Model Order Reduction by Parameter-Varying Oblique Projection

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***Abstract*— A method to reduce the dynamic order of linear parameter-varying (LPV) systems in grid representation is developed in this paper. It approximates balancing and truncation by an oblique projection onto a dominant subspace. The approach is novel in its use of a parameter-varying kernel to define the direction of this projection. Parameter-varying state transformations in general lead to parameter rate dependence in the model. The proposed projection avoids this dependence and maintains a consistent state space basis for the reduced-order system. The method is compared with LPV balancing and truncation for a nonlinear mass-spring-damper system. It is shown to yield similar accuracy, while the required computation time is reduced by a factor of almost 100,000.**

I. Introduction

Linear parameter-varying (LPV) models are particularly useful for the design of gain-scheduled controllers due to the availability of powerful synthesis techniques and computational tools [1], [2]. These techniques produce controllers that are at least of the same dynamic order as the plant model. One limitation is that the synthesis requires the solution of linear matrix inequalities (LMIs). The computation required for this synthesis grows rapidly with increasing state dimension. For many physically motivated models, directly obtaining models with a low number of states is not easy. For instance, structural mechanics models are often obtained from finite element analysis with a dense grid of nodes and hence these models have a large number of states. Similarly, unsteady aerodynamic models often have several thousands of states. The method proposed in this paper can be used to obtain low-order models and hence reduce the computation required for LPV synthesis.

The foundation of LPV model order reduction was established in [3], [4] by extending the concept of balancing and truncation [5] to LPV systems. Balancing and truncation consists of a state transformation followed by removing states that are considered negligible in the new coordinates. The extension to LPV systems requires the solution of LMIs to obtain generalized Gramians. Hence, this approach suffers from the same computational limitations as the LPV synthesis problem. In addition, LPV balancing in general uses parameter-varying transformations. As a result, the reduced order model depends not only on the parameter but also on the parameter rate of change. These two aspects are among the major obstacles in model order reduction for LPV systems. Consequently, several approaches have been proposed that use linear time invariant (LTI) techniques for frozen-parameter models and then seek to interpolate the reduced-order models for time-varying parameters, e. g., [6], [7], [8], [9]. In recent years, the problem of *parametric model reduction* has also received considerable attention, e. g., [10], [11], [12]. Parametric model reduction considers only constant parameter values and the goal is to approximate a family of parameterized LTI models. This differs substantially from the *LPV model order reduction* problem studied in this paper, which considers time-varying parameter values and whose goal is to approximate an LPV model, see Section II.

In this paper, an approximation to balancing and truncation is proposed. It defines a reduced state space from projection onto a dominant subspace. The approach is novel in its use of a parameter-varying kernel to define the direction of this projection. It is shown in Section III that no rate dependence is introduced by this projection and that a consistent state space basis is maintained throughout the entire parameter space. The proposed procedure only requires the solution of two Lyapunov equations at each grid point. It can thus be applied to models with up to a few thousand states, whereas the LMI approach is currently limited to models with up to about 50–100 states. When the original model is stable, the reduced-order model has all its poles in the complex left half plane for all frozen parameter values. This property guarantees stability for "slowly" varying parameter trajectories, see [13]. Further, an estimate of the full state vector can be recovered from the projection. This is a crucial feature for simulation and control if the LPV model is meant to approximate a nonlinear dynamic system. The effectiveness of the approach is compared to LPV balancing and truncation in Section IV on a nonlinear spring-mass-damper system with 100 states. The proposed method is shown to achieve similar accuracy with a major reduction in computational effort. It is further shown to accurately reduce a system with 1,000 states, where balancing becomes computationally intractable.

II. LPV Model Order Reduction

In this section, LPV systems are introduced and the model order reduction problem is formulated. The problems arising from parameter-varying transformations are highlighted, and it is shown how model order reduction can be stated as an oblique projection.

A. LPV Modeling and Reduced Order Models

LPV systems are dynamic systems whose state space matrices are continuous functions of a time-varying parameter

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vector $\rho(t) \in \mathbb{R}^{n_\rho}$. Based on physical considerations, the admissible parameter trajectories are confined to a compact set $\mathcal{P} \subset \mathbb{R}^{n_\rho}$. This infinite dimensional set is commonly approximated by a finite dimensional subset $\{\rho_k\}_{k=1}^{g} \subset \mathcal{P}$, called a grid. The state space equations for an LPV system with state vector $x(t) \in \mathbb{R}^{n_x}$ and input vector $u(t) \in \mathbb{R}^{n_u}$ are

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))\,x(t) + B(\rho(t))\,u(t) - \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}(\rho(t)) \\ y(t) &= C(\rho(t))\,x(t) + D(\rho(t))\,u(t), \end{aligned} \tag{1}$$

where the term $\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}(\rho(t))$ is included to allow a parameter-varying equilibrium point $\bar{x}(\rho(t))$. Such a term naturally arises if an LPV is obtained as the linearization of a nonlinear system with respect to a parameter-varying trim condition, see [14] for details. For a constant $\bar{x}$, the notion commonly encountered in the literature is recovered.

