Robust Synthesis for Linear Parameter Varying Systems Using Integral Quadratic Constraints

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Abstract

A robust synthesis algorithm is developed for a class of uncertain, linear parameter varying (LPV) systems. The uncertain system is described as an interconnection of a nominal LPV system and a block structured uncertainty. The nominal part is a "gridded" