INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust. Nonlinear Control 0000; **00**:1–15
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc

Less Conservative Robustness Analysis of Linear Parameter Varying Systems Using Integral Quadratic Constraints

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SUMMARY

This paper considers the robustness of a feedback connection of a known linear parameter varying (LPV) system and a perturbation. A sufficient condition is derived to bound the worst-case gain and ensure robust asymptotic stability. The input/output behavior of the perturbation is described by multiple integral quadratic constraints (IQCs). The analysis condition is formulated as a dissipation inequality. The standard approach requires a non-negative definite storage function and the use of "hard" IQCs. The term "hard" means the IQCs can be specified as time-domain integral constraints that hold over all finite horizons. The main result demonstrates that the dissipation inequality condition can be formulated requiring neither a non-negative storage function nor hard IQCs. A key insight used to prove this result is that the multiple IQCs, when combined, contain hidden stored energy. This result can lead to less conservative robustness bounds. Two simple examples are presented to demonstrate this fact.

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1. INTRODUCTION

This paper considers the robustness of an uncertain linear parameter varying (LPV) system. The uncertain system is represented as a feedback connection of a known LPV system and a perturbation. The class of "gridded" LPV systems is considered in this paper. For this class the state matrices are arbitrary functions of the scheduling parameter, e.g as in [29, 30]. This is more general than LFT-type LPV systems whose state matrices have a rational dependence on the scheduling parameter [1, 13, 18]. Integral quadratic constraints (IQCs) are used to model the uncertain and/or nonlinear components. IQCs, introduced in [11], provide a general framework for robustness analysis. An IQC stability theorem was formulated in [11] for a feedback interconnection of a linear time-invariant (LTI) system and a perturbation. The stability theorem involved with frequency domain conditions and was proved using a homotopy method.

The uncertain systems in this paper involve a nominal LPV system and hence the frequency domain conditions are not applicable. Instead, a time-domain approach is used. This time-domain approach involves "hard" IQCs that are specified as integral constraints that hold over all finite time intervals. These hard IQCs can be used to formulate an alternative time domain stability theorem based on dissipativity theory [27,28]. This approach was used in [15] to derive a sufficient condition for robust performance of an uncertain LPV system. This result, summarized in Section 2, requires a nonnegative storage function. In addition, a combined IQC is parameterized as a conic combination of individual hard IQCs. This approach is correct but can lead to unnecessary conservatism for

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two reasons. First, the frequency domain condition derived in [11] for nominal LTI systems can be converted to a related dissipation inequality type constraint by the KYP Lemma but without the non-negativity constraint on the storage function. Second, conic combinations of hard IQCs is not the most general parameterization, e.g. it does not include the alternative parameterizations in [19].

The main contribution of this paper is to provide a less conservative condition to assess the worst case gain and robust asymptotic stability of an uncertain LPV system. This result, stated as Theorem 1 in Section 3, differs in two respects from the prior result in [15]. First, the main result involves a dissipation inequality but it does not enforce the storage function to be non-negative. Second, the main result allows for more general IQC parameterizations. In particular, the IQC need not be hard, i.e. it need not specify a valid finite-horizon integral constraint. Instead, the main result replaces the standard dissipation inequality assumptions with a milder technical assumption on the combined multiplier. This technical assumption essentially implies that the combined multiplier has a special J-spectral factorization [3]. This is used to show that the combined multiplier has some hidden stored energy. As a result, the analysis condition can be reformulated into a valid dissipation inequality with a single hard IQC and a non-negative storage function.

This main result Theorem 1 directly generalizes the result in [15]. This can lead to less conservative analysis results as demonstrated by examples in Section 4. This extends the results in [21] developed for a nominal LTI system and a single IQC. It was shown in [21] that the non-negativity constraint on the storage function can be dropped in the time domain approach if a J-spectral factorization is used for the IQC multiplier. The main result here extends these results to LPV systems and multiple IQCs. Another closely related prior work is [23] which provides an IQC dissipation inequality condition after a loop transformation. Moreover, it uses a specific, unique factorization. This paper avoids such a transformation and focuses on non-unique factorizations.

2. BACKGROUND

2.1. Notation

$\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. $\mathbb{RL}_\infty$ denotes the set of rational functions with real coefficients that are proper and have no poles on the imaginary axis. $\mathbb{RH}_\infty$ is the subset of functions in $\mathbb{RL}_\infty$ that are analytic in the closed right half of the complex plane. $\mathbb{C}^{m\times n}$, $\mathbb{RL}_\infty^{m\times n}$ and $\mathbb{RH}_\infty^{m\times n}$ denote the sets of $m \times n$ matrices whose elements are in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{RL}_\infty$, $\mathbb{RH}_\infty$, respectively. A single superscript index is used for vectors, e.g. $\mathbb{R}^n$ denotes the set of $n \times 1$ vectors whose elements are in $\mathbb{R}$. $\mathcal{S}^n$ denotes the set of $n \times n$ symmetric matrices. $\mathbb{R}^+$ describes the set of nonnegative real numbers. For $z \in \mathbb{C}$, $\bar{z}$ denotes the complex conjugate of z. For a matrix $M \in \mathbb{C}^{m\times n}$, M^T denotes the transpose and M^* denotes the complex conjugate transpose. The para-Hermitian conjugate of $G \in \mathbb{RL}_\infty^{m\times n}$, denoted as $G^\sim$, is defined by $G^\sim(s) := G(-\bar{s})^*$. Note that on the imaginary axis, $G^\sim(j\omega) = G(j\omega)^*$. $L_2^n[0,\infty)$ is the space of functions $v : [0,\infty) \to \mathbb{R}^n$ satisfying $\|v\|_2 < \infty$ where

$$\|v\|_2 := \left[\int_0^\infty v(t)^T v(t)\, dt\right]^{0.5} \tag{1}$$

Given $v \in L_2^n[0,\infty)$, v_T denotes the truncated function:

$$v_T(t) := \begin{cases} v(t) & \text{for } t \le T \\ 0 & \text{for } t > T \end{cases} \tag{2}$$

The extended space, denoted L_{2e}, is the set of functions v such that $v_T \in L_2$ for all $T \ge 0$.

2.2. Problem Statement

This paper considers the robustness of uncertain LPV systems. The uncertain system is described by the feedback interconnection of an LPV system G and an uncertainty Δ as shown in Figure 1.

This feedback interconnection with Δ wrapped around the top of G is denoted $F_u(G,\Delta)$. The LPV system G is a linear system whose state space matrices depend on a time-varying parameter vector $\rho : \mathbb{R}^+ \to \mathbb{R}^{n_\rho}$ as follows:

$$\begin{aligned} \dot{x}_G(t) &= A_G(\rho(t))\, x_G(t) + B_G(\rho(t)) \begin{bmatrix} w(t) \\ d(t) \end{bmatrix} \\ \begin{bmatrix} v(t) \\ e(t) \end{bmatrix} &= C_G(\rho(t))\, x_G(t) + D_G(\rho(t)) \begin{bmatrix} w(t) \\ d(t) \end{bmatrix} \end{aligned} \tag{3}$$

where $x_G \in \mathbb{R}^{n_G}$ is the state, $w \in \mathbb{R}^{n_w}$ and $d \in \mathbb{R}^{n_d}$ are inputs, and $v \in \mathbb{R}^{n_v}$ and $e \in \mathbb{R}^{n_e}$ are outputs. The state matrices of G have dimensions compatible with these signals, e.g. $A_G(\rho) \in \mathbb{R}^{n_G \times n_G}$. In addition, the state matrices are assumed to be continuous functions of ρ. The state matrices at time t depend on the parameter vector at time t. Hence, LPV systems represent a special class of time-varying systems. The explicit dependence on t is occasionally suppressed to shorten the notation. Moreover, it is important to emphasize that the state matrices are allowed to have an arbitrary dependence on the parameters. This is called a “gridded” LPV system and is more general than “LFT” LPV systems whose state matrices are restricted to have a rational dependence on the parameters [1, 13, 18].

