large class of uncertainties Δ have a multiplicative form $(\Delta v)(t) = \delta(t)v(t)$, where $\delta(t)$ is the uncertain source term. Some examples of δ include, but are not limited to:

- Constant real scalar: $\delta \in \mathbb{R}$
- Time-varying real scalar: $\delta(t) \in \mathbb{R}$
- Time-varying real matrix: $\delta(t) \in \mathbb{R}^{n_w \times n_v}$
- Coefficients from a polytope: $\delta(t)$ is a measurable matrix in a polytope of matrices with the extremal points $\delta_1, \ldots, \delta_N$
- Periodic real scalar: $\delta(t)$ is a scalar function with period T
- Multiplication by a harmonic oscillation: $\delta(t) = \cos(\omega_0 t)$
- Rate-bounded, time-varying scalar: $\delta(t)$ satisfies $|\dot{\delta}(t)| \leq d$

For all the above cases, Δ and the scaling operator $\mathcal{S}_{\alpha\pm}$ commute: $\Delta\mathcal{S}_{\alpha\pm} = \mathcal{S}_{\alpha\pm}\Delta$. Therefore, the scaling relationship directly leads to $w_\alpha(t) = \delta(t)v_\alpha(t)$, and $\Delta_\alpha = \Delta$. The boundedness of Δ guarantees that Δ_α is a bounded operator, and any IQCs on Δ are directly IQCs on Δ_α. The frequency domain IQCs on Δ are well documented in [13, Section VI]. All these frequency domain IQCs can be directly applied to describe the input/output behavior of Δ_α.

B. Uncertain Delay

An uncertain (constant) delay Δ is defined as $(\Delta v)(t) = 0$ for $t < \tau$ and $(\Delta v)(t) = v(t-\tau)$ for $t \geq \tau$, where $\tau \in [0,\tau_0]$. When $t \geq \tau$, one can use the scaling relationship to get:

$$w_\alpha(t) = w(t)e^{\alpha t} = v(t-\tau)e^{\alpha t} = v_\alpha(t-\tau)e^{\alpha\tau} \qquad (12)$$

When $t < \tau$, one trivially gets $w_\alpha(t) = 0$. Therefore, $(\Delta_\alpha v_\alpha)(t) = 0$ for $t < \tau$ and $(\Delta_\alpha v_\alpha)(t) = v_\alpha(t-\tau)e^{\alpha\tau}$ for $t \geq \tau$. It is straightforward to verify that Δ_α is bounded and $\|\Delta_\alpha\| \leq e^{\alpha\tau_0}$.

Δ_α is the product of the original delay Δ and a constant uncertain real scalar $\delta = e^{\alpha\tau}$. The scaled system $F_u(G_\alpha,\Delta_\alpha)$ can be transformed into a system with block diagonal uncertainty $diag(\Delta,\delta)$. This discussion can be extended to the case with time-varying delays $(\tau(t))$. There exist standard IQCs for time delays Δ [13], [5], [9], [15] and uncertain real parameters [13]. This approach decouples Δ_α into two operators and constructs separate IQCs for Δ and δ. It is also possible to derive new IQCs directly for Δ_α. When τ is an uncertain (constant) delay then Δ_α satisfies any multiplier of the form:

$$\begin{bmatrix} e^{2\alpha\tau_0}X(j\omega) & 0 \\ 0 & -X(j\omega) \end{bmatrix}$$

where $X(j\omega) \geq 0$ is a bounded, measurable function of ω. A similar multiplier holds for Δ_α when τ is time-varying but with $X(j\omega)$ replaced by the constant matrix $X = X^T \geq 0$. Other existing multipliers for time delays (e.g. Section VI.H in [13]) can be extended to directly develop multipliers for the scaled delay $\Delta_\alpha = \Delta e^{\alpha\tau}$.

C. Memoryless Nonlinearity in a Sector

If $(\Delta v)(t) = \phi(v(t),t)$ and ϕ is in a sector: $\beta_1 v^2 \leq \phi(v(t),t)v \leq \beta_2 v^2$, then $(\Delta_\alpha v_\alpha)(t) = e^{\alpha t}\phi(v_\alpha(t)e^{-\alpha t},t)$. One can check that $\beta_1 v_\alpha^2 \leq e^{\alpha t}\phi_\alpha\left(v_\alpha(t)e^{-\alpha t},t\right)v_\alpha \leq \beta_2 v_\alpha^2$. Therefore, Δ_α is a bounded operator and $\|\Delta_\alpha\| \leq \max(|\beta_1|,|\beta_2|)$. Moreover, Δ_α satisfies the IQC defined by the multiplier $\begin{bmatrix} -2\beta_1\beta_2 & \beta_1+\beta_2 \\ \beta_1+\beta_2 & -2 \end{bmatrix}$.

