# LPV filter design for discrete-time systems with time-domain IQCs

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***Abstract*— This paper is concerned with the filter design problem for linear parameter varying (LPV) discrete-time systems with nonlinearities described by integral quadratic constraints. The LPV system is considered to have an arbitrary dependence on the time-varying parameters, not necessarily rational. The filter designed assures a bound to the worst case gain from the disturbance input to the filtering error. The development of the proposed approach is based on the existence of a quadratic storage function. The conditions are given in terms of linear matrix inequalities and are derived through congruence transformations and change of variables. Numerical examples illustrate the applicability and performance of the proposed method.**

## I. INTRODUCTION

There has been a considerable study of uncertain systems mainly because mathematical models cannot describe real systems exactly. In this context, several tools have been developed to represent the uncertainties in control systems. One may cite, the linear fractional transformation (LFT), the norm-bounded uncertainties, the polytopic uncertainties and the integral quadratic constraints (IQC) [1]. One advantage of the IQC approach is the fact that IQCs can be used to model uncertain and nonlinear components, such as saturation and time-delay. The IQC approach was originally introduced in [2] based upon the frequency domain. This approach requires the nominal part of the uncertain system to be linear time-invariant (LTI). Hence, it is not directly applicable for time-varying uncertain systems, e.g. uncertain linear parameter varying (LPV) systems. In this case, a time domain IQC interpretation combined with dissipativity theory is required as discussed in [3].

An important topic in the control literature that is affected by the presence of nonlinearities in the plant is the filtering problem. In [4] the problem of filter design had been considered for linear time-invariant (LTI) continuous and discrete-time uncertain systems. The approach was based on the use of IQCs and the solution was obtained with the aid of the S-procedure applied in the $\mathcal{H}_\infty$ problem. In [5], the $\mathcal{H}_\infty$ filter design problem was considered for continuous-time linear systems with uncertainties described by IQCs. In this case, a frequency-dependent linear matrix inequality (LMI) condition was obtained and the solution was determined with a dense frequency grid. In [6] the optimal filter for the nominal system was computed via LMIs and in [7] the presence of polytopic uncertainties in the plant have been considered in the filter design. Representing the uncertainties with LFT, the method in [8] provided conditions for filter design extending the well-known results for polytopic systems. In [9] the robust estimation problem for continuous-time LTI systems with uncertainties have been investigated using dynamic IQCs with a frequency domain formulation. The connections between frequency domain IQCs and time-domain IQCs with dissipation inequalities were first introduced in [10]. Then, the robustness problem for continuous-time LPV systems was addressed in [3] (worst case gain) and the control synthesis problem was considered in [11] with time-domain IQCs. In the discrete-time case, however, the use of time-domain IQC remains less studied.

This paper presents new conditions for filter design of LPV (with unbounded rates of variation in the parameters) discrete-time systems, considering a time-domain interpretation for the IQC. The proposed method differs from existing LPV filter synthesis results, for instance, [12] and [13], that do not take into account the presence of uncertainties in the plant. Moreover, thanks to the grid approach employed in this paper, the LPV system can present arbitrary dependence on the time-varying parameters. The worst case gain from the input disturbance to the filtering error have been used as the performance criteria. First, an augmented system considering the dynamics of the plant, the IQC and the filter are described. Then, the bounded real lemma for discrete-time systems with time-domain IQC is introduced. The filter design condition is obtained through classical congruence transformations and change of variables. The approach is based on the use of a quadratic storage matrix and dissipation inequalities. The conditions are given in terms of LMIs. If the time-varying parameters can be measured or estimated on line a parameter dependent filter can be obtained, if this is not the case, a robust filter can be provided as well. Numerical examples illustrate the applicability of the proposed method.

The paper is organized as follows. Section II present some preliminary results concerning the time-domain IQC and the problem formulation, describing the interconnection of the plant, the uncertainties and the filter to be designed. The robustness analysis for discrete-time systems with IQCs is presented in Section III. The main results are presented in Section IV. Section V is devoted to numerical examples and Section VI concludes the paper.