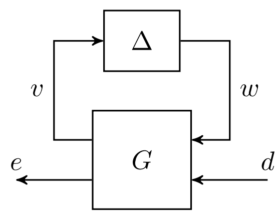

Figure 1. Feedback Interconnection

The parameter ρ is assumed to be a continuously differentiable function of time and admissible trajectories are restricted to a known compact set $\mathcal{P} \subset \mathbb{R}^{n_\rho}$. In addition, the parameter rates of variation $\dot{\rho} : \mathbb{R}^+ \to \dot{\mathcal{P}}$ are assumed to lie within a hyperrectangle $\dot{\mathcal{P}} := \{q \in \mathbb{R}^{n_\rho} \,|\, \underline{\nu}_i \le q_i \le \bar{\nu}_i,\ i = 1, \ldots, n_\rho\}$. The set of admissible trajectories is defined as

$$\mathcal{T} := \left\{\rho : \mathbb{R}^+ \to \mathbb{R}^{n_\rho} \ :\ \rho \in \mathcal{C}^1,\ \rho(t) \in \mathcal{P} \text{ and } \dot{\rho}(t) \in \dot{\mathcal{P}}\ \forall t \ge 0\right\} \tag{4}$$

The parameter trajectory is said to be rate unbounded if $\dot{\mathcal{P}} = \mathbb{R}^{n_\rho}$.

Throughout the paper it is assumed that the uncertain system has a form of nominal stability. Specifically, G is assumed to be parametrically-dependent stable as defined in [29].

Definition 1. G is <u>parametrically-dependent stable</u> if there is a continuously differentiable function $P : \mathbb{R}^{n_\rho} \to \mathcal{S}^{n_G}$ such that $P(p) \ge 0$ and

$$A_G(p)^T P(p) + P(p) A_G(p) + \sum_{i=1}^{n_\rho} \frac{\partial P}{\partial p_i} q_i < 0 \tag{5}$$

hold for all $p \in \mathcal{P}$ and all $q \in \dot{\mathcal{P}}$.

As discussed in [29], parametric-stability implies G has a strong form of robustness. In particular, the state $x_G(t)$ of the autonomous response ($w = 0$, $d = 0$) decays exponentially to zero for any initial condition $x_G(0) \in \mathbb{R}^{n_G}$ and allowable trajectory $\rho \in \mathcal{T}$ (Lemma 3.2.2 of [29]). Moreover, the state $x_G(t)$ of the forced response decays asympotically to zero for any initial condition $x_G(0) \in \mathbb{R}^{n_G}$, allowable trajectory $\rho \in \mathcal{T}$, and inputs $w, d \in L_2$ (Lemma 3.3.2 of [29]). The parameter-dependent Lyapunov function $V(x_G, \rho) := x_G^T P(\rho) x_G$ plays a key role in the proof of these results. To shorten the notation, a differential operator $\partial P : \mathcal{P} \times \dot{\mathcal{P}} \to \mathbb{R}^{n_x}$ is introduced as in [20]. ∂P is defined as $\partial P(p, q) := \sum_{i=1}^{n_\rho} \frac{\partial P}{\partial \rho_i}(p)\, q_i$. This simplifies the expression of Lyapunov-type inequalities similar to Equation (5).

The uncertainty $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ is a bounded, causal operator. The notation $\mathbf{\Delta}$ is used to denote the set of bounded, causal uncertainties Δ. The input/output behavior of the uncertain set is bounded using quadratic constraints as described further in the next section. At this point it is sufficient to state that Δ can have block-structure as is standard in robust control modeling [32]. Δ can include blocks that are hard nonlinearities (e.g. saturations) and infinite dimensional operators (e.g. time delays) in addition to true system uncertainties. The term uncertainty is used for simplicity when referring to the perturbation Δ.

The objective of this paper is to assess the robustness of the uncertain system $F_u(G,\Delta)$. For a given $\Delta \in \mathbf{\Delta}$, the induced L_2 gain from d to e is defined as:

$$\|F_u(G,\Delta)\| := \sup_{\substack{0\neq d\in L_2^{n_d}[0,\infty) \\ \rho\in\mathcal{T},\, x_G(0)=0}} \frac{\|e\|_2}{\|d\|_2} \tag{6}$$

Two forms of robustness are considered. First, the worst-case induced L_2 gain from input d to the output e is defined as

$$\sup_{\Delta\in\mathbf{\Delta}} \|F_u(G_\rho,\Delta)\|. \tag{7}$$

This is the worst-case gain over all uncertainties $\Delta \in \mathbf{\Delta}$ and admissible trajectories $\rho \in \mathcal{T}$. Second, the system has robust asymptotic stability if $x_G(t) \to 0$ for any initial condition $x_G(0) \in \mathbb{R}^{n_G}$, allowable trajectory $\rho \in \mathcal{T}$, disturbance $d \in L_2$ and uncertainty $\Delta \in \mathbf{\Delta}$. The main result provides a sufficient condition for the uncertain LPV system to have both robust asymptotic stability and bounded worst-case gain.

2.3. Integral Quadratic Constraints (IQCs)

A frequency-domain IQC is defined in [11] using a multiplier Π. In particular, let $\Pi : j\mathbb{R} \to \mathbb{C}^{(n_v+n_w)\times(n_v+n_w)}$ be a measurable Hermitian-valued function. Two signals $v \in L_2^{n_v}[0,\infty)$ and $w \in L_2^{n_w}[0,\infty)$ satisfy the IQC defined by the multiplier Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} V(j\omega) \\ W(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} V(j\omega) \\ W(j\omega) \end{bmatrix} d\omega \geq 0 \tag{8}$$

where $V(j\omega)$ and $W(j\omega)$ are Fourier transforms of v and w, respectively. A bounded, causal operator $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ satisfies the IQC defined by Π if Equation (8) holds for all $v \in L_2^{n_v}[0,\infty)$ and $w = \Delta(v)$.

IQCs were introduced in [11] to assess the robustness of the feedback interconnection $F_u(G,\Delta)$ for the case that G is LTI. The stability conditions in [11] were expressed in the frequency domain. In this paper, the nominal part G is LPV and hence the frequency-domain conditions cannot be applied. In addition, the stability condition used in this paper does not require a homotopy type condition as in [11]. An alternative time-domain stability condition can be constructed using (time-domain) IQCs and dissipation theory. Specifically, any $\Pi \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ can be factorized as $\Pi = \Psi^\sim M \Psi$ where $M \in \mathcal{S}^{n_z}$ and $\Psi \in \mathbb{RH}_\infty^{n_z\times(n_v+n_w)}$. Such factorizations are not unique but can be computed with state-space methods [20]. Appendix A provides two specific factorizations. Let (Ψ, M) be any such factorization of Π. Then $v, w \in L_2$ satisfy the IQC in Equation (8) if and only if $Z(j\omega) := \Psi(j\omega) \begin{bmatrix} V(j\omega) \\ W(j\omega) \end{bmatrix}$ satisfies $\int_{-\infty}^{\infty} Z(j\omega)^* M Z(j\omega)\, d\omega \geq 0$. By Parseval's theorem, this frequency-domain constraint on z can be equivalently expressed in the time-domain as:

$$\int_0^\infty z(t)^T M z(t)\, dt \geq 0 \tag{9}$$

where $z = \Psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$ is the output of the linear system Ψ starting from zero initial conditions:

$$\begin{aligned} \dot{\psi}(t) &= A_\psi \psi(t) + B_{\psi 1} v(t) + B_{\psi 2} w(t), \quad \psi(0) = 0 \\ z(t) &= C_\psi \psi(t) + D_{\psi 1} v(t) + D_{\psi 2} w(t) \end{aligned} \tag{10}$$