D. Static Nonlinearity

If $(\Delta v)(t) = \phi(v(t))$, where ϕ is a continuous function, then $(\Delta_\alpha v_\alpha)(t) = e^{\alpha t}\phi(e^{-\alpha t}v_\alpha(t))$. It is assumed that ϕ lies within a sector $[\beta_1,\beta_2]$ for finite β_1 and β_2. Hence Δ_α is bounded based on Section IV-C. Clearly, the multiplier in Section IV-C can be applied to Δ_α. Under certain circumstances, two other sets of IQCs can also be used. First, a Popov IQC will be presented. The following lemma modifies the procedure in [8, Example 1] for the scaled operator Δ_α.

Lemma 1 (Popov IQC)**.** *Assume $\phi : \mathbb{R} \to \mathbb{R}$ satisfies $\phi(0) = 0$ and $\phi(z)z \geq 0\ \forall z \in \mathbb{R}$. In addition, assume ϕ lies in a finite sector so that $w_\alpha = \Delta_\alpha(v_\alpha)$ (as defined above) is bounded. If $\alpha \geq 0$, then*

$$\int_0^\infty (\alpha v_\alpha(t) - \dot{v}_\alpha(t))w_\alpha(t)dt \geq 0, \forall v_\alpha, \dot{v}_\alpha \in L_2^{n_v}[0,\infty) \qquad (13)$$

Hence Δ_α satisfies the IQC defined by $\begin{bmatrix} 0 & \alpha+j\omega \\ \alpha-j\omega & 0 \end{bmatrix}$.

Proof: First notice:

$$-\int_0^\infty (\alpha v_\alpha(t) - \dot{v}_\alpha(t))\,w_\alpha(t)dt = \int_0^\infty e^{2\alpha t}\dot{v}(t)\phi(v(t))dt \quad (14)$$

Boundedness of Δ_α implies that $w_\alpha \in L_2^{n_w}$. It follows from Cauchy-Schwartz inequality that the integral on the left side (and hence also the right side) is finite. Since $e^{2\alpha t} = 1 + \int_0^t 2\alpha e^{2\alpha t_0}dt_0$, the right side of (14) can be manipulated as:

$$\begin{aligned} \int_0^\infty e^{2\alpha t}\dot{v}(t)\phi(v(t))dt &= \int_0^\infty \dot{v}(t)\phi(v(t))dt \\ &+ \int_0^\infty \left(\int_0^t 2\alpha e^{2\alpha t_0}dt_0\right)\dot{v}(t)\phi(v(t))dt \end{aligned} \qquad (15)$$

The double integral (second term on right side) is finite. This follows because both the left side and the first term on the right are finite integrals. Hence Fubini's theorem can be used to swap the double integral so that the right side of (15) can be expressed as:

$$\int_0^\infty \dot{v}(t)\phi(v(t))dt + \int_0^\infty 2\alpha e^{2\alpha t_0}\left(\int_{t_0}^\infty \dot{v}(t)\phi(v(t))dt\right)dt_0$$

Notice that $v_\alpha, \dot{v}_\alpha \in L_2^{n_v}[0,\infty)$ implies $\lim_{T\to\infty} v_\alpha(T) = 0$ [8, Lemma 1]. Hence, $\lim_{T\to\infty} v(T) = 0$ and the following holds:

$$\int_{t_0}^\infty \dot{v}(t)\phi(v(t))dt = \lim_{T\to\infty}\int_{v(t_0)}^{v(T)} \phi(\sigma)d\sigma = -\int_0^{v(t_0)} \phi(\sigma)d\sigma \le 0$$

Therefore, $\int_0^\infty e^{2\alpha t}\dot{v}(t)\phi(v(t))dt \le 0$. Based on (14), one can see $\int_0^\infty (\alpha v_\alpha(t)w_\alpha(t) - \dot{v}_\alpha(t)w_\alpha(t))\,dt \ge 0$. From Parseval's theorem, Δ_α satisfies the IQC defined by the multiplier $\left[\begin{smallmatrix} 0 & \alpha+j\omega \\ \alpha-j\omega & 0\end{smallmatrix}\right]$. ■