**Notation.** Capital letters refer to matrices and lowercase letters indicate vectors. For matrices and vectors $(^T)$ indi-

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cates the transpose. $\mathbb{S}^n$ denotes the set of $n \times n$ symmetric matrices. For a symmetric matrix $A$, $A > 0$ ($A \geq 0$) means that $A$ is positive definite (positive semi-definite). Identity matrices are denoted by $I$ and null matrices are denoted by $0$. The block diagonal matrix is expressed by $\mathrm{diag}(X_1, X_2, \ldots, X_n)$. The symbol $\star$ means a symmetric block in matrices. $\ell_2^n[0,\infty)$ is the space of sequences $v : \mathbb{N} \to \mathbb{R}^n$ satisfying $||v||_2 < \infty$ where

$$||v||_2 := \left[\sum_{k=0}^{\infty} v(k)^T v(k)\right]^{\frac{1}{2}}$$

Let $v_T$ denote the truncation of the signal $v$ by setting $v_T = 0$ for $k \geq T$. Moreover, let $\ell_{2e}$ denote the extended space of signals $v$ such that $v_T$ is in $\ell_2$ for all $T \geq 0$.

## II. Preliminaries and problem formulation

Before introducing the main problem let us present some useful results on the time-domain IQC that will be used to represent the uncertainties in this paper.

### A. Time-domain IQC

In the time-domain an IQC is defined by a symmetric matrix $M$ and a stable linear system $\Psi$. A bounded causal operator $\Delta : \ell_{2e}^n \to \ell_{2e}^m$ satisfies an IQC defined by $(\Psi, M)$ if the following inequality holds for all $v \in \ell_{2e}^n[0,\infty)$, $w = \Delta(v)$ and $T \geq 0$:

$$\sum_{k=0}^{T} z(k)^T M z(k) \geq 0 \tag{1}$$

where $z(k)$ is the output of the linear system $\Psi$:

$$\begin{aligned} x_\psi(k+1) &= A_\psi x_\psi(k) + B_{\psi 1} v(k) + B_{\psi 2} w(k) \\ z(k) &= C_\psi x_\psi(k) + D_{\psi 1} v(k) + D_{\psi 2} w(k) \end{aligned} \tag{2}$$

with $x_\psi \in \mathbb{R}^{n_\psi}$ and $z \in \mathbb{R}^{n_z}$. Figure 1 provides a graphic interpretation of the IQC. The input and output signals of $\Delta$ are filtered through $\Psi$. If $\Delta \in \mathrm{IQC}(\Psi, M)$ then the output signal $z(k)$ satisfies the (time-domain) constraint in (1) for any finite horizon $T \geq 0$.

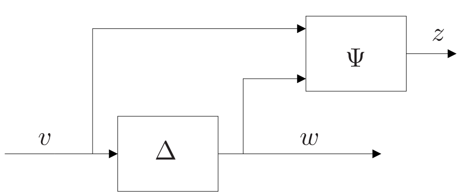


Fig. 1. IQC interconnection.

*Example 1.* Consider a causal nonlinear operator $\Delta$ satisfying the bound $||\Delta|| \leq b$. The norm bound on $\Delta$ implies that $||w||_2 \leq b||v||_2$, for $w = \Delta(v)$ and $v \in \ell_2^{n_v}$. This constraint can be rewritten as

$$\sum_{k=0}^{\infty} \begin{bmatrix} v(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(k) \\ w(k) \end{bmatrix} \geq 0$$

Note that the constraint above is over an infinite horizon. It follows from the causality of $\Delta$ that this constraint holds over all finite horizons $T \geq 0$ as defined in (1). In this way, a norm bounded uncertainty can be handled by the IQC approach considering (2) as

$$z(k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} v(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(k)$$

and $M = \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix}$.

### B. Problem formulation

Consider the following discrete-time LPV plant $\mathcal{P}_\rho$:

$$\begin{aligned} x_P(k+1) &= A_P(\rho) x_P(k) + B_{P_w}(\rho) w(k) + B_{P_d}(\rho) d(k) \\ g(k) &= C_{P_g}(\rho) x_P(k) + D_{P_{gw}}(\rho) w(k) + D_{P_{gd}}(\rho) d(k) \\ y(k) &= C_{P_y}(\rho) x_P(k) + D_{P_{yw}}(\rho) w(k) + D_{P_{yd}}(\rho) d(k) \\ v(k) &= C_{P_v}(\rho) x_P(k) + D_{P_{vw}}(\rho) w(k) + D_{P_{vd}}(\rho) d(k) \end{aligned} \tag{3}$$