Thus Δ satisfies the IQC defined by $\Pi = \Psi^{\sim} M \Psi$ if and only if the time domain constraint in Equation (9) holds for all $v \in L_2^{n_v}[0,\infty)$ and $w = \Delta(v)$. The constraint in Equation (9) holds, in general, only over infinite time. The term hard IQC in [11] refers to the more restrictive property: $\int_0^T z(t)^T M z(t)\, dt \geq 0$ holds $\forall T \geq 0$. In contrast, IQCs for which the time domain constraint need not hold for all finite times are called soft IQCs. This distinction is important because the dissipation theorem presented below requires the use of hard IQCs. A more precise definition is now given.

Definition 2. Let Π be factorized as $\Psi^{\sim} M \Psi$ with Ψ stable. (Ψ, M) is a hard factorization of Π if for any bounded, causal operator Δ satisfying the (frequency domain) IQC defined by Π (Equation 8) the following (time-domain) inequality holds

$$\int_0^T z(t)^T M z(t)\, dt \geq 0 \tag{11}$$

for all $T \geq 0$, $v \in L_2^{n_v}[0,\infty)$, $w = \Delta(v)$ and $z = \Psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$.

As noted above, the factorization of Π is not unique. Thus the hard/soft property is not inherent to the multiplier Π but instead depends on the factorization (Ψ, M). One particularly useful factorization is the J-spectral factorization defined as follows: $(\hat{\Psi}, J_{n_v,n_w})$ is a J-spectral factor of Π if $\Pi = \hat{\Psi}^{\sim} J_{n_v,n_w} \hat{\Psi}$, $J_{n_v,n_w} = \left[\begin{smallmatrix} I_{n_v} & 0 \\ 0 & -I_{n_w} \end{smallmatrix}\right]$ and $\hat{\Psi}, \hat{\Psi}^{-1} \in \mathbb{RH}_\infty^{(n_v+n_w)\times(n_v+n_w)}$. In other words, the factor $\hat{\Psi}$ is square, stable, and stably inverible. It follows from Lemma 1 in Appendix A that a J-spectral factorization is a hard factorization. Lemma 1 also provides a simple frequency domain condition that is sufficient for the existence of a J-spectral factor. A J-spectral factor of Π (if one exists) can be constructed using the solution of a related algebraic Riccati equation. Note that depending on the type of uncertainty/nonlinearity it is either more natural to start with a time domain or frequency domain constraint. An example for the natural usage of frequency domain IQCs are time delays, see [16] for details. A J-spectral factorization can then be used to obtain a hard time domain IQC starting from a frequency domain condition.

A strength of the IQC framework is that many IQCs for a single Δ can be incorporated into the analysis. A simple example is provided below to demonstrate that classes of IQCs can be parameterized in different ways.

Example 1. Consider the SISO, LTI uncertainty $\Delta \in \mathbb{RH}_\infty$ with $\|\Delta\|_\infty := \sup_\omega |\Delta(j\omega)| \leq 1$. In [11] an IQC for the operator Δ is defined by $\Pi := \left[\begin{smallmatrix} X & 0 \\ 0 & -X \end{smallmatrix}\right]$ where $X(j\omega) = X(j\omega)^* \geq 0\ \forall \omega$. This IQC multiplier can be parameterized by picking a collection of transfer functions $\{X_i\}_{i=1}^{n_\pi}$ that are ≥ 0 on the imaginary axis. Then Δ satisfies the IQCs defined by multipliers of the form $\Pi_i := \left[\begin{smallmatrix} X_i & 0 \\ 0 & -X_i \end{smallmatrix}\right]$. Moreover, Δ satisfies any conic combination of these multipliers defined by $\Pi(\lambda) := \sum_{i=1}^{n_\pi} \lambda_i \Pi_i$ where $\lambda_i \geq 0$ for $i = 1, \ldots, n_\pi$. A factorization of this conic combination can also be parameterized. By the spectral factorization theorem [31, 32] it is possible to construct stable, minimum phase systems D_i such that $X_i = D_i^{\sim} D_i$. The D_i are called D-scales in the robust control literature [6, 14, 17]. Define the filter $\Psi_i := D_i I_2$ and matrix $M_i := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. This yields a factorization for each individual multiplier as $\Pi_i := \Psi_i^{\sim} M_i \Psi_i$. Note that each Ψ_i is square, stable and stably invertible. Hence each (Ψ_i, M_i) is a J-spectral (and thus a hard) factorization of Π_i. A factorization for the conic combination of multipliers can be parameterized as:

$$\Pi(\lambda) = \underbrace{\begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_{n_\pi} \end{bmatrix}^{\sim}}_{\Psi^{\sim}} \underbrace{\begin{bmatrix} \lambda_1 M_1 & & \\ & \ddots & \\ & & \lambda_{n_\pi} M_{n_\pi} \end{bmatrix}}_{M(\lambda)} \underbrace{\begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_{n_\pi} \end{bmatrix}}_{\Psi} \tag{12}$$

Note that Ψ as defined in Equation (12) is not square, in general, and therefore $(\Psi, M(\lambda))$ is not a J-spectral factorization for $\Pi(\lambda)$.

A more general parameterization for IQCs satisfied by Δ is given in [24]. Select a column of stable systems and stack into a vector as $\bar{\Psi} \in \mathbb{RH}_\infty^{k\times 1}$. The IQC multiplier is parameterized as:

$$\Pi(\Lambda) = \underbrace{\begin{bmatrix} \bar{\Psi} & 0 \\ 0 & \bar{\Psi} \end{bmatrix}^\sim}_{\Psi^\sim} \underbrace{\begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}}_{M(\lambda)} \underbrace{\begin{bmatrix} \bar{\Psi} & 0 \\ 0 & \bar{\Psi} \end{bmatrix}}_{\Psi} \tag{13}$$

with $\Lambda = \Lambda^T \in \mathbb{R}^{k\times k}$ constrained to satisfy $\bar{\Psi}^\sim(j\omega)\Lambda\bar{\Psi}(j\omega) \geq 0$ for all ω and λ denoting the vector of unique entires in the matrix variable Λ. This parameterization is more general as it is not simply a conic combination of multipliers. Details, including the MIMO case, are provided in [24]. Again, the Ψ as defined in Equation (13) is not square, in general, and therefore $(\Psi, M(\lambda))$ is not a J-spectral factorization for $\Pi(\Lambda)$.