Remark 2. *The Popov IQC requires $\dot{v}_\alpha \in L_2$. This is satisfied when G_α is stable and $D_G = 0$. The application of the Popov IQC requires careful justifications on the properties of $\dot{v}_\alpha$, e.g. see examples in [8]. Moreover, the multiplier given in Lemma 1 is not bounded on the imaginary axis. This issue is usually fixed by a loop transformation.*

Finally, the Zames-Falb IQCs [20], [6] will be constructed for the operator Δ_α. The scalar nonlinearity ϕ is bounded and monotone nondecreasing if $\phi(0) = 0$, $[\phi(y_1) - \phi(y_2)](y_1 - y_2) \ge 0$, and $\|\phi(y_1)\| \le c\|y_1\|$ for some c and all y_1, y_2.

Lemma 2 (Zames-Falb IQCs)**.** *Assume $\phi : \mathbb{R} \to \mathbb{R}$ is bounded and monotone nondecreasing so that Δ_α is bounded. Let $f \in L_1$ satisfy $\int_0^\infty |f(t)|e^{\alpha t}dt \le 1$ for some $\alpha \ge 0$ and $f(t) \ge 0$ for all t. Then Δ_α satisfies the multiplier $\left[\begin{smallmatrix} 0 & 1-F^* \\ 1-F & 0\end{smallmatrix}\right]$ where F denotes the Laplace transform of f.*

Proof: Let $v_\alpha \in L_2$. It suffices to show that

$$\int_0^\infty w_\alpha(t)(f * v_\alpha)(t)dt \le \int_0^\infty w_\alpha(t)v_\alpha(t)dt \qquad (16)$$

Since $w_\alpha(t) = w(t)e^{\alpha t}$ and $v_\alpha(t) = v(t)e^{\alpha t}$, one can directly check that (16) is equivalent to

$$\int_0^\infty e^{2\alpha t}w(t)(g * v)(t)dt \le \int_0^\infty e^{2\alpha t}w(t)v(t)dt \qquad (17)$$

where $g(\tau) := e^{-\alpha\tau}f(\tau) \in L_1$. Since $v_\alpha \in L_2$ and Δ_α is bounded, one has $w_\alpha \in L_2$. Moreover $f \in L_1$ implies $f * v_\alpha \in L_2$. It follows from Cauchy-Schwartz inequality that the left side of (16) (and hence the left side of (17)) is finite. Hence, Fubini's Theorem can be used to rewrite the left side of (17) as

$$\begin{aligned}&\int_0^\infty\int_0^\infty e^{2\alpha t}w(t)g(\tau)v(t-\tau)d\tau dt\\ &= \int_0^\infty e^{2\alpha\tau}g(\tau)\left(\int_0^\infty e^{2\alpha(t-\tau)}w(t)v(t-\tau)dt\right)d\tau\end{aligned} \qquad (18)$$

Since $v_\alpha, w_\alpha \in L_2$, Statement (2) of Lemma 4 in the appendix can be directly applied to show the first inequality below:

$$\begin{aligned}&\int_0^\infty e^{2\alpha t}w(t)(g * v)(t)dt\\ &\le \left(\int_0^\infty e^{2\alpha\tau}g(\tau)d\tau\right)\left(\int_0^\infty e^{2\alpha t}w(t)v(t)dt\right)\\ &\le \int_0^\infty e^{2\alpha t}w(t)v(t)dt\end{aligned} \qquad (19)$$

The second inequality follows from the definition of g and the assumptions on f. Thus (17) holds. This completes the proof. ■

Remark 3. *Following the procedure in [6], the above result can be extended to odd or slope-restricted or multi-input multi-output nonlinearities. Another important related result is the discrete-time α-IQC construction of Zames-Falb multipliers [1].*

V. Related Work

This section discusses connections to the results in [11]. The time domain α-IQC introduced in [11] is defined as follows.

Definition 6. *Let Ψ be an $n_z \times (n_v + n_w)$ LTI system, and $M = M^T \in \mathbb{R}^{n_z\times n_z}$. A causal operator $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ satisfies the time domain α-IQC defined by (Ψ, M), if the following inequality holds for all $v \in L_{2e}^{n_v}[0,\infty)$, $w = \Delta(v)$ and $T \ge 0$*

$$\int_0^T e^{2\alpha t}z^T(t)Mz(t)\,dt \ge 0 \qquad (20)$$

where z is the output of Ψ driven by inputs (v, w) with zero initial conditions.

Suppose the uncertainty Δ satisfies multiple time domain α-IQCs defined by $\{(\Psi_k, M_k)\}_{k=1}^N$. All $\{\Psi_k\}_{k=1}^N$ are aggregated into a filter Ψ governed by Equation (4). Let $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ denote the state space realization of $H(G, \Psi)$. A trivial extension of [11, Theorem 4] from discrete to continuous time yields the following result:

Theorem 4. *Let G be a LTI system defined by (1) and $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ be a causal operator such that $F_u(G,\Delta)$ is well-posed. Assume Δ satisfies the time domain α-IQCs defined by $\{(\Psi_i, M_i)\}_{i=1}^N$. If one of the following condition holds*

(a) $\exists$ a matrix $P = P^T > 0$ and scalars $\lambda_i \ge 0$ such that

$$\begin{bmatrix}\mathcal{A}^TP + P\mathcal{A} + 2\alpha P & P\mathcal{B}\\ \mathcal{B}^TP & 0\end{bmatrix} + \begin{bmatrix}\mathcal{C}^T\\ \mathcal{D}^T\end{bmatrix} M_\lambda \begin{bmatrix}\mathcal{C} & \mathcal{D}\end{bmatrix} \le 0 \qquad (21)$$

(b) $\exists$ a matrix $P = P^T \ge 0$ and scalars $\lambda_i \ge 0$ such that the left side of the LMI (21) is strictly less than $(<)$ 0.

Then $F_u(G,\Delta)$ is exponentially stable with rate α.

Proof: Assume Condition (a) holds. The discrete-time version of this case has been proved in [11]. The proof is sketched as follows. Set $V(x) = x^TPx$. Left and right multiply LMI (21) by $[x^T, w^T]$ and $[x^T, w^T]^T$ to show that V satisfies: $\frac{d}{dt}\left(e^{2\alpha t}V(x(t))\right) + \sum_{i=1}^N \lambda_i e^{2\alpha t}z_i(t)^TM_iz_i(t) \le 0$. Integrating this inequality from $t = 0$ to $t = T$ with initial condition $x(0) = \left[\begin{smallmatrix} x_G(0)\\ 0\end{smallmatrix}\right]$ and applying the time domain α-IQC conditions yields $V(x(T)) \le V(x(0))\,e^{-2\alpha T}$. Therefore, $\|x_G(t)\| \le \sqrt{\text{cond}(P)}\|x_G(0)\|e^{-\alpha t}$ and $F_u(G,\Delta)$ is exponentially stable with rate α. A perturbation argument can be used to complete the proof when Condition (b) holds. ■

As pointed out in [11, Section 3.1], quadratic constraints that hold pointwise in time, e.g. constraints on sector nonlinearities, lead to time-domain α-IQCs for any $\alpha \ge 0$. The use of time domain α-IQCs is more general than the well-known GEVP formulations using time-domain pointwise quadratic constraints, since one can construct this type of α-IQCs for the gradient of strongly convex functions to analyze the convergence rates of optimization algorithms [11].

The next lemma provides a connection between IQCs for the scaled perturbation Δ_α and α-IQCs for the original perturbation Δ. The lemma statement involves the scaled filter $\Psi_\alpha = \mathcal{S}_{\alpha-}^{-1}\Psi\mathcal{S}_{\alpha+}^{-1}$. As discussed in Section III, Ψ_α will denote the specific LTI state-space realization $(A_\Psi + \alpha I, [\,B_{\Psi1}\ B_{\Psi2}\,], C_\Psi, [\,D_{\Psi1}\ D_{\Psi2}\,])$. Similarly, $H_\alpha(G,\Psi)$ denotes the specific state-space realization for $\mathcal{S}_{\alpha-}^{-1}H(G,\Psi)\mathcal{S}_{\alpha+}^{-1}$ based on shifting the state matrix of $H(G,\Psi)$. The use of $\mathcal{S}_{\alpha\pm}$ here involves a slight abuse of notation because Ψ and $H(G,\Psi)$ have different input/output dimensions than G.