where $x_P \in \mathbb{R}^{n_P}$ is the plant state vector, $d \in \mathbb{R}^{n_d}$ is the disturbance input vector, $w \in \mathbb{R}^{n_w}/v \in \mathbb{R}^{n_v}$ are the inputs/outputs associated with the uncertainty, $y \in \mathbb{R}^{n_y}$ is the measured output available to the filter and $g \in \mathbb{R}^{n_g}$ is the signal to be estimated. The parameter $\rho$ is time-varying and belongs to a known compact set $\mathcal{C}_\rho \in \mathbb{R}^{n_\rho}$. The parameters can vary arbitrarily in $\mathcal{C}_\rho$, i.e., there is no bound on the rates of variation of the parameters. The plant (3) is assumed to be asymptotically stable for $d = 0$. Consider the IQC dynamics $\Psi$ with state-space realization given as in (2). Define the augmented state vector $x_G = \begin{bmatrix} x_P^T & x_\psi^T \end{bmatrix}^T \in \mathbb{R}^{n_G}$, $n_G = n_P + n_\psi$ to write the system $\mathcal{G}_\rho$ :

$$\begin{aligned} x_G(k+1) &= A_G x_G(k) + B_{G_w} w(k) + B_{G_d} d(k) \\ z(k) &= C_{G_z} x_G(k) + D_{G_{zw}} w(k) + D_{G_{zd}} d(k) \\ y(k) &= C_{G_y} x_G(k) + D_{G_{yw}} w(k) + D_{G_{yd}} d(k) \\ g(k) &= C_{G_g} x_G(k) + D_{G_{gw}} w(k) + D_{G_{gd}} d(k) \end{aligned} \tag{4}$$

with

$$A_G = \begin{bmatrix} A_p(\rho) & 0 \\ B_{\psi 1} C_{P_v}(\rho) & A_\psi \end{bmatrix}, B_{G_d} = \begin{bmatrix} B_{P_d}(\rho) \\ B_{\psi 1} D_{P_{vd}}(\rho) \end{bmatrix},$$
$$B_{G_w} = \begin{bmatrix} B_{P_w}(\rho) \\ B_{\psi 1} D_{P_{vw}}(\rho) + B_{\psi 2} \end{bmatrix},$$
$$C_{G_z} = \begin{bmatrix} D_{\psi 1} C_{P_v}(\rho) & C_\psi \end{bmatrix}, \quad D_{G_{zd}} = D_{\psi 1} D_{P_{vd}}(\rho),$$
$$D_{G_{zw}} = D_{\psi 1} D_{P_{vw}}(\rho) + D_{\psi 2}, \quad C_{G_y} = \begin{bmatrix} C_{P_y}(\rho) & 0 \end{bmatrix},$$
$$D_{G_{yd}} = D_{P_{yd}}(\rho),\ D_{G_{yw}} = D_{P_{yw}}(\rho),\ D_{G_{gd}} = D_{P_{gd}}(\rho)$$
$$C_{G_g} = \begin{bmatrix} C_{P_g}(\rho) & 0 \end{bmatrix}, \quad D_{G_{gw}} = D_{P_{gw}}(\rho).$$

If the parameters can be estimated or measured on line a LPV filter can be designed. Let us consider the interconnection of the augmented system $\mathcal{G}_\rho$ in (4) with the LPV filter $\mathcal{F}_\rho$:

$$\begin{aligned} x_f(k+1) &= A_f(\rho) x_f(k) + B_f(\rho) y(k) \\ q(k) &= C_f(\rho) x_f(k) + D_f(\rho) y(k) \end{aligned} \tag{5}$$

where $x_f \in \mathbb{R}^{n_f}$ is the filter state vector and $q \in \mathbb{R}^{n_q}$ ($n_q = n_g$), is the estimated output. By defining

$x = \begin{bmatrix} x_G^T & x_f^T \end{bmatrix}^T \in \mathbb{R}^{2n_G}$ and the estimator error $e = g - q$ one can write the augmented system

$$\begin{aligned} x(k+1) &= Ax(k) + B_w w(k) + B_d d(k) \\ e(k) &= C_e x(k) + D_{ew} w(k) + D_{ed} d(k) \\ z(k) &= C_z x(k) + D_{zw} w(k) + D_{zd} d(k) \end{aligned} \tag{6}$$

with

$$A = \begin{bmatrix} A_G & 0 \\ B_f C_{G_y} & A_f(\rho) \end{bmatrix}, \quad B_d = \begin{bmatrix} B_{G_d} \\ B_f(\rho) D_{G_{yd}} \end{bmatrix},$$
$$B_w = \begin{bmatrix} B_{G_w} \\ B_f(\rho) D_{G_{yw}} \end{bmatrix},$$
$$C_e = \begin{bmatrix} C_{G_g} - D_f(\rho) C_{G_y} & -C_f(\rho) \end{bmatrix},$$
$$D_{ed} = D_{G_{gd}} - D_f(\rho) D_{G_{yd}}, \; D_{ew} = D_{G_{gw}} - D_f(\rho) D_{G_{yw}},$$
$$C_z = \begin{bmatrix} C_{G_z} & 0 \end{bmatrix}, \quad D_{zd} = D_{G_{zd}}, \quad D_{zw} = D_{G_{zw}}.$$

Figure 2 summarizes the connected augmented system (6), composed by the LPV plant $\mathcal{P}_\rho$, the IQC dynamics $\Psi$, the LPV filter $\mathcal{F}_\rho$ and the uncertainties $\Delta$. The system has inputs $w(k)$ and $d(k)$ with outputs $e(k)$ and $z(k)$.

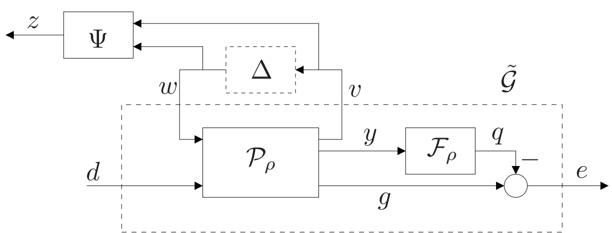


Fig. 2. Connected system composed by the LPV plant $\mathcal{P}_\rho$, the IQC dynamics $\Psi$, the LPV filter $\mathcal{F}_\rho$ and the uncertainties $\Delta$.

## III. Robustness Analysis

The worst case induced $\ell_2$ gain from the disturbance $d$ to the output error $e$ is defined as

$$\sup_{\Delta \text{ satisfying IQC}(\Psi, M)} ||\mathcal{F}_u(\widetilde{\mathcal{G}}, \Delta)|| \tag{7}$$

where $\mathcal{F}_u$ represents an upper LFT and $\widetilde{\mathcal{G}}$ denotes the interconnection of the LPV plant $\mathcal{P}_\rho$ with the LPV filter $\mathcal{F}_\rho$.

*Theorem 1 (Bounded real lemma):* Assume $\mathcal{F}_u(\widetilde{\mathcal{G}}, \Delta)$ is well posed for all $\Delta$ satisfying the IQC$(\Psi, M)$. If there exists a matrix $P \in \mathbb{S}^{n_x}$ and a scalar $\lambda > 0$ such that

$$P > 0 \tag{8}$$

$$\begin{bmatrix} A^T P A - P & A^T P B_w & A^T P B_d \\ \star & B_w^T P B_w & B_w^T P B_d \\ \star & \star & B_d^T P B_d - I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C_e^T \\ D_{ew}^T \\ D_{ed}^T \end{bmatrix} \begin{bmatrix} C_e & D_{ew} & D_{ed} \end{bmatrix} + \lambda \begin{bmatrix} C_z^T \\ D_{zw}^T \\ D_{zd}^T \end{bmatrix} M \begin{bmatrix} C_z & D_{zw} & D_{zd} \end{bmatrix} < 0 \tag{9}$$

Then, the worst case gain of the augmented system (6) is bounded by $\gamma$.

*Proof:* Equation (9) is a strict inequality, so there exist $\epsilon > 0$ such that the following perturbed matrix inequality holds

$$\begin{bmatrix} A^T P A - P & A^T P B_w & A^T P B_d \\ \star & B_w^T P B_w & B_w^T P B_d \\ \star & \star & B_d^T P B_d - (1-\epsilon) I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C_e^T \\ D_{ew}^T \\ D_{ed}^T \end{bmatrix} \begin{bmatrix} C_e & D_{ew} & D_{ed} \end{bmatrix} + \lambda \begin{bmatrix} C_z^T \\ D_{zw}^T \\ D_{zd}^T \end{bmatrix} M \begin{bmatrix} C_z & D_{zw} & D_{zd} \end{bmatrix} \leq 0$$

Multiplying by $\begin{bmatrix} x(k)^T & w(k)^T & d(k)^T \end{bmatrix}$ on the left and by $\begin{bmatrix} x(k)^T & w(k)^T & d(k)^T \end{bmatrix}^T$ on the right, yields

$$\begin{aligned} x(k+1)^T P x(k+1) - x(k)^T P x(k) - (1-\epsilon) d(k)^T d(k) \\ + \frac{1}{\gamma^2} e(k)^T e(k) + \lambda z(k)^T M z(k) \leq 0 \end{aligned} \tag{10}$$

A storage function can be defined as $V(k) = x(k)^T P x(k)$ and condition (8) implies that this storage function is positive definite. Moreover, one can rewrite (10) as

$$V(k+1) - V(k) - (1-\epsilon) d(k)^T d(k) + \frac{1}{\gamma^2} e(k)^T e(k) + \lambda z(k)^T M z(k) \leq 0$$

By summing the last inequality

$$\sum_{k=0}^{T-1} V(k+1) - \sum_{k=0}^{T-1} V(k) - (1-\epsilon) \sum_{k=0}^{T-1} d(k)^T d(k) + \frac{1}{\gamma^2} \sum_{k=0}^{T-1} e(k)^T e(k) + \lambda \sum_{k=0}^{T-1} z(k)^T M z(k) \leq 0$$

considering that $x(0) = 0$, one has

$$V(T) - (1-\epsilon) \sum_{k=0}^{T-1} d(k)^T d(k) + \frac{1}{\gamma^2} \sum_{k=0}^{T-1} e(k)^T e(k) + \lambda \sum_{k=0}^{T-1} z(k)^T M z(k) \leq 0$$

$V(T) > 0$ implies that

$$\sum_{k=0}^{T-1} e(k)^T e(k) \leq (1-\epsilon) \gamma^2 \sum_{k=0}^{T-1} d(k)^T d(k), \quad \forall T \geq 0$$

Let $T \to \infty$ to obtain the bound $||e||_2 < \gamma \, ||d||_2$ ■

*Remark 1:* It is important to remember that the matrices that appear in condition (9) can depend arbitrarily on the time-varying parameters $\rho$ and for this reason the conditions must be satisfied for all $\rho \in \mathcal{C}_\rho$. If $\rho$ enters affinely in the conditions, it suffices to check the LMI in the vertices of

$\mathcal{C}_\rho$. In other cases, the grid approach can be used to evaluate the conditions in a finite set of points that is a subset of the space of the parameters. ⋆

Theorem 1 presents a LMI condition that can be extended to include more than one IQC, for this end it suffices to include the dynamic of the IQCs in (2) and modify the last term of condition (9) as

$$\sum_{k=1}^{N} \lambda_k z_k^T M_k z_k \tag{11}$$

where $N$ is the number of IQCs. Moreover, in some cases, as for instance in Example 1, a more general structure [14], [15] can be considered. It is possible to proof that $||\Delta|| \leq b$ in Example 1 satisfies IQCs defined by

$$M = \begin{bmatrix} b^2 X & 0 \\ 0 & -X \end{bmatrix} \tag{12}$$

where $X \in \mathbb{S}^N$ is a positive definite matrix. For this end, one must choose $\Psi$ as

$$\Psi = \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}$$

where $\Gamma = \begin{bmatrix} \Psi_1 & \dots & \Psi_k \end{bmatrix}^T$.

## IV. Main Results

*Theorem 2:* Assume $\mathcal{F}_u(\widetilde{\mathcal{G}}_\rho, \Delta)$ is well posed for all $\Delta$ satisfying the IQC$(\Psi, M)$. If there exist matrices $P_1 \in \mathbb{S}^{n_G}$, $Y \in \mathbb{S}^{n_G}$, $P_3 \in \mathbb{S}^{n_G}$, $P_2 \in \mathbb{R}^{n_G \times n_G}$, $\bar{A}_f(\rho) \in \mathbb{R}^{n_G \times n_G}$, $\bar{B}_f(\rho) \in \mathbb{R}^{n_G \times n_y}$, $\bar{C}_f(\rho) \in \mathbb{R}^{n_g \times n_G}$, $D_f(\rho) \in \mathbb{R}^{n_g \times n_y}$ and a scalar $\lambda > 0$ such that the LMIs

$$P_1 - Y > 0 \tag{13}$$

and (14) are satisfied for all $\rho \in \mathcal{C}_\rho$, then, the worst case gain is bounded by $\gamma$ and the filtering matrices are given by

$$\begin{bmatrix} A_f(\rho) & B_f(\rho) \\ C_f(\rho) & D_f(\rho) \end{bmatrix} = \begin{bmatrix} P_2^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}_f(\rho) & \bar{B}_f(\rho) \\ \bar{C}_f(\rho) & D_f(\rho) \end{bmatrix} \begin{bmatrix} -P_2^{-T} P_3 & 0 \\ C_{G_y} P_2^{-T} P_3 & I \end{bmatrix}$$

*Proof:* The matrix $P$ is considered to have the following structure:

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \tag{16}$$

By applying Schur complement in (9) one has

$$\begin{bmatrix} -P & 0 & 0 & A^T P & C_e^T \\ 0 & 0 & 0 & B_w^T P & D_{ew}^T \\ 0 & 0 & -I & B_d^T P & D_{ed}^T \\ PA & PB_w & PB_d & -P & 0 \\ C_e & D_{ew} & D_{ed} & 0 & -\gamma^2 I \end{bmatrix} + \lambda \begin{bmatrix} C_z^T \\ D_{zw}^T \\ D_{zd}^T \\ 0 \\ 0 \end{bmatrix} M \begin{bmatrix} C_z & D_{zw} & D_{zd} & 0 & 0 \end{bmatrix} < 0$$

Then, multiplying it by $Q$ on the right and by $Q^T$ on the left, with $Q = \text{diag}\,(T, I, I, T, I)$,

$$T = \begin{bmatrix} I & I \\ -P_3^{-1} P_2^T & 0 \end{bmatrix}$$

and defining

$$\begin{aligned} Y &= P_1 - P_2 P_3^{-1} P_2^T \\ \bar{A}_f(\rho) &= P_2 \left( B_f(\rho) C_{G_y} - A_f(\rho) P_3^{-1} P_2^T \right) \\ \bar{B}_f(\rho) &= P_2 B_f(\rho) \\ \bar{C}_f(\rho) &= -C_f(\rho) P_3^{-1} P_2^T + D_f(\rho) C_{G_y} \end{aligned} \tag{17}$$

one has condition (14). The first step to get the filtering matrices is to reconstruct the storage matrix (16). It can be made by assigning values for the matrices $P_2$ or $P_3$. After that, the relations in (17) can be used. For details about filter reconstruction the reader is referred to [16]. ■

If the parameters are not available to be measured on line a robust filter can be designed. Theorem 2 can be adapted to deal with this case. It suffices to consider the robust filter in the following form

$$\begin{aligned} x_f(k+1) &= A_f x_f(k) + B_f y(k) \\ q(k) &= C_f x_f(k) + D_f y(k) \end{aligned} \tag{18}$$

The next corollary states a condition for the robust filter design.

*Corollary 1:* Assume $\mathcal{F}_u(\widetilde{\mathcal{G}}_\rho, \Delta)$ is well posed for all $\Delta$ satisfying the IQC$(\Psi, M)$. If there exist matrices $P_1 \in \mathbb{S}^{n_G}$, $Y \in \mathbb{S}^{n_G}$, $P_3 \in \mathbb{S}^{n_G}$, $P_2 \in \mathbb{R}^{n_G \times n_G}$, $\bar{A}_f \in \mathbb{R}^{n_G \times n_G}$, $\bar{B}_f \in \mathbb{R}^{n_G \times n_y}$, $\bar{C}_f \in \mathbb{R}^{n_g \times n_G}$, $D_f \in \mathbb{R}^{n_g \times n_y}$ and a scalar $\lambda > 0$ such that (13) and (14) are satisfied for all $\rho \in \mathcal{C}_\rho$, then, the worst case gain is bounded by $\gamma$ and the filtering matrices are given by

$$\begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} = \begin{bmatrix} P_2^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}_f & \bar{B}_f \\ \bar{C}_f & D_f \end{bmatrix} \begin{bmatrix} -P_2^{-T} P_3 & 0 \\ C_{G_y} P_2^{-T} P_3 & I \end{bmatrix}$$

*Proof:* Follows straightforward the steps in the proof of Theorem 2. ■

*Remark 2:* The conditions presented in Theorem 2 and in Corollary 1 deal with LPV systems using time-domain IQCs. The frequency domain approach has been used in [17] to handle the problem of designing gain scheduling estimators for continuous-time LPV systems. However, in their formulation, the LPV system is restricted to have a linear fractional dependence on uncertainties. ⋆

## V. Numerical Examples

The objective of the examples is to illustrate the possible scenarios where the proposed approach can be employed. The routines were implemented in MATLAB, version 8.3.0.532 (R14) using the packages Yalmip [18] and SeDuMi [19].