2.4. Dissipation Inequality Condition

This section describes a dissipation inequality condition to assess the robustness of the uncertain LPV system $F_u(G,\Delta)$. The result in this section is a minor extension of that contained in [15]. Assume the uncertainty Δ satisfies a collection of IQCs $\{\Pi_i\}_{i=1}^{n_\pi}$ with corresponding factorizations $\{(\Psi_i, M_i)\}_{i=1}^{n_\pi}$. The Ψ_i can be stacked into a single filter:

$$\Psi := \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_{n_\pi} \end{bmatrix} \tag{14}$$

Ψ has a state space realization as in Equation (10). The robustness of the uncertain LPV system $F_u(G,\Delta)$ can be analyzed using the interconnection structure shown in Fig. 2. The feedback interconnection including Ψ is described by $w = \Delta(v)$ and

$$\begin{aligned} \dot{x} &= A(\rho)x + B_1(\rho)w + B_2(\rho)d \\ z &= C_1(\rho)x + D_{11}(\rho)w + D_{12}(\rho)d \\ e &= C_2(\rho)x + D_{21}(\rho)w + D_{22}(\rho)d, \end{aligned} \tag{15}$$

where $x := \left[\begin{smallmatrix} x_G \\ \psi \end{smallmatrix}\right] \in \mathbb{R}^{n_G+n_\psi}$ is the extended state. The state matrices of the extended system in Equation (15) can be constructed from the state matrices of G (Equation 3) and Ψ (Equation 10). The output z has block structure corresponding to the outputs of Ψ_i:

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_{n_\pi} \end{bmatrix} = \begin{bmatrix} C_{1,1}(\rho) \\ \vdots \\ C_{1,n_\pi}(\rho) \end{bmatrix} x + \begin{bmatrix} D_{11,1}(\rho) \\ \vdots \\ D_{11,n_\pi}(\rho) \end{bmatrix} w + \begin{bmatrix} D_{12,1}(\rho) \\ \vdots \\ D_{12,n_\pi}(\rho) \end{bmatrix} d \tag{16}$$

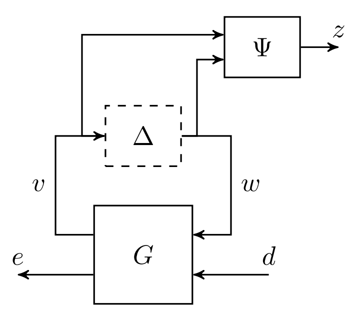

Figure 2. Analysis Interconnection

The next theorem provides an analysis condition using IQCs and a standard dissipation argument. The analysis replaces the precise relation $w = \Delta(v)$ with integral quadratic constraints on z_i. The sufficient condition uses a quadratic storage function that is defined using a symmetric, parameter-dependent matrix $P : \mathcal{P} \to \mathcal{S}^{n_x}$.

Theorem 1. Let G be a parametrically stable LPV system defined by Equation (3) and $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ be a bounded, causal operator such that $F_u(G,\Delta)$ is well-posed. Assume Δ satisfies the IQCs defined by the multipliers $\{\Pi_i\}_{i=1}^{n_\pi}$. If

1. Each Π_i has a hard factorization (Ψ_i, M_i).
2. There exists a continuously differentiable $P : \mathcal{P} \to \mathcal{S}^{n_x \times n_x}$, scalars $\lambda_i \geq 0$, and a scalar $\gamma > 0$ such that $P(p) \geq 0$ and

$$\begin{bmatrix} A^T P + PA + \partial P & PB_1 & PB_2 \\ B_1^T P & 0 & 0 \\ B_2^T P & 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C_2^T \\ D_{21}^T \\ D_{22}^T \end{bmatrix} \begin{bmatrix} C_2^T \\ D_{21}^T \\ D_{22}^T \end{bmatrix}^T + \sum_{i=1}^{n_\pi} \lambda_i \begin{bmatrix} C_{1,i}^T \\ D_{11,i}^T \\ D_{12,i}^T \end{bmatrix} M_i \begin{bmatrix} C_{1,i}^T \\ D_{11,i}^T \\ D_{12,i}^T \end{bmatrix}^T < 0 \tag{17}$$

   hold for all $p \in \mathcal{P}$ and all $q \in \dot{\mathcal{P}}$.

Then

a) For any $x(0) \in \mathbb{R}^{n_G + n_\psi}$ and $d \in L_2$, $\lim_{T\to\infty} x(T) = 0$

b) $\|F_u(G,\Delta)\| \leq \gamma$

In Equation (17) the dependence of the matrices on p and q has been omitted to shorten the notation.

Proof
To show b), define a parameter-dependent storage function $V : \mathbb{R}^{n_G+n_\psi} \times \mathbb{R}^{n_\rho} \to \mathbb{R}^+$ by $V(x,\rho) = x^T P(\rho) x$ and let $d \in L_2^{n_d}[0,\infty)$ be any input signal and $\rho \in \mathcal{T}$ any allowable parameter trajectory. From well-posedness, the interconnection $F_u(G,\Delta)$ has a solution that satisfies the dynamics in Equation (15). Left and right multiply Equation (17) by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ to show that V satisfies:

$$\dot{V}(t) + \sum_{i=1}^{n_\pi} \lambda_i z_i(t)^T M_i z_i(t) \leq \gamma^2 d(t)^T d(t) - e(t)^T e(t) \tag{18}$$

The dissipation inequality Equation (18) can be integrated from $t = 0$ to $t = T$ with the initial condition $x(0) = 0$ to yield:

$$V(x(T)) + \sum_{i=1}^{n_\pi} \lambda_i \int_0^T z_i(t)^T M_i z_i(t)\, dt \leq \gamma^2 \int_0^T d(t)^T d(t)\, dt - \int_0^T e(t)^T e(t)\, dt \tag{19}$$

Apply the hard IQC conditions, $\lambda_i \geq 0$, and $V \geq 0$ to show Equation (19) implies $\int_0^T e(t)^T e(t)\, dt \leq \gamma^2 \int_0^T d(t)^T d(t)\, dt$. Hence $\|F_u(G,\Delta)\| \leq \gamma$.

The proof for a) is more subtle but follows similar arguments to those given in [10]. First, note that Equation (17) still holds if the term $\epsilon \begin{bmatrix} I_{n_G+n_\psi} & 0 \\ 0 & 0_{n_w+n_d} \end{bmatrix}$ is added to the left hand side with $\epsilon > 0$ sufficiently small. Left and right multiply the modified Equation (17) to yield:

$$\dot{V}(t) + \sum_{i=1}^{n_\pi} \lambda_i z_i(t)^T M_i z_i(t) + \epsilon x^T(t) x(t) \leq \gamma^2 d(t)^T d(t) - e(t)^T e(t) \tag{20}$$

Consider now the response for any initial condition $x(0)$, input $d \in L_2$, and allowable trajectory $\rho \in \mathcal{T}$. Integrate Equation 20 from $t = 0$ to $t = T$ and apply the hard IQC conditions, $\lambda_i \geq 0$, and $V \geq 0$ to show

$$\epsilon \int_0^T x^T x\, dt \leq \gamma^2 \int_0^T d^T d\, dt - \int_0^T e^T e\, dt + V(x(0), \rho(0)) \tag{21}$$

As $T \to \infty$ this gives $\epsilon \|x\|_2^2 \leq \gamma^2 \|d\|_2^2 + V(x(0),\rho(0)) < \infty$. It follows that $x \in L_2$. A similar perturbation argument can be used to show that $v \in L_2$ and hence $w = \Delta(v) \in L_2$ by the assumed boundedness of Δ. The time derivative of x is given by $\dot{x} = A(\rho)x + B_1(\rho)w + B_2(\rho)d$. Therefore $\dot{x} \in L_2$ since $(x, w, d) \in L_2$ and A, B_1 and B_2 are bounded on $\mathcal{P}$. Finally, $(x, \dot{x}) \in L_2$ implies that $x(T) \to 0$ as $T \to \infty$ (e.g. see Appendix B of [5]). □