The problem of LPV model order reduction consists of finding an approximation for (1) as

$$\begin{aligned} \dot{z}(t) &= A_{\text{red}}(\rho(t))\,z(t) + B_{\text{red}}(\rho(t))\,u(t) - \frac{\mathrm{d}}{\mathrm{d}t}\bar{z}(\rho(t)) \\ y(t) &= C_{\text{red}}(\rho(t))\,z(t) + D_{\text{red}}(\rho(t))\,u(t)\ . \end{aligned} \tag{2}$$

The reduced state $z(t) \in \mathbb{R}^{n_z}$ should be of much lower dimension than $x(t) \in \mathbb{R}^{n_x}$, while the input-output behavior from u to y should be as similar as possible to that of the original model. Further, stability of the original model should be preserved in the reduced-order model. Finally, the equilibrium $\bar{x}$ needs to be sufficiently well approximated by $\bar{z}$ so that the results can be related back to the original nonlinear system.

In the remainder of this paper, time dependence is dropped and parameter dependence is denoted by the subscript ρ, i. e., $A_\rho := A(\rho(t))$.

B. Balancing and Truncation for LTI Model Reduction

For a fixed parameter $\rho = \rho_k$, system (1) simplifies to the standard LTI system

$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C\,x + D\,u\ . \end{aligned} \tag{3}$$

A standard model reduction method for LTI systems is balancing and truncation [5]. The first step is obtaining the controllability Gramian X_{c} and the observability Gramian X_{o} as solutions to the Lyapunov equations

$$A\,X_{\mathrm{c}} + X_{\mathrm{c}}A^T + B\,B^T = 0\ , \tag{4}$$

$$A^T X_{\mathrm{o}} + X_{\mathrm{o}}\,A + C^T C = 0\ . \tag{5}$$

Given a state x_0, the minimum energy required to steer the system from $x = 0$ to $x = x_0$ is $\epsilon_{\mathrm{c}} = x_0^T X_{\mathrm{c}}^{-1} x_0$. Further, $\epsilon_{\mathrm{o}} = x_0^T X_{\mathrm{o}}\,x_0$ is the energy of the free response to the initial condition x_0 [5]. The ratio $\epsilon_{\mathrm{o}}/\epsilon_{\mathrm{c}}$ thus measures how much a state is affected by the input and how much it affects the output. A transformation $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = T\,x$ can be calculated so that $T\,X_{\mathrm{c}}\,T^T = (T^{-1})^T X_{\mathrm{o}}\,T^{-1} = \Sigma_H^{1/2}$, where Σ_H is diagonal and contains the eigenvalues of the product $X_{\mathrm{c}}\,X_{\mathrm{o}}$ in descending order of magnitude. These singular values are exactly the ratios $\epsilon_{\mathrm{o}}/\epsilon_{\mathrm{c}}$ for each state in the new coordinates. System (3) can hence be partitioned as

$$\begin{aligned} \dot{x}_1 &= A_{11}\,x_1 + A_{12}\,x_2 + B_1\,u \\ \dot{x}_2 &= A_{21}\,x_1 + A_{22}\,x_2 + B_2\,u \\ y &= \ C_1\,x_1 + \ C_2\,x_2 + \ D\,u\ . \end{aligned} \tag{6}$$

The states that are both highly controllable and observable are represented by $z := x_1$. The states x_2 contribute little to the input-output behavior and are removed from the state vector by truncation, leading to a reduced-order model

$$\begin{aligned} \dot{z} &= A_{11}\,z + B_1\,u \\ y &= \ C_1\,z + \ D\,u\ . \end{aligned} \tag{7}$$

C. Balancing for LPV Models

For parameter-varying systems as defined by (1), balancing was extended in [3] by introducing parameter-varying generalized Gramians $X_{\mathrm{c},\rho}$ and $X_{\mathrm{o},\rho}$ that satisfy the LMIs

$$-\frac{\mathrm{d}}{\mathrm{d}t}X_{\mathrm{c},\rho} + A_\rho\,X_{\mathrm{c},\rho} + X_{\mathrm{c},\rho}A_\rho^T + B_\rho\,B_\rho^T \prec 0\ , \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}X_{\mathrm{o},\rho} + A_\rho^T X_{\mathrm{o},\rho} + X_{\mathrm{o},\rho}\,A_\rho + C_\rho^T C_\rho \prec 0\ . \tag{9}$$

Minimization of $\operatorname{trace}(X_{\mathrm{c},\rho}X_{\mathrm{o},\rho})$ subject to (8) and (9) can be used to calculate a parameter-varying balancing transformation so that $T_\rho\,X_{\mathrm{c},\rho}\,T_\rho^T = (T_\rho^{-1})^T X_{\mathrm{o},\rho}\,T_\rho^{-1} = \Sigma_{H,\rho}^{1/2}$, see [4] for an iterative approach to this nonconvex problem. The diagonal matrix $\Sigma_{H,\rho}$ in this case contains the parameter-varying eigenvalues of $X_{\mathrm{c},\rho}\,X_{\mathrm{o},\rho}$ ordered by decreasing magnitude along its diagonal. Such a transformation implies

$$\left[\begin{smallmatrix} \dot{x}_1 \\ \dot{x}_2 \end{smallmatrix}\right] = \frac{\mathrm{d}}{\mathrm{d}t}T_\rho\,x = \frac{\partial T_\rho}{\partial \rho}\,\dot{\rho}\,x + T_\rho\,\dot{x} \tag{10}$$

and consequently the resultant system

$$\begin{aligned} \left[\begin{smallmatrix} \dot{x}_1 \\ \dot{x}_2 \end{smallmatrix}\right] &= \left(T_\rho A_\rho + \frac{\partial T_\rho}{\partial \rho}\dot{\rho}\right) T_\rho^{-1} \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] + T_\rho B_\rho u - T_\rho \frac{\mathrm{d}}{\mathrm{d}t}\bar{x}_\rho \\ y &= C_\rho\,T_\rho^{-1}\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] + D_\rho\,u\ , \end{aligned} \tag{11}$$

depends on the parameter rate $\dot{\rho}$ in addition to the original parameter ρ. A parameter-varying transformation thus inevitably increases the complexity of the model since the parameter space is enlarged. A reduced-order LPV model without additional rate dependence can be obtained as described in II-B only if solutions to (8) and (9) are restricted to parameter independent matrices. In this case, more conservative solutions are to be expected. Further, even such parameter independent solutions require extensive computational effort to be calculated by numerical methods.