$$\begin{bmatrix} -Y & -Y & 0 & 0 & A_G^T Y & A_G^T P_1 + \bar{A}_f^T & C_{G_g}^T - \bar{C}_f^T \\ \star & -P_1 & 0 & 0 & A_G^T Y & A_G^T P_1 + C_{G_y}^T \bar{B}_f^T & C_{G_g}^T - C_{G_y}^T D_f^T \\ \star & \star & 0 & 0 & B_{G_w}^T Y & B_{G_w}^T P_1 + D_{G_{yw}}^T \bar{B}_f^T & D_{G_{gw}}^T - D_{G_{yw}}^T D_f^T \\ \star & \star & \star & -I & B_{G_d}^T Y & B_{G_d}^T P_1 + D_{G_{yd}}^T \bar{B}_f^T & D_{G_{gd}}^T - D_{G_{yd}}^T D_f^T \\ \star & \star & \star & \star & -Y & -Y & 0 \\ \star & \star & \star & \star & \star & -P_1 & 0 \\ \star & \star & \star & \star & \star & \star & -\gamma^2 I \end{bmatrix} + X^T M(\lambda) X < 0 \tag{14}$$

$$X = \begin{bmatrix} C_{G_z} & C_{G_z} & D_{G_{zw}} & D_{G_{zd}} & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

*Example 2.* Consider the discrete-time LPV system described by

$$\begin{aligned} A_P(\rho) &= \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \rho, \quad |\rho| \leq 0.4 \\ B_{P_d} &= \begin{bmatrix} -6 & 0 \\ 1 & 0 \end{bmatrix}, \; B_{P_w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ C_{P_g} &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D_{P_{gd}} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \; D_{P_{gw}} = 0, \\ C_{P_y} &= \begin{bmatrix} -100 & 10 \end{bmatrix}, \; D_{P_{yd}} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \; D_{P_{yw}} = 0, \\ C_{P_v} &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \; D_{P_{vd}} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \; D_{P_{vw}} = 0, \end{aligned} \tag{19}$$

and a LTI norm bounded uncertainty $||\Delta|| \leq b$. Let us consider three different IQCs $(\Psi_1, M_1)$, $(\Psi_2, M_2)$, $(\Psi_3, M_3)$ described by

$$\Psi_1 = I, \; , \Psi_2 = \begin{bmatrix} \frac{1}{z-\alpha} & 0 \\ 0 & \frac{1}{z-\alpha} \end{bmatrix}, \; \Psi_3 = \begin{bmatrix} \frac{1}{z-\beta} & 0 \\ 0 & \frac{1}{z-\beta} \end{bmatrix} \tag{20}$$

$$M_1 = M_2 = M_3 = \begin{bmatrix} b^2 & 0 \\ 0 & -1 \end{bmatrix}$$

All the experiments in this example have been performed with $\alpha = 0.1$ and $\beta = 0.4$. First, the design of strictly proper robust filters ($D_f = 0$) for LPV system is considered. The bounds for the worst case gain obtained using different configurations are presented in the first part of Table I. It can be seen that the use of more than one IQC connected as in (12) provide better results than considering each IQC separately as in (11). In this case the solver did not find feasible solution for $b > 0.06$. It is important to remember that the parameters can vary arbitrarily fast and this can be one of the reasons why it is not possible to include more uncertainty in the problem.

The second part of Table I presents the bounds obtained by employing a parameter dependent filter. A filter with an affine structure have been used

$$Z_f(\rho) = Z_{f0} + \rho Z_{f1} \tag{21}$$

where $Z_f$ represent the filtering matrices. As expected, the LPV filter can further reduce the conservativeness of the bounds obtained, indicating that whenever the time-varying parameters are available to be measured on line this is the wisest choice to be made. In this situation, the use of three IQCs (20) connected as in (12) proved to be the best choice to reduce the conservatism of the worst case bounds.

Last, the design of proper LPV filters as in (21) have been performed. The bounds are shown in the third part of Table I.