Conclusion (b) of Theorem 1 is essentially Theorem 2 in [15]. Conclusion (a) is the minor extension provided here. It provides a condition for internal, asymptotic stability in addition to the classical input/output stability result in (b). Theorem 1 is correct but there are two key issues. First, it parameterizes the IQC as conic combinations of individual hard factorizations, i.e. $\sum \lambda_i \Pi_i = \Psi^\sim M(\lambda)\Psi$ with a λ independent Ψ, see appendix A. The hard IQCs $\int_0^T z_i(t)^T M_i z_i(t) dt \geq 0$ are clearly used in the dissipation inequality proof. However, this approach cannot incorporate more general parameterizations $\Psi^\sim M(\lambda)\Psi$, e.g. as in [24] where Ψ is stable (possibly non-square) and $M(\lambda)$ is an affine function of λ. Second, Theorem 1 requires $P(p) \geq 0$ for all $p \in \mathcal{P}$. This is a natural assumption given the dissipation inequality approach used in the proof. However, the constraint $P(p) \geq 0$ can lead to conservative results. For example, the frequency domain condition in [11] can be applied when G is LTI. This frequency domain condition is equivalent to a similar LMI (by the KYP Lemma) but without the constraint $P(p) \geq 0$. A key insight is that the constraint $P(p) \geq 0$ neglects additional energy stored in the combined IQC multiplier. The main result in the next section addresses both of these key issues.

3. MAIN RESULT

In Theorem 1, it was assumed that the storage function $V(x,\rho) = x^T P(\rho) x$ is non-negative definite and the (Ψ_i, M_i) are hard factorizations. As noted above, this neglects the additional (hidden) energy stored in the combined IQC and also prevents the use of more general IQC parameterizations. A related result in [21] shows that, for a single IQC Π and LTI plant G, the constraint $P \geq 0$ can be dropped if a J-spectral factorization is used for the multiplier. The main result (Theorem 2 below) extends this for multiple IQCs and LPV plants G. This provides less conservative analysis results for the uncertain system $F_u(G,\Delta)$ by not enforcing $P(\rho) \geq 0$ and allowing for more general (not necessarily hard) IQC parameterizations. Theorem 2 again uses the interconnection of G and Ψ as shown in Fig. 2. It is assumed that the IQC has the form $\Pi = \Psi^\sim M(\lambda)\Psi$ so that the interconnection of G and Ψ again has a state-space representation as in Equation (15).

Theorem 2. Let G be a parametrically stable LPV system defined by Equation (3) and $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ be a bounded, causal operator such that $F_u(G,\Delta)$ is well-posed. Assume Δ satisfies the IQC parameterized by $\Pi(\lambda) = \Psi^\sim M(\lambda)\Psi$ with Ψ stable. If

1. The combined multiplier, partitioned as $\Pi(\lambda) = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\sim & \Pi_{22} \end{bmatrix}$, satisfies $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0 \; \forall \omega \in \mathbb{R} \cup \{\infty\}$ where Π_{11} is $n_v \times n_v$ and Π_{22} is $n_w \times n_w$.
2. There exists a continuously differentiable $P : \mathcal{P} \to \mathcal{S}^{n_x \times n_x}$, and a scalar $\gamma > 0$ such that

$$\begin{bmatrix} A^T P + PA + \partial P & PB_1 & PB_2 \\ B_1^T P & 0 & 0 \\ B_2^T P & 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} C_2^T \\ D_{21}^T \\ D_{22}^T \end{bmatrix} \begin{bmatrix} C_2^T \\ D_{21}^T \\ D_{22}^T \end{bmatrix}^T + \begin{bmatrix} C_1^T \\ D_{11}^T \\ D_{12}^T \end{bmatrix} M(\lambda) \begin{bmatrix} C_1^T \\ D_{11}^T \\ D_{12}^T \end{bmatrix}^T < 0 \qquad (22)$$

   hold for all $p \in \mathcal{P}$ and all $q \in \dot{\mathcal{P}}$.

Then

a) For any $x(0) \in \mathbb{R}^{n_G + n_\psi}$ and $d \in L_2$, $\lim_{T\to\infty} x(T) = 0$

b) $\|F_u(G,\Delta)\| \leq \gamma$

In Equation (22) the dependence of the matrices on p and q has been omitted to shorten the notation.

Proof
Define a parameter-dependent storage function $V : \mathbb{R}^{n_G+n_\psi} \times \mathbb{R}^{n_\rho} \to \mathbb{R}^+$ by $V(x,\rho) := x^T P(\rho) x$. Left and right multiply Equation (22) by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ to show that V satisfies:

$$\dot{V}(t) + z(t)^T M(\lambda) z(t) \leq \gamma^2 d(t)^T d(t) - e(t)^T e(t) \qquad (23)$$

This is not a valid dissipation inequality as neither $P(p) \geq 0$ nor $\int_0^T z(t)^T M(\lambda) z(t) dt \geq 0$ hold, in general. The proof is based on converting Equation (22) into an equivalent formulation with only a single, hard IQC $(\hat{\Psi}, J_{n_v,n_w})$ and a new matrix $\hat{P}(p) \geq 0$.

First, note that the state space representation of Ψ (Equation 10) can be used to express $z^T M z$ in terms of (ψ, v, w) as follows:

$$z^T M z = \begin{bmatrix} \psi \\ \hline v \\ w \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} \psi \\ \hline v \\ w \end{bmatrix} \tag{24}$$

where $Q := C_\psi^T M(\lambda) C_\psi$, $S := C_\psi^T M(\lambda) D_\psi$ and $R := D_\psi^T M(\lambda) D_\psi$. By assumption 1 and Lemma 1 in the appendix, it follows that the combined multiplier $\Pi(\lambda)$ has a J-spectral factorization. This can be shown using Game Theory conditions as has been detailed in [21]. The state space representation realization of the J-spectral factor $\hat{\Psi}$ is given by Lemma 1 as $(A_\psi, B_\psi, \hat{C}_\psi, \hat{D}_\psi)$ with $\hat{C}_\psi = J_{n_v,n_w} \hat{D}_\psi^{-T}(B_\psi^T X + S^T)$ and $\hat{D}_\psi$ satisfying $R = \hat{D}_\psi^T J_{n_v,n_w} \hat{D}_\psi$. The matrix $X = X^T$ is the unique, stabilizing solution of the algebraic Riccati equation (ARE) (Equation 38). This ARE can be re-written in terms of the state matrices of the J-spectral factor as:

$$Q = -A_\psi^T X - X A_\psi + \hat{C}_\psi^T J_{n_v,n_w} \hat{C}_\psi \tag{25}$$

Substitute for Q in Equation (24) using the ARE and use $S^T = \hat{D}_\psi^T J_{n_v,n_w} \hat{C}_\psi - B_\psi^T X$ to obtain

$$\begin{aligned} z^T M z = &-(A_\psi \psi + B_\psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right])^T X \psi - \psi^T X (A_\psi \psi + B_\psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]) \\ &+ (\hat{C}_\psi \psi + \hat{D}_\psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right])^T J_{n_v,n_w} (\hat{C}_\psi \psi + \hat{D}_\psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]) \end{aligned}$$

This can be simplified to the following expression:

$$z^T M z = -\dot{\psi}^T X \psi - \psi^T X \dot{\psi} + \hat{z}^T J_{n_v,n_w} \hat{z} \tag{26}$$

where ψ and $\hat{z}$ are the state and the output, respectively of the J-spectral factor $\hat{\Psi}$.