Lemma 3. *Let G be an $n_v \times n_w$ LTI system described by Equation (1). Δ satisfies the time-domain α-IQC defined by (Ψ, M) if and only if Δ_α satisfies the time-domain IQC defined by (Ψ_α, M). Moreover, $H_\alpha(G,\Psi) = H(G_\alpha, \Psi_\alpha)$.*

Proof: The proof follows by simply tracking the various signal definitions. The key of the proof is the following fact. Let z be the output of Ψ driven by (v, w) with zero initial condition. Set $z_\alpha(t) := e^{\alpha t} z(t)$. Then z_α will be the output of Ψ_α driven by (v_α, w_α) with zero initial condition. The details of the proof are omitted. ■

Remark 4. *The frequency domain α-IQCs introduced in [1] can be connected to the frequency domain IQCs on Δ_α in a similar manner.*

Lemma 3 states that Theorem 4 and Theorem 3 are equivalent. Both theorems use time-domain proofs and can be extended to other linear systems which do not have frequency domain interpretations, e.g. LPV systems. Note that both theorems require non-negativity constraints on P. If G_α is stable and Δ_α is bounded then this non-negativity constraint can be dropped in Theorem 3 using the approach in [16, Theorem 2]. This approach constructs a non-negative storage function using additional energy stored in the IQC. It is possible to similarly modify the α-IQC proof in Theorem 4 to drop the non-negativity constraints.

In [11], there are cases where the specified $\{(\Psi_i, M_i)\}_{i=1}^N$ do not depend on α. Then LMI (21) directly leads to a GEVP. Similarly, for the IQCs specified in Sections IV-A and IV-C, the associated multipliers $\{\Pi_i\}_{i=1}^N$ do not depend on α. Then Theorem 3 leads to a GEVP. However, Theorem 4 and Theorem 3 do not always lead to GEVPs. When Δ is an uncertain constant time delay, the multipliers for Δ_α depend on α via an exponential relationship. To find the smallest exponential rate bound for $F_u(G, \Delta)$ in this case, a bisection algorithm is required. At each iteration, a J-spectral factorization has to be performed for the α-dependent multipliers $\{\Pi\}_i^N$. Similarly, when Popov IQCs are used, the resultant LMIs do not lead to GEVPs.

VI. Conclusion

This paper develops a unified IQC-based approach to test the α-exponential stability of uncertain LTI systems. LMI conditions are derived based on connections between α-exponential stability of the original loop and linear stability of an exponentially scaled loop. The proposed approach requires the construction of IQCs for a scaled perturbation operator. A library of IQCs for the scaled perturbation operator is derived for several important types of perturbations.

References

[1] R. Boczar, L. Lessard, and B. Recht. Exponential convergence bounds using integral quadratic constraints. In *IEEE Conf. on Decision and Control*, pages 7516–7521, 2015.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, 1994.

[3] M. Cantoni, U. Jönsson, and S.Z. Khong. Robust stability analysis for feedback interconnections of time-varying linear systems. *SIAM J. of Control Optim.*, 51(1):353–379, 2013.

[4] M. Corless and G. Leitmann. Bounded controllers for robust exponential convergence. *Journal of Optimization Theory and Applications*, 76(1):1–12, 1993.

[5] K. Gu, V. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Birkhauser, 2002.

[6] W. Heath and A. Wills. Zames-Falb multipliers for quadratic programming. In *IEEE Conf. on Decision and Control*, pages 963–968, 2005.

[7] U. Jönsson. A nonlinear Popov criterion. In *IEEE Conf. on Decision and Control*, pages 3523–3527, 1997.

[8] U. Jönsson. Stability analysis with Popov multipliers and integral quadratic constraints. *Systems & Control Letters*, 31(2):85 – 92, 1997.

[9] C. Kao and A. Rantzer. Stability analysis of systems with uncertain time-varying delays. *Automatica*, 43(6):959–970, 2007.

[10] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, third edition, 2001.

[11] L. Lessard, B. Recht, and A. Packard. Analysis and design of optimization algorithms via integral quadratic constraints. *SIAM Journal on Optimization*, 26(1):57–95, 2016.

[12] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints: Part I. Technical Report TFRT-7531-SE, Lund Inst. of Technology, 1995.

[13] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. on Automatic Control*, 42:819–830, 1997.

[14] I. Petersen, V. Ugrinovskii, and A. Savkin. *Robust Control Design Using $\mathcal{H}_\infty$ Methods*. Springer-Verlag, 2000.