TABLE I

BOUNDS $\gamma$ FOR THE WORST CASE GAIN OBTAINED FOR THE DESIGN OF ROBUST AND LPV FILTERS BY USING MULTIPLE IQCS CONNECTED AS IN (11) AND (12). THE PARAMETERS $\alpha = 0.1$ AND $\beta = 0.4$ HAVE BEEN USED.

| $b$ | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 |
|---|---|---|---|---|---|---|
| IQC | Robust filter - strictly proper | | | | | |
| $1, 2$ as in (11) | 11.12 | 11.72 | 12.54 | 13.88 | 16.45 | 27.36 |
| $1, 2$ as in (12) | 11.11 | 11.69 | 12.49 | 13.74 | 16.24 | 26.28 |
| $1, 2, 3$ as in (12) | 11.10 | 11.66 | 12.43 | 13.65 | 16.07 | 25.87 |
| | LPV filter - strictly proper | | | | | |
| $1, 2$ as in (11) | 9.16 | 9.49 | 9.92 | 10.54 | 11.72 | 16.48 |
| $1, 2$ as in (12) | 9.16 | 9.48 | 9.89 | 10.48 | 11.48 | 15.01 |
| $1, 2, 3$ as in (12) | 9.08 | 9.30 | 9.55 | 9.87 | 10.34 | 11.85 |
| | LPV filter - proper | | | | | |
| $1, 2, 3$ as in (12) | 1.60 | 1.64 | 1.69 | 1.74 | 1.82 | 2.09 |

The use of different basis functions in the IQC formulation as $\frac{1}{(z-\alpha)^n}$ for instance, could provide different results.

*Example 3.* In order to compare the proposed conditions with existing approaches that can deal with LTI systems let us consider a discrete-time LTI plant with matrices as in (19), $\rho = 0$ and a LTI norm bounded uncertainty $||\Delta|| \leq b$ with IQC $(\Psi_1, M_1)$ as in (20).

For $b = 0.35$ and the IQC $(\Psi_1, M_1)$ as in (20), Corollary 1 yields a bound to the worst case $\gamma = 4.26$, that is the same bound provided by the conditions in [7], [20] and [21] when considering quadratic Lyapunov matrices in their approaches.

Figure 3 presents the behavior of the worst case gain $\gamma$ obtained by Corollary 1 with $b = 0.35$, IQCs $(\Psi_1, M_1)$ and $(\Psi_2, M_2)$ as in (20) for different values of $\alpha$. Two different situations have been considered. First the IQCs have been considered separately as in (11) (black dashed line) and then, a more general formulation for the IQCs as in (12) was employed (blue solid line).

The D-K synthesis [22] can also deal with the presented problem, and a robust performance can be guaranteed, that

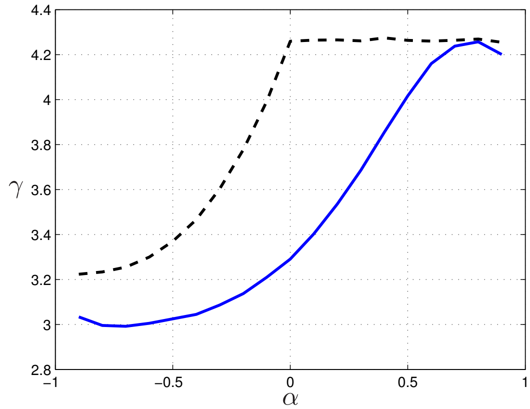


Fig. 3. Behavior of the worst case gain $\gamma$ obtained by Corollary 1 with $b = 0.35$, IQCs $(\Psi_1, M_1)$ and $(\Psi_2, M_2)$ as in (20) for different values of $\alpha$. IQCs gathered as in (11) (black dashed line) and IQCS gathered as in (12) (blue solid line).

is, the worst case gain of the closed loop system is less than $\gamma$ for all $||\Delta|| \leq 1/\gamma$. For this example, a sixth order robust filter can be computed by the D-K approach with a bound $\gamma = 2.86$, that is less conservative then the ones in Figure 3. However, by using the more general formulation in (12) with $\Psi_1$, $\Psi_2$ and $\Psi_3$ as in (20), with $\alpha = 0.1$ and $\beta = 0.4$, one has $\gamma = 2.17$ that is approximately $25\%$ smaller than the bound obtained with the D-K synthesis.

## VI. Conclusions

This paper presented conditions for filter design of LPV (with unbounded rates of variation) uncertain systems. Time-domain IQCs have been used to represent the uncertainties which are present in the LPV plant, since the frequency is not adequate to treat LPV systems. The filter design conditions were derived by using a quadratic storage function. This strategy allowed designing both LPV and robust filters for LPV systems. Two different strategies to connect the IQCs have been employed. It has been shown that the use of multiple IQCs and the design of LPV filters can reduce the conservativeness of the worst case gain bounds. Moreover, the proposed strategy can outperform the bounds obtained with the DK-synthesis for the LTI case. As future research, the authors are investigating how to consider LPV systems with bounded rates of variation.

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