D. Projection Perspective on Model Order Reduction

The truncation operation applied to turn (6) into (7) can be expressed as replacing $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} I_{n_z} & 0_{n_z \times (n_x - n_z)} \end{smallmatrix}\right]^T x_1$ and multiplying the state equation from the left by $\left[\begin{smallmatrix} I_{n_z} & 0_{n_z \times (n_x - n_z)} \end{smallmatrix}\right]$. An equivalent representation of the reduced-order system (7) is thus

$$\begin{aligned} \dot{z} &= \overbrace{W^T A\,V}^{A_{\text{red}}}\ z + \overbrace{W^T B}^{B_{\text{red}}}\ u \\ y &= \underbrace{C\,V}_{C_{\text{red}}}\ z + D\,u \end{aligned} \tag{12}$$

with $V = T^{-1}\left[I_{n_z}\ 0_{n_z\times(n_x-n_z)}\right]^T$, $W^T = \left[I_{n_z}\ 0_{n_z\times(n_x-n_z)}\right]T$. It is shown in this section that the reduced-order model (12) obtained from balancing and truncation is a Petrov-Galerkin approximation of the original system, i. e., an approximation obtained by oblique projection. Taking this perspective makes it possible to extend model order reduction by projection in Section III-A to LPV systems and to consequently construct an approximation to LPV balancing and truncation in Section III-B.

An oblique projection is a linear operation defined by a matrix $\Pi = V\,(W^TV)^{-1}W^T$ with $V \in \mathbb{R}^{n_x\times n_z}$, $W \in \mathbb{R}^{n_x\times n_z}$ and $\operatorname{rank}(W^TV) = n_z$. Hence, a projection is idempotent, i. e., $\Pi = \Pi^2$. It is completely characterized by its range space $\operatorname{span}(\Pi) = \operatorname{span}(V)$ and its nullspace $\ker(\Pi) = \operatorname{span}\big(\Pi^T\big)^\perp = \operatorname{span}(W)^\perp$. This fact is easy to prove by replacing V and W with their respective thin QR-factorizations. A vector space is said to be projected by Π *along* the orthogonal complement of the subspace spanned by the columns of W and *onto* a subspace spanned by the columns of V. The projection thus restricts a given vector space $\mathcal{X} \subseteq \mathbb{R}^{n_x}$ to a lower dimensional subspace $\operatorname{span}(V) \subseteq \mathbb{R}^{n_z}$. Reference [15] shows that any projection can be parameterized by V and a symmetric positive definite matrix $S \in \mathbb{R}^{n_x\times n_x}$ as

$$\Pi = V\,\underbrace{(V^TS\,V)^{-1}\,V^TS}_{W^T}\ . \tag{13}$$

Any W constructed in this way is biorthogonal to V, i. e., $W^TV = I_{n_z}$. Thus, from this point on, biorthogonality of V and W is assumed without loss of generality.

It helps to apply some geometrical interpretation at this point. Given V and W with $W^TV = I_{n_z}$ and a point $x \in \mathcal{X}$, the projection of x lies in the span of V and can hence be written as $V\,z$ with some coefficient vector $z \in \mathbb{R}^{n_z}$. The component of x that is eliminated by the projection is in the nullspace of Π and hence orthogonal to W. This can be stated as $W^T(x - V\,z) = 0$. Consequently, the coefficient vector is uniquely determined by $z = W^Tx$. The projection $\Pi\,x$ can thus be seen as an approximation to x in $\operatorname{span}(V)$ with zero error within $\operatorname{span}(W)$. The subspace $\operatorname{span}(V)$ is consequently termed *basis space* of the approximation and $\operatorname{span}(W)$ is called *test space*.

Model order reduction requires the approximation of a dynamic system given by a differential equation rather than an approximation for a single point in the state space. The goal is thus to find an approximate solution $x_{\text{approx}} = V\,z$ to the state equation in (3), i. e.

$$\underbrace{\dot{x}_{\text{approx}}}_{V\,\dot{z}} \approx A\ \underbrace{x_{\text{approx}}}_{V\,z} + B\,u\ . \tag{14}$$

From the previous discussion, $V\,z$ is uniquely determined for given V, W, and x. Hence, the right hand side of (14) is known for a given state x and input u. This does however not immediately provide a solution for $\dot{z}$. In fact, (14) imposes n_x equations to determine the n_z-dimensional vector $\dot{z}$ and consequently no $\dot{z}$ exists that exactly satisfies (14). The residual of the approximation (14) is

$$r := V\dot{z} - (A\,V\,z + B\,u)\ . \tag{15}$$

If $\dot{z}$ is now selected such that the residual (15) is restricted to be orthogonal to the test space $\operatorname{span}(W)$, i. e.