[15] H. Pfifer and P. Seiler. Integral quadratic constraints for delayed nonlinear and parameter-varying systems. *Automatica*, 56:36 – 43, 2015.

[16] H. Pfifer and P. Seiler. Less conservative robustness analysis of linear parameter varying systems using integral quadratic constraints. *To appear in International Journal of Robust and Nonlinear Control*, 2016.

[17] P. Seiler. Stability analysis with dissipation inequalities and integral quadratic constraints. *IEEE Trans. on Automatic Control*, 60(6):1704–1709, 2015.

[18] J. Veenman and C. W. Scherer. Stability analysis with integral quadratic constraints: A dissipativity based proof. In *IEEE Conf. on Decision and Control*, pages 3770–3775, 2013.

[19] J. Veenman and C.W. Scherer. IQC-synthesis with general dynamic multipliers. *International Journal of Robust and Nonlinear Control*, 24:3027–3056, 2012.

[20] G. Zames and P.L. Falb. Stability conditions for systems with monotone and slope-restricted nonlinearities. *SIAM Journal of Control*, 6(1):89–108, 1968.

Appendix

Lemma 4. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be bounded and monotone nondecreasing. In addition, let $w(t) = \phi(v(t))$. Then*

1) *For any $v \in L_2$, $\tau \geq 0$, and $t_0 \geq 0$. one has*

$$\int_{t_0}^{\infty} v(t)\left(w(t) - w(t+\tau)\right) dt \geq 0 \tag{22}$$

2) *If $\alpha \geq 0$, $v(t)e^{\alpha t}, w(t)e^{\alpha t} \in L_2$ and $\tau \geq 0$, then*

$$\int_0^{\infty} e^{2\alpha(t-\tau)} w(t)v(t-\tau)dt \leq \int_0^{\infty} e^{2\alpha t} w(t)v(t)dt \tag{23}$$

Proof: To prove Statement (1), first let $t_0 = 0$. It follows from a traditional Zames-Falb result [6, Lemma 1] that:

$$\int_0^{\infty} w(t)v(t)dt \geq \int_0^{\infty} w(t)v(t-\tau)dt \tag{24}$$

The right side can be re-written with a change of variables as:

$$\int_0^{\infty} w(t)v(t)dt \geq \int_0^{\infty} w(t+\tau)v(t)dt \tag{25}$$

This proves (22) for $t_0 = 0$. For $t_0 \geq 0$, set $\tilde{v} := v - P_{t_0} v$. Then

$$\int_0^{\infty} \tilde{v}(t)\left(\tilde{w}(t) - \tilde{w}(t+\tau)\right) dt \geq 0, \forall \tau \geq 0 \tag{26}$$

This proves Statement (1) for $t_0 \geq 0$.

To prove Statement (2), rewrite the left side of (23) with a change of variables as $\int_0^{\infty} e^{2\alpha t} w(t+\tau)v(t)dt$. Thus (23) is equivalent to:

$$\int_0^{\infty} e^{2\alpha t} v(t)\left(w(t) - w(t+\tau)\right) dt \geq 0 \tag{27}$$

The integral is finite (Cauchy-Schwartz), since $v(t)e^{\alpha t}, w(t)e^{\alpha t} \in L_2$. Since $e^{2\alpha t} = 1 + \int_0^t 2\alpha e^{2\alpha t_0} dt_0$, the left side of (27) equals

$$\begin{aligned} &\int_0^{\infty} v(t)\left(w(t) - w(t+\tau)\right) dt \\ &+ \int_0^{\infty} \left(\int_0^t 2\alpha e^{2\alpha t_0} dt_0\right) v(t)\left(w(t) - w(t+\tau)\right) dt \end{aligned} \tag{28}$$

The first integral on the left side is finite because $v, w \in L_2$ (Cauchy-Schwartz) and hence the double integral is also finite. From Fubini's theorem, this double integral can be re-arranged as

$$\int_0^{\infty} 2\alpha e^{2\alpha t_0} \left(\int_{t_0}^{\infty} v(t)\left(w(t) - w(t+\tau)\right) dt\right) dt_0 \tag{29}$$

Statement (1) implies the inner integral in (29) is ≥ 0 $\forall t_0 \geq 0$. Thus the double integral in (28) is ≥ 0. By Statement (1), the first term in (28) is also ≥ 0. Hence (27) holds and Statement (2) is true. ■