Define the modified matrix $\hat{P}(\rho) := P(\rho) - \left[\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right]$. This yields a modified storage function $\hat{V} : \mathbb{R}^{n_G + n_\psi} \times \mathbb{R}^{n_\rho} \to \mathbb{R}^+$ defined as $\hat{V}(x, \rho) = x^T \hat{P}(\rho) x$. This modified storage has the form $\hat{V}(x, \rho) = V(x, \rho) - \psi^T X \psi$ where the second term can be interpreted as hidden energy stored in the combined IQC multiplier. Substitute Equation (26) into Equation (23) to get

$$\dot{\hat{V}}(t) + \hat{z}(t)^T J_{n_v,n_w} \hat{z}(t) \leq \gamma^2 d(t)^T d(t) - e(t)^T e(t), \tag{27}$$

This dissipation inequality is equivalent to the linear matrix inequality (Equation 17) in Theorem 1 with a single IQC $(\hat{\Psi}, J_{n_v,n_w})$. It remains to show that $\hat{P}(p) \geq 0$ so that $\hat{V} \geq 0$ is a valid storage function.

Use the J-spectral factorization $(\hat{\Psi}, J_{n_v,n_w})$ to define the cost $\underline{J}(\psi_0)$ of the following max/min game:

$$\begin{aligned} \underline{J}(\psi_0) := \sup_{w \in L_2^{n_w}[0,\infty)} \inf_{v \in L_2^{n_v}[0,\infty)} &\int_0^\infty \hat{z}(t)^T J_{n_v,n_w} \hat{z}(t)\, dt \\ \text{subject to:} & \\ \dot{\psi} &= A_\psi \psi + B_{\psi 1} v(t) + B_{\psi 2} w(t), \quad \psi(0) = \psi_0 \\ \hat{z} &= \hat{C}_\psi \psi + \hat{D}_{\psi 1} v(t) + \hat{D}_{\psi 2} w(t) \end{aligned} \tag{28}$$

By Lemma 1 the cost of this max/min game is $\underline{J}(\psi_0) = 0$. Note that $\underline{J}(\psi_0)$ involves a max over w followed by a min over v. Hence the choice of v may depend on w. Choose v to be the output of the nominal LPV plant G generated by w with some initial condition $x_{G,0}$ and allowable parameter trajectory $\rho \in \mathcal{T}$. This specific choice of v yields a value that is no lower than the infimum over all possible $v \in L_2$. Hence, $\hat{V}^*(x_0) \geq \underline{J}(\psi_0) = 0$, where $\hat{V}^*(x_0)$ is defined as:

$$\hat{V}^*(x_0) := \sup_{\substack{w \in L_2^{n_w} \\ \rho \in \mathcal{T}}} \int_0^\infty \hat{z}(t)^T J_{n_v,n_w} \hat{z}(t)\, dt \tag{29}$$

$$\begin{aligned} &\text{subject to:} \\ &\dot{x}(t) = A(\rho(t))x(t) + B_1(\rho(t))w(t), \quad x(0) = x_0 \\ &\hat{z}(t) = C(\rho(t))x(t) + D_{11}(\rho(t))w(t) \end{aligned}$$

where (A, B_1, C, D_{11}) are the state matrices obtained by connecting G and Ψ as in the analysis interconnection (Figure 2) but neglecting the disturbance d and error e signals. As before $x := \left[\begin{smallmatrix} x_G \\ \psi \end{smallmatrix}\right] \in \mathbb{R}^{n_G+n_\psi}$ denotes the extended state of this interconnection.

The last step of the proof is to show $\hat{V}(x_0) \geq \hat{V}^*(x_0)$ for all x_0. This follows along the lines of Theorems 2 and 3 in [26] and hence the proof is only sketched. Let $x(t)$ and $z(t)$ be the resulting solutions of the interconnection G and $\hat{\Psi}$ for a given $w(t)$, $\rho(t)$ and x_0. Disregarding the performance inputs and outputs d and e, Equation (27) can be integrated from $t = 0$ to $t = T$ resulting in

$$\hat{V}(x(T), \rho(T)) + \int_0^T \hat{z}(t)^T J_{n_v,n_w} \hat{z}(t)\, dt \leq \hat{V}(x_0, \rho(0)) \tag{30}$$

By assumption G is parametrically stable (Definition 1) and hence for any $w \in L_2$ and initial condition $x_G(0)$ it follows that $\lim_{T\to\infty} x_G(T) = 0$ and $v \in L_2$. Moreover, the stability of $\hat{\Psi}$ and $w, v \in L_2$ together imply that $\lim_{T\to\infty} \psi(T) = 0$. Hence, $\lim_{T\to\infty} x(T) = 0$ and therefore $\lim_{T\to\infty} \hat{V}(x(T), \rho(T)) = 0$. Maximizing the left hand side of Equation (30) over $w \in L_2^{n_w}[0, \infty)$ for $T = \infty$ yields $\hat{V}(x_0, \rho(0)) \geq \hat{V}^*(x_0)$.

To summarize, it has been shown that $\hat{V}(x, \rho) = x^T \hat{P}(\rho) x$ satisfies the dissipation inequality in Equation (27). This dissipation inequality is equivalent to the linear matrix inequality (Equation 17) in Theorem 1 with a single IQC $(\hat{\Psi}, J_{n_v,n_w})$. Moreover, $\hat{V}(x_0) \geq \hat{V}^*(x_0) \geq \underline{J}(\psi_0) = 0$ for all x_0. Hence $\hat{P}(\rho) \geq 0$. Finally, the J-spectral factorization $(\hat{\Psi}, J_{n_v,n_w})$ is a hard IQC by Lemma 1. Hence Theorem 1 can be applied to conclude that statements (a) and (b) hold. □

Theorem 2 has two main benefits as compared to Theorem 1. First, it drops the constraint $P(p) \geq 0$. Second, it allows for more general IQC parameterizations that are not necessarily hard factorizations. This can reduce the conservatism in the analysis as demonstrated in the examples below. Theorem 2 adds one technical restriction (Condition 1) on the combined multiplier $\Pi(\lambda)$. This restriction is related to the one presented in [23, 25]. The proof uses this technical condition to obtain a J-spectral factorization for the combined multiplier. This allows the analysis condition to be converted into a valid dissipation inequality with a non-negative storage function and a single (hard) IQC. Note that this is primarily a practical advantage. From a theoretical point of view both theorems are equivalent but Theorem 2 allows searching over a broader class of IQC parameterizations, as the ones presented in [24].

The implementation of Theorem 2 involves some numerical issues. These are briefly described here and more details can be found in [15]. If the IQC is parameterized such that $M(\lambda)$ is an affine function of λ then Theorem 2 involves parameter dependent LMI conditions in the variables $P(\rho)$ and λ. Note that the entries of λ do not have to be nonnegative as is required in Theorem 1. λ only needs to satisfy condition 1 in Theorem 2, i.e. $\Pi_{11} > 0$ and $\Pi_{22} < 0$. These are infinite dimensional (one LMI for each $(p, q) \in \mathcal{P} \times \dot{\mathcal{P}}$) and they are typically approximated with finite-dimensional LMIs evaluated on a grid of parameter values. Additionally, the main decision variable is the function $P(\rho)$ which must be restricted to a finite dimensional subspace. A common practice [2, 30] is to restrict $P(\rho)$ to be a linear combination of user-specified basis functions. The analysis can then be performed as a finite-dimensional SDP [4], e.g. minimizing γ subject to the approximate finite-dimensional LMI conditions. This paper focused on gridded LPV systems whose state matrices have an arbitrary dependence on the parameter. If the LPV system has a rational dependence on ρ then finite dimensional LMI conditions can be derived (with no gridding) using the techniques in [1, 13].