$$W^T\,(V\dot{z} - (A\,V\,z + B\,u)) = 0\ , \tag{16}$$

the procedure is known as *Petrov-Galerkin approximation*, see e. g. [16], [17], [18]. The unique solution to (16) is

$$\dot{z} = W^TA\,V\,z + W^TB\,u\ . \tag{17}$$

The desired approximation is hence given by $x_{\text{approx}} = V\,z$, where z is the solution to (17). Adding the output equation $y = C\,x_{\text{approx}} + D\,u$ to (17) then immediately yields the reduced-order model (12). This shows, that balancing and truncation is indeed a Petrov-Galerkin approximation. Left multiplication of (17) by V further shows that this approximation is in fact obtained by projecting the dynamic system (3) along $\operatorname{span}(W)^\perp$ onto $\operatorname{span}(V)$, i. e.

$$\dot{x}_{\text{approx}} = V\,W^T\,(A\,x_{\text{approx}} + B\,u)\ . \tag{18}$$

There is a rich geometric interpretation for this projection, see [18] for details.

III. Dominant Subspace Approximation by Parameter-Varying Oblique Projection

Section II-C revealed that a parameter-varying state transformation introduces an additional parameter rate dependence. The same is true if an oblique projection is constructed as in Section II-D from a parameter-varying transformation and the truncation operator. This section shows as the main result of this paper that it is possible to construct a parameter-varying projection that does not introduce rate dependence. It is then shown how a dominant subspace approximation for LPV systems is obtained.

A. Main Result: Parameter-Varying Oblique Projections

Constructing a reduced-order model for an LPV system essentially requires the same steps as in Section II-D. Replacing x in (1) with $V_\rho z$ and left multiplying the resulting equation by W_ρ^T shows that any parameter-varying projection $\Pi_\rho = V_\rho\,W_\rho^T$ with $W_\rho^TV_\rho = I_{n_x}$ leads to a reduced-order model

$$\begin{aligned} \dot{z} &= W_\rho^T\left(A_\rho V_\rho - \frac{\partial V_\rho}{\partial \rho}\dot{\rho}\right)z + W_\rho^TB_\rho u - W_\rho^T\frac{\mathrm{d}}{\mathrm{d}t}(V_\rho\bar{z}_\rho) \\ y &= C_\rho\,V_\rho\,z + D_\rho\,u\ . \end{aligned} \tag{19}$$

System (19) depends on $\dot{V}_\rho$ but not on the time derivative of W_ρ. Hence, rate dependence can be avoided by restricting parameter dependence in the projection to the kernel. Such a parameter-varying oblique projection $\Pi_\rho = V\,W_\rho^T$ is obtained from the parameterization (13) when only the symmetric matrix S is parameter dependent, i. e.,

$$W_\rho^T = (V^TS_\rho\,V)^{-1}\,V^TS_\rho\ . \tag{20}$$

Since V is now constant and $W_\rho^T V = I_{n_z}$ still holds for all parameters, the projected state space equations (19) simplify to

$$\begin{aligned} \dot{z} &= \overbrace{W_\rho^T A_\rho V}^{A_{\text{red},\rho}} z + \overbrace{W_\rho^T B_\rho}^{B_{\text{red},\rho}} u - \frac{\mathrm{d}}{\mathrm{d}t}\bar{z}_\rho \\ y &= \underbrace{C_\rho V}_{C_{\text{red},\rho}} z + D_\rho u \,. \end{aligned} \tag{21}$$

System (21) has exactly the structure of the desired reduced-order system (2). The key is that V is constant. State consistency for the LPV system is hence preserved with both $x \approx V z$ and $\dot{x} \approx V \dot{z}$. The direction along which the full-order model is projected onto this constant subspace is, on the other hand, allowed to vary with the parameter. From the perspective of a Petrov-Galerkin approximation, this is equivalent to enforcing (16) over a varying test space $\operatorname{span}(W_\rho)$.

If the intent is to simulate or control a nonlinear system using the reduced-order LPV model, the equilibrium reduced-order state can be calculated by the Petrov-Galerkin approximation $W_\rho^T (V\bar{z}_\rho - \bar{x}_\rho) = 0$, i. e., as $\bar{z}_\rho = W_\rho^T \bar{x}_\rho$.

B. Dominant Subspace Approximation

Recall from Section II-D that balancing and truncation can be interpreted as an oblique projection. Given the controllability and observability Gramians, either as solutions to (4) and (5) for LTI models or as parameter independent solutions that satisfy (8) and (9) for LPV models, this projection can be directly constructed from what is known as the *square root algorithm* [19]. Doing so requires the Cholesky factorizations $X_\text{o} = L_\text{o} L_\text{o}^T$ and $X_\text{c} = L_\text{c} L_\text{c}^T$, as well as the singular value decomposition (SVD) of the product

$$L_\text{c}^T L_\text{o} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} N_1 & N_2 \end{bmatrix}^T \,. \tag{22}$$

The singular values are ordered by descending magnitude, such that the diagonal matrix Σ_1 contains the largest n_z singular values. The orthogonal matrices $[\, U_1 \; U_2 \,]$ and $[\, N_1 \; N_2 \,]$ contain the corresponding left and right singular vectors. The oblique projection for balancing and truncation is

$$\Pi_\text{bal} = \underbrace{L_\text{c} U_1 \Sigma_1^{-1/2}}_{V} \underbrace{\Sigma_1^{-1/2} N_1^T L_\text{o}^T}_{W^T} \,, \tag{23}$$

i. e., a projection onto $\operatorname{span}(L_\text{c} U_1)$ along $\ker\big(N_1^T L_\text{o}^T\big)$, see [17], [19] for details.

The goal of this section is to extend (23) to a parameter-varying projection as introduced in Section III-A. As a first step, projection (23) is rewritten in terms of the parameterization (13). Since $\Sigma_1^{-1} = (\Sigma_1 N_1^T N_1 \Sigma_1)^{-1} \Sigma_1$,

$$\Pi_\text{bal} = L_\text{c} U_1 (\Sigma_1 N_1^T N_1 \Sigma_1)^{-1} \Sigma_1 N_1^T L_\text{o}^T .$$

It further follows from (22) that $\Sigma_1 N_1^T = U_1^T L_\text{c}^T L_\text{o}$ and thus

$$\Pi_\text{bal} = L_\text{c} U_1 \left(U_1^T L_\text{c}^T L_\text{o} L_\text{o}^T L_\text{c} U_1 \right)^{-1} U_1^T L_\text{c}^T L_\text{o} L_\text{o}^T .$$

Replacing finally $L_\text{c} U_1 = Q R$ by its thin QR-factorization and $L_\text{o} L_\text{o}^T$ by X_o yields

$$\Pi_\text{bal} = Q \left(Q^T X_\text{o} Q \right)^{-1} Q^T X_\text{o}. \tag{24}$$

Equation (24) has the desired form (13) with $V = Q$ and $S = X_\text{o}$.

This suggests the following approximation of LPV balancing and truncation by a parameter-varying projection: Find a constant basis $\bar{Q}$ such that $\operatorname{span}(\bar{Q})$ approximates the parameter-varying dominant subspace $\operatorname{span}(L_{\text{c},\rho} U_{1,\rho})$ and replace X_o by the parameter-varying observability Gramian $X_{\text{o},\rho}$, i. e.,

$$\Pi_\rho = \underbrace{\bar{Q}}_{V} \underbrace{(\bar{Q}^T X_{\text{o},\rho} \bar{Q})^{-1} \bar{Q}^T X_{\text{o},\rho}}_{W_\rho^T} \,. \tag{25}$$

This method only requires calculation of the Cholesky factors L_{c,ρ_k}, L_{o,ρ_k}, and the SVD (22) at each grid point ρ_k over the grid $\{\rho_1, \ldots, \rho_g\}$. The matrix $X_{\text{o},\rho}$ is then formed from interpolation of $L_{\text{o},\rho_k} L_{\text{o},\rho_k}^T$. An approximation for $\operatorname{span}(L_{\text{c},\rho} U_{1,\rho})$ could be obtained by adopting a prevalent approach from parametric model reduction to build a common basis from the SVD of the orthonormal bases calculated at each grid point [12]. That approach only considers the directions but not their individual importance. Thus, a direction that varies little or not at all and hence appears at all grid points would be given priority over a varying direction regardless how difficult it might be to reach. Instead, $\operatorname{span}(L_{\text{c},\rho} U_{1,\rho})$ is approximated by $\bar{Q} = \bar{U}_1 \in \mathbb{R}^{n_x \times n_z}$ from the SVD

$$\begin{bmatrix} L_{\text{c},\rho_1} U_{1,\rho_1} \cdots L_{\text{c},\rho_g} U_{1,\rho_g} \end{bmatrix} = \begin{bmatrix} \bar{U}_1 & \bar{U}_2 \end{bmatrix} \begin{bmatrix} \bar{\Sigma}_1 \bar{N}_1^T \\ \bar{\Sigma}_2 \bar{N}_2^T \end{bmatrix} \,. \tag{26}$$

The singular values $\bar{\Sigma}_1$ in (26) still provide a measure of how easy the subspace spanned by $\bar{U}_1$ is to reach. Thus, they provide a meaningful threshold to decide on the order of the approximation.

There are several noteworthy properties of projection (25). First, it results in a reduced-order LPV model (21) without additional rate dependence. Second, since $X_{\text{o},\rho}$ is a continuous function of ρ, interpolation is accurate for a sufficiently dense grid. Further, $W_\rho^T V = I_{n_z}$ is exactly enforced by (25) for all parameter values regardless of the method used to interpolate $X_{\text{o},\rho}$. Third, when applied to an LTI system, $\bar{Q}$ is simply an orthonormal basis for $\operatorname{span}(L_c U_1)$ and hence (25) exactly coincides with balancing and truncation, or more precisely, with the *balancing-free square-root algorithm* [20]. Finally, a result from Reference [15] is invoked to show that for frozen parameters, all poles of the reduced-order system are in the left half plane. Multiplying the Lyapunov equation (5) for the original system from the left by $\bar{Q}^T$ and from the right by $\bar{Q}$ results in

$$\bar{Q}^T A^T X_\text{o} \bar{Q} + \bar{Q}^T X_\text{o} A \bar{Q} + \bar{Q}^T C^T C \bar{Q} = 0 \,. \tag{27}$$

Using $A_\text{red} = (\bar{Q}^T X_\text{o} \bar{Q})^{-1} \bar{Q}^T X_\text{o} A \bar{Q}$ and $C \bar{Q} = C_\text{red}$, it can be shown that (27) is equivalent to

$$A_\text{red}^T (\bar{Q}^T X_\text{o} \bar{Q}) + (\bar{Q}^T X_\text{o} \bar{Q}) A_\text{red} + C_\text{red}^T C_\text{red} = 0. \tag{28}$$

Since X_o is symmetric positive definite, so is $\bar{Q}^T X_\text{o} \bar{Q}$ and consequently A_red has all its eigenvalues in the left half plane. This guarantees stability of the reduced-order model for "slowly" varying parameters, see [13] for details.