4. NUMERICAL EXAMPLE

4.1. Simple LTI Example

The first example is a simple nominal LTI system G under a perturbation Δ described by two IQC multipliers Π_1 and Π_2. The nominal system G is given by

$$\begin{aligned}\dot{x}_G &= -0.5\,x_G + \begin{bmatrix}0.1 & -1\end{bmatrix}\begin{bmatrix}w\\ d\end{bmatrix}\\ \begin{bmatrix}v\\ e\end{bmatrix} &= \begin{bmatrix}-1\\ 0.1\end{bmatrix} x_G + \begin{bmatrix}-1 & 0\\ -1 & 0\end{bmatrix}\begin{bmatrix}w\\ d\end{bmatrix}.\end{aligned} \tag{31}$$

The first IQC multiplier is $\Pi_1 = \Psi_1^\sim M \Psi_1$ with $M = \left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$ and $\Psi_1 = \left[\begin{smallmatrix}-3 & 1\\ 1 & 6\end{smallmatrix}\right]$. This is a static multiplier with no dynamics in Ψ_1. The second IQC multiplier is $\Pi_2 = \Psi_2^\sim M \Psi_2$ with Ψ_2 given by

$$\begin{aligned}\dot{\psi} &= -0.2\,\psi + \begin{bmatrix}-1 & -10\end{bmatrix}\begin{bmatrix}v\\ w\end{bmatrix}\\ z &= \begin{bmatrix}0\\ -0.1\end{bmatrix}\psi + \begin{bmatrix}1 & 0\\ 0 & 0.5\end{bmatrix}\begin{bmatrix}v\\ w\end{bmatrix}.\end{aligned} \tag{32}$$

Both Ψ_1 and Ψ_2 are stable with stable inverses. Thus $\{(\Psi_i, M)\}_{i=1}^2$ are both J-spectral factorizations and hence hard factorizations (Lemma 1 in Appendix A). Invoking Theorem 1 with the IQC parameterization $\Pi(\lambda) = \lambda_1\Pi_1 + \lambda_2\Pi_2$ yields a worst case gain bound of $\gamma_1 = 6.8302$. This is solved by minimizing γ subject to the LMI conditions $P \geq 0$ and Equation (17). Using instead Theorem 2, i.e. dropping the constraint $P \geq 0$, results in $\gamma_2 = 5.1983$. The optimal decision variables in this case are $P^* = \left[\begin{smallmatrix}17.2867 & 0.5897\\ 0.5897 & 0.0108\end{smallmatrix}\right]$, $\lambda_1^* = 0.1986$, and $\lambda_2^* = 0.878$. The resulting P has eigenvalues at 17.3068 and -0.0093 and is therefore indefinite. By Lemma 1, a J-spectral factorization of the combined multiplier $\Pi(\lambda^*) = \lambda_1^*\Pi_1 + \lambda_2^*\Pi_2$ can be constructed. The stabilizing solution of the ARE for $\Pi(\lambda^*)$ is $X = -0.0126$. This yields a modified storage function $\hat{P} = P - \left[\begin{smallmatrix}0 & 0\\ 0 & X\end{smallmatrix}\right]$, as described in the proof of Theorem 2. As expected, $\hat{P} \geq 0$ with eigenvalues at 17.3068 and 0.0033. This simple example demonstrates that enforcing $P \geq 0$ with multiple IQCs will yield conservative results. Theorem 2 provides a valid dissipation inequality proof (under additional technical assumptions on $\Pi(\lambda)$) even if the constraint $P \geq 0$ and the hard IQC assumption are dropped.

4.2. LPV Example

The second example is the same example used in [15] except that the uncertainty is assumed to be a real parameter in addition to norm bounded. The example represents a feedback interconnection of a first-order LPV system with a gain-scheduled proportional-integral controller as shown in Fig. 3. The system H, taken from [22], is first order with dependence on a single parameter ρ. It can be written as

$$\begin{aligned}\dot{x}_H &= -\frac{1}{\tau(\rho)}x_H + \frac{1}{\tau(\rho)}u_H\\ y_H &= K(\rho)x_H\end{aligned} \tag{33}$$

with the time constant $\tau(\rho)$ and output gain $K(\rho)$ depending on the scheduling parameter as follows:

$$\begin{aligned}\tau(\rho) &= \sqrt{133.6 - 16.8\rho},\\ K(\rho) &= \sqrt{4.8\rho - 8.6}.\end{aligned} \tag{34}$$

The scheduling parameter and rate are restricted to $\rho \in [2, 7]$ and $|\dot{\rho}| \leq 1$. For the following analysis a grid of six points is used that span the parameter space equidistantly. A time-delay of

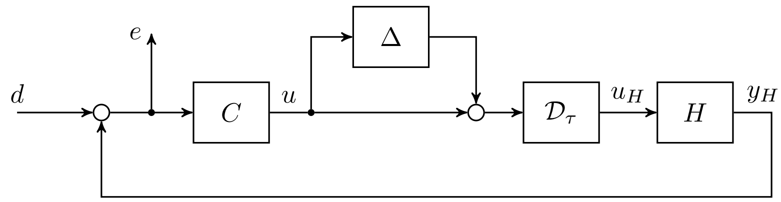

Figure 3. Closed Loop Interconnection with Parametric Uncertainty

0.5 seconds is included at the control input. The time delay is represented by a second order Pade approximation, denoted $\mathcal{D}_\tau$:

$$\mathcal{D}_\tau(s) = \frac{\frac{(T_d s)^2}{12} - \frac{T_d s}{2} + 1}{\frac{(T_d s)^2}{12} + \frac{T_d s}{2} + 1}, \tag{35}$$

where $T_d = 0.5$. A gain-scheduled PI-controller C is designed that guarantees a closed loop damping $\zeta_{cl} = 0.7$ and a closed loop frequency $\omega_{cl} = 0.25$ at each frozen value of ρ. The controller gains that satisfy these requirements are given by

$$\begin{aligned} K_p(\rho) &= -\frac{2\zeta_{cl}\omega_{cl}\tau(\rho) - 1}{K(\rho)}, \\ K_i(\rho) &= -\frac{\omega_{cl}^2 \tau(\rho)}{K(\rho)}. \end{aligned} \tag{36}$$

The controller is realized in the following state space form:

$$\begin{aligned} \dot{x}_c &= K_i(\rho) e \\ u &= x_c + K_p(\rho) e \end{aligned} \tag{37}$$

The uncertainty Δ is assumed to be real, constant scalar satisfying $|\Delta| \le 0.5$. Δ satisfies IQC multipliers of the form $\Pi(\lambda) := \Psi^\sim M(\lambda)\Psi$ where $\Psi(j\omega) := \begin{bmatrix} \bar\Psi(j\omega) & 0 \\ 0 & \bar\Psi(j\omega) \end{bmatrix}$, $M := \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2^T & -\Lambda_1 \end{bmatrix}$, $\Lambda_1 = \Lambda_1^T$, $\Lambda_2 = -\Lambda_2^T$, ψ stable, and $\Psi(j\omega)^\sim \Lambda_1 \Psi(j\omega) > 0$ for all ω. See [19] for details. In this example, $\bar\Psi$ is chosen as $\bar\Psi = \begin{bmatrix} 1 & \frac{1}{s+1.5} & \frac{0.5}{s+1} \end{bmatrix}^T$.