IV. Application Example

A nonlinear mass-spring-damper system, taken from [21], is used to demonstrate the approach. It represents the interconnection of M blocks with mass $m = 1\,\mathrm{kg}$, that are each connected both to their neighboring blocks and the initial system by a linear damper with damping constant $d = 1\,\frac{\mathrm{N\,s}}{\mathrm{m}}$ and a nonlinear spring with stiffness $k(q) = k_1 + k_2\,q^2$, $k_1 = 0.5\frac{\mathrm{N}}{\mathrm{m}}$, $k_2 = 1\frac{\mathrm{N}}{\mathrm{m}^3}$. An external force ρ and a controlled force u are acting on the M^{th} block. The equations of motion for the i^{th} block in terms of its displacement q_i are thus

$$m\,\ddot{q}_i = \begin{cases} -F_1 - F_{1,2}, & i = 1 \\ -F_i - F_{i,i-1} - F_{i,i+1}, & i = 2,\ldots,M-1 \\ -F_M - F_{M,M-1} + \rho + u & i = M\,. \end{cases}$$

The force $F_{i,j} = d\,(\dot{q}_i - \dot{q}_j) + k_1\,(q_i - q_j) + k_2\,(q_i - q_j)^3$ is caused by the relative motion of neighboring blocks and $F_i = d\,\dot{q}_i + k(q_i)\,q_i$ is due to the connection with the initial system. With state vector $\xi := \begin{bmatrix} q_1,\ldots,q_M,\dot{q}_1,\ldots,\dot{q}_M\end{bmatrix}^T$, the system is written as $\dot{\xi} = f(\xi,u,\rho)$. The output of the system is the displacement $q_M = h(\xi,u,\rho)$ of the M^{th} block. For each value of $\rho \in \mathcal{P}$, an equilibrium point $\bar{x}$ is defined by $0 = f(\bar{x}(\rho),0,\rho)$ with a corresponding equilibrium output $\bar{q}_M = h(\bar{x}(\rho),0,\rho)$. Linearization around $(\bar{x}(\rho(t)),0,\bar{q}_M(\rho(t)))$ yields an LPV system of the form (1) in the perturbation variables $x := \xi - \bar{x}(\rho)$ and $y = q_M - \bar{q}_M(\rho)$, see [14], [21] for details.

A. Comparison to LPV Balanced Truncation

The number of blocks for the mass-spring-damper model is selected as $M = 50$. The admissible parameter range is restricted to $\mathcal{P} = [0\;\;2]$ and the nonlinear system is linearized on a grid $\{\rho_k\}_{k=1}^{g} = \{0,1,2\}$. For this example, the proposed approach is compared to the standard LPV balancing and truncation method [4]. The function `lpvbalreal` of the LPVTools toolbox [2] is used to solve for a constant balancing transformation and parameter independent generalized Gramians by minimizing $\mathrm{trace}(X_\mathrm{c}\,X_\mathrm{o})$ subject to (8) and (9). An oblique projection is then formed as described in Section II-D. Obtaining a reduced-order model with 4 states requires vastly different computational effort with the two methods. The proposed projection takes 0.1 seconds, whereas solving the LMIs for balancing takes more than 2.5 hours on a standard desktop computer. The resulting reduced-order models, on the other hand, are very similar. An upper bound $\bar{\gamma}$ for the induced $\mathcal{L}_2$-norm of the error system

$$\begin{bmatrix}\dot{\tilde{x}}\\ \dot{z}\end{bmatrix} = \begin{bmatrix} A_\rho & (I_{n_x}{-}\Pi_\rho)\,A_\rho V \\ 0_{n_z\times n_x} & W_\rho^T A_\rho V\end{bmatrix}\begin{bmatrix}\tilde{x}\\ z\end{bmatrix} + \begin{bmatrix}(I_{n_x}{-}\Pi_\rho)\,B_\rho \\ \Pi_\rho\,B_\rho\end{bmatrix} u$$

$$e = \begin{bmatrix} C & 0_{n_y\times n_z}\end{bmatrix}\begin{bmatrix}\tilde{x}\\ z\end{bmatrix},$$

with $\tilde{x} = x - V\,z$ is calculated for both methods using the LPVTools function `lpvnorm`. This guarantees $\|e\|_2 < \bar{\gamma}\,\|u\|_2$ and certifies stability of the reduced-order system for arbitrary fast parameter variations. Balancing and truncation yields a slightly better error bound $\bar{\gamma} = 3.9\text{e-}03$

TABLE I

Comparison of reduced-order models

	LPV Balancing	Proposed Projection
Computation Time†	9170 s	0.1 s
$\mathcal{L}_2$-norm error bound	3.9e-03	12.1e-03
$\mathcal{H}_\infty$-norm error‡	2.2e-03	1.5e-03
Output MSE	1.5e-03	1.5e-03

† on a 64 bit desktop PC with 3.4 GHz 8-core CPU and 8 GB RAM
‡ for frozen parameters at grid points

compared to $\bar{\gamma} = 12.1\text{e-}03$ for the proposed method. Additionally, the maximum $\mathcal{H}_\infty$-norm error $\underline{\gamma}$ for "frozen" parameters at all grid points is calculated as a lower bound for the induced $\mathcal{L}_2$-norm and the frequency responses are shown in Fig. 1. The proposed method results in $\underline{\gamma} = 1.5\text{e-}03$ compared to $\underline{\gamma} = 2.2\text{e-}03$ for balancing. The time-domain responses for an external force $\rho(t) = (1 + \cos(0.5\,t))\,\mathrm{N}$ and a step input $u = 0.5\,\mathrm{N}$ after 25 s are compared in Fig. 2. Both methods approximate the output of the nonlinear system very well and result in an identical mean square error of 1.5e-03. Table I summarizes these results.

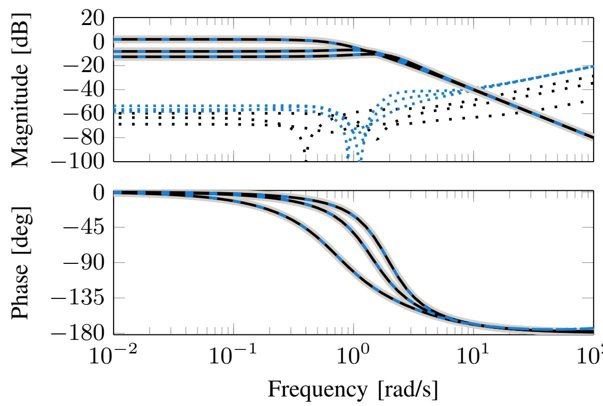

Fig. 1. Frequency response at frozen parameters $\rho = 0,1,2$ for original model (100 states) and reduced-order model from balancing and truncation (4 states) and parameter-varying projection (4 states). The relative error is shown in the magnitude plot for balancing and truncation (······) and parameter-varying projection (· · · ·).

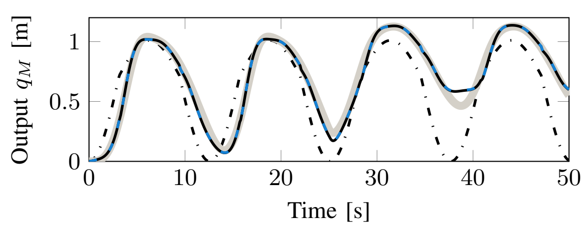

Fig. 2. Nonlinear simulation of original model (100 states), equilibrium output (– · –), and reduced-order model from balancing and truncation (4 states) and parameter-varying projection (4 states).

B. Example for Larger System

In a second example, $M = 500$ blocks are used and the admissible parameter range is increased to $\mathcal{P} = [0\;\;10]$. The system is linearized on a grid $\{\rho_k\}_{k=1}^{g} = \{0,1,\ldots,10\}$. The LPV model considered in Section IV-A is about the maximum size that is currently tractable for balancing and truncation on a desktop computer. Consequently, no balanced reduced-order model could be obtained for this larger LPV system. Computation with the proposed approach, on the other hand, takes 47 seconds and results in a fifth order model. Since calculation of the $\mathcal{L}_2$-norm error bound is also intractable, the "frozen" parameter frequency responses are used as surrogates and shown in Fig. 3. They again match very well for all grid points. The nonlinear simulation for an external force $\rho(t) = (5 + 5\,\cos(0.5\,t))\,\mathrm{N}$ and a step input $u = 0.5\,\mathrm{N}$ after $25\,\mathrm{s}$ is shown in Figure 4 and also indicates excellent agreement.

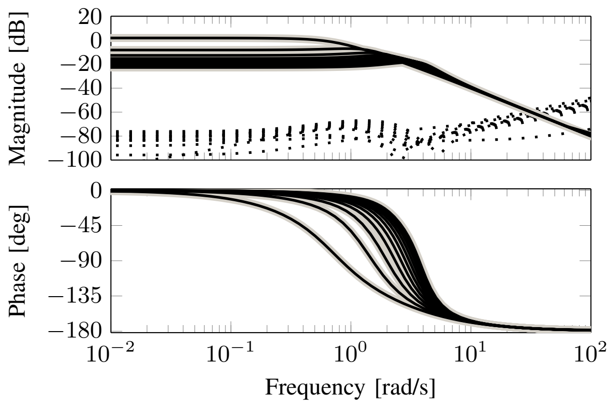

Fig. 3. Frequency response at frozen parameters $\rho = 1, \ldots, 10$ for original model (1000 states), reduced-order model (5 states), and relative error (· · · ·).

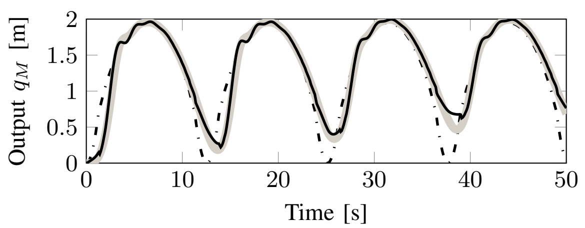

Fig. 4. Nonlinear simulation of original model (1000 states), equilibrium output ( ), and reduced-order model (5 states).

V. Conclusion and Extensions

A. Conclusion

A model order reduction method for LPV systems is developed in this paper. It is shown to approximate balancing and truncation within a fraction of its required computation time and is hence also applicable to systems where balancing and truncation becomes intractable.