Using an affine parameter dependence for P, i.e. $P(p) = P_0 + pP_1$, and restricting $P(p) \ge 0$ yields a (smallest) bound on the worst case gain of $\gamma_1 = 10.28$. Applying the analysis condition in Theorem 2, i.e. removing the positivity constraint $P(p) \ge 0$, improves the bound to $\gamma_2 = 9.06$. The minimum eigenvalue of the optimal $P^*(p)$ is between -1636.9 at $p = 2$ and -1183.8 at $p = 7$. The optimal decision variables Λ_1^* and Λ_2^* in this case are

$$\Lambda_1^* = \begin{bmatrix} 12.6 & -56.26 & 50.17 \\ -56.26 & 2560.9 & -3317.6 \\ 50.17 & -3317.6 & 4353 \end{bmatrix}, \qquad \Lambda_2^* = \begin{bmatrix} 0 & -639.6 & 1027.7 \\ 639.6 & 0 & -1257.8 \\ -1027.7 & 1257.8 & 0 \end{bmatrix}.$$

The stabilizing solution of the ARE for $\Pi(\Lambda^*)$ is

$$X = \begin{bmatrix} 18.99 & -24.51 & 0 & -17935 \\ -24.51 & 32 & 17935 & 0 \\ 0 & 17935 & -75.96 & 98.04 \\ -17935 & 0 & 98.04 & -127.98 \end{bmatrix}.$$

This yields a modified storage function $\hat{P}(p) = P(p) - \left[\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right]$ with minimum eigenvalue between 1.12 at $p = 2$ and 6.76 at $p = 7$. This example demonstrates that Theorem 2 enables the use of more general IQC parameterizations and also reduces the conservatism by dropping the constraint $P(p) \ge 0$.

5. CONCLUSIONS

This paper derived new robustness analysis conditions for uncertain LPV systems using dissipativity theory and IQCs. Unlike previous results, the new conditions require neither a hard factorization of the IQC nor a non-negative definite storage function. The proof of this new result used a time-domain characterization that included the additional energy that is implicitly stored by the combined IQC. Simple numerical examples demonstrated that the new conditions are less conservative than previous results.

ACKNOWLEDGMENTS

This work was partially supported by the AFOSR under the grant entitled A Merged IQC/SOS Theory for Analysis of Nonlinear Control Systems, Dr. Fahroo technical monitor. This work was also partially supported by the National Science Foundation under Grant No. NSF-CMMI-1254129 entitled CAREER: Probabilistic Tools for High Reliability Monitoring and Control of Wind Farms.

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APPENDIX

A. J-SPECTRAL FACTORIZATION

This appendix provides numerical procedures to factorize $\Pi = \Pi^\sim \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ as $\Psi^\sim M \Psi$. Such factorizations are not unique and this appendix provides two specific factorizations. The second of these factorizations (Lemma 1) is particularly useful for use in time-domain dissipation inequality results. First, let $(A_\pi, B_\pi, C_\pi, D_\pi)$ be a minimal state-space realization for Π. Separate Π into its stable and unstable parts $\Pi = G_S + G_U$. Let (A, B, C, D_π) denote a state space realization for the stable part G_S so that A is Hurwitz. The assumptions on Π imply that G_U has a state space realization of the form $(-A^T, -C^T, B^T, 0)$ (Section 7.3 of [9]). Thus $\Pi = G_S + G_U$ can be written as $\Pi = \Psi^\sim M \Psi$ where $\Psi(s) := \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix}$ and $M := \begin{bmatrix} 0 & C^T \\ C & D_\pi \end{bmatrix}$. This provides a factorization $\Pi = \Psi^\sim M \Psi$ where $M = M^T \in \mathbb{R}^{n_z \times n_z}$ and $\Psi \in \mathbb{RH}_\infty^{n_z \times (n_v+n_w)}$. For this factorization Ψ is, in general, non-square ($n_z \neq (n_v + n_w)$).

The stability theorems in this paper require a special J-spectral factorization [3] such that Ψ is square ($n_z = (n_v + n_w)$), stable, and with a stable inverse. More precisely, given non-negative integers p and q, let $J_{p,q}$ denote the signature matrix $\begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$. $\hat{\Psi}$ is called a J-spectral factor of Π if $\Pi = \hat{\Psi}^\sim J_{p,q} \hat{\Psi}$ and $\hat{\Psi}, \hat{\Psi}^{-1} \in \mathbb{RH}_\infty^{(n_v+n_w)\times(n_v+n_w)}$. Lemma 1 provides a simple frequency domain condition that is sufficient for the existence of a J-spectral factor. In addition, this lemma provides several useful properties of J-spectral factorizations.

Lemma 1. Let $\Pi(s) := \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix}^\sim \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix} \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ be given with A Hurwitz, $Q = Q^T$ and $R = R^T$. Partition Π as $\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\sim & \Pi_{22} \end{bmatrix}$ where $\Pi_{11} \in \mathbb{RL}_\infty^{n_v \times n_v}$ and $\Pi_{22} \in \mathbb{RL}_\infty^{n_w \times n_w}$. If $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ then

1. R is nonsingular and there exists a unique real solution $X = X^T$ to the Algebraic Riccati Equation
$$A^T X + XA - (XB + S)R^{-1}(B^T X + S^T) + Q = 0 \tag{38}$$
such that $A - BR^{-1}\left(B^T X + S^T\right)$ is Hurwitz.

2. Π has a J-spectral factorization $(\hat{\Psi}, J_{n_v,n_w})$. Moreover, $\hat{\Psi}$ is a J-spectral factor of Π if and only if it has a state-space realization
$$\left[\begin{array}{c|c} \hat{A}_\psi & \hat{B}_\psi \\ \hline \hat{C}_\psi & \hat{D}_\psi \end{array}\right] := \left[\begin{array}{c|c} A & B \\ \hline J_{n_v,n_w} W^{-T}\left(B^T X + S^T\right) & W \end{array}\right] \tag{39}$$
where W is a solution of $R = W^T J_{n_v,n_w} W$.

3. $(\hat{\Psi}, J_{n_v,n_w})$ is a hard factorization of Π.

4. The cost of the max/min game defined in Equation 40 based on $(\hat{\Psi}, J_{n_v,n_w})$ is $\underline{J}(\psi_0) = 0$.

$$\begin{aligned} \underline{J}(\psi_0) := & \sup_{w \in L_2^{n_w}[0,\infty)} \inf_{v \in L_2^{n_v}[0,\infty)} \int_0^\infty \hat{z}(t)^T J_{n_v,n_w} \hat{z}(t)\, dt \\ & \text{subject to:} \\ & \dot{\psi} = \hat{A}_\psi \psi + \hat{B}_\psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right], \quad \psi(0) = \psi_0 \\ & \hat{z} = \hat{C}_\psi \psi + \hat{D}_\psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] \end{aligned} \tag{40}$$

where $(\hat{A}_\psi, \hat{B}_\psi, \hat{C}_\psi, \hat{D}_\psi)$ is a state space realization for $\hat{\Psi}$.

Proof

This lemma follows from results in [21]. Briefly, the sign definite conditions on Π_{11} and Π_{22} can be used to show that Π has no equalizing vectors as defined in [12]. Thus the Riccati Equation 38 has a unique stabilizing solution (Theorem 2.4 in [12]). Statements 3 and 4 follow from known results on linear quadratic games [7, 8]. $\square$