B. Future Extensions

In case the Lyapunov equations become intractable to solve by standard means, low-rank approximations of Gramians can be used. Such approximations can be calculated even for large-scale systems with tens of thousands of states, e. g., by Krylov subspace methods [22]. The use of frequency-weighted or frequency-limited Gramians also lends itself well to the approach and could allow to specify a frequency region of interest. Finally, the basis space can be calculated by entirely different means. For instance, V can be calculated by Krylov moment matching algorithms, while a parameter-varying W_ρ is still obtained from the observability Gramian. This provides an extension of the SVD-Krylov algorithm proposed in [23] to LPV systems.

Acknowledgment

This work was supported by the NASA Langley NRA Cooperative Agreement under Grant no. NNX12AM55A entitled "Analytical Validation Tools for Safety Critical Systems Under Loss-of-Control Conditions". Dr. Christine Belcastro is the technical monitor.

References

[1] F. Wu, X. Yang, A. Packard, and G. Becker, "Induced $\mathcal{L}_2$-norm control for LPV systems with bounded parameter variation rates," *Int. J. Robust Nonlinear Control*, vol. 6, pp. 2379–2383, 1996.

[2] A. Hjartarson, A. Packard, and P. Seiler, "LPVTools: A toolbox for modeling, analysis, and synthesis of parameter varying control systems," in *1st IFAC Workshop Linear Parameter Varying Syst.*, 2015.

[3] G. D. Wood, "Control of parameter-dependent mechanical systems," Ph.D. dissertation, Univ. Cambridge, 1995.

[4] G. D. Wood, P. J. Goddard, and K. Glover, "Approximation of linear parameter-varying systems," in *IEEE CDC*, vol. 1, 1996, pp. 406–411.

[5] B. C. Moore, "Principal component analysis in linear systems: Controllability, observability, and model reduction," *IEEE Trans. Autom. Control*, vol. 26, no. 1, pp. 17–32, 1981.

[6] D. Amsallem and C. Farhat, "An online method for interpolating linear parametric reduced-order models," *SIAM J. Sci. Comp.*, vol. 33, no. 5, pp. 2169–2198, 2011.

[7] C. Poussot-Vassal and C. Roos, "Generation of a reduced-order LPV/LFT model from a set of large-scale MIMO LTI flexible aircraft models," *Control Eng. Pract.*, vol. 20, pp. 919–930, 2012.

[8] F. Adegas, I. Sonderby, M. Hansen, and J. Stoustrup, "Reduced-order LPV model of flexible wind turbines from high fidelity aeroelastic codes," in *IEEE Conf. Control Applications*, 2013, pp. 424–429.

[9] J. Theis, B. Takarics, H. Pfifer, G. Balas, and H. Werner, "Modal matching for LPV model reduction of aeroservoelastic vehicles," in *AIAA Conf. Atmospheric Flight Mechanics*, 2015.

[10] P. Benner, S. Gugercin, and K. Willcox, "A survey of model reduction methods for parametric systems," Max Planck Institute, Magdeburg, Germany, Tech. Rep. MPIMD/13-14, Aug. 2013. [Online]. Available: http://www2.mpi-magdeburg.mpg.de/preprints/2013/14/

[11] U. Baur, C. Beattie, P. Benner, and S. Gugercin, "Interpolatory projection methods for parameterized model reduction," *SIAM J. Sci. Comp.*, vol. 33, no. 5, pp. 2489–2518, 2011.

[12] H. Panzer, J. Mohring, R. Eid, and B. Lohmann, "Parametric model order reduction by matrix interpolation," *Automatisierungstechnik*, vol. 8, pp. 475–484, 2010.

[13] C. Desoer, "Slowly varying system x= A(t) x," *IEEE Trans. Autom. Control*, vol. 14, no. 6, pp. 780–781, 1969.

[14] B. Takarics and P. Seiler, "Gain scheduling for nonlinear systems via integral quadratic constraints," in *American Control Conf.*, 2015, pp. 811–816.

[15] C. de Villemagne and R. Skelton, "Model reductions using a projection formulation," *Int. J. Control*, vol. 46, no. 6, pp. 2141–2169, 1987.

[16] Y. Saad, *Iterative methods for sparse linear systems*. SIAM, 2000.

[17] A. Antoulas, *Approximation of Large-Scale Dynamical Systems*. Philadelphia, PA: SIAM, 2005.

[18] M. Rathinam and L. Petzold, "A new look at proper orthogonal decomposition," *SIAM J. Numer. Anal.*, vol. 41, no. 5, pp. 1893–1925, 2003.

[19] A. J. Laub, M. T. Heath, C. C. Paige, and R. C. Ward, "Computation of system balancing transformations and other applications of simultaneous diagonalization algorithms," *IEEE Trans. Autom. Control*, vol. 32, no. 2, pp. 115–122, 1987.

[20] A. Varga, "Balancing free square-root algorithm for computing singular perturbation approximations," in *IEEE Conf. Decision and Control*. IEEE, 1991, pp. 1062–1065.

[21] J. Annoni and P. Seiler, "A method to construct reduced-order parameter varying models," *submitted to J. Robust Nonlinear Control*, 2015.

[22] I. M. Jaimoukha and E. M. Kasenally, "Krylov subspace methods for solving large Lyapunov equations," *SIAM J. Numer. Anal.*, vol. 31, no. 1, pp. 227–251, 1994.

[23] S. Gugercin, "An iterative SVD-Krylov based method for model reduction of large-scale dynamical systems," in *IEEE Conf. Decision and Control*, 2005, pp. 5905–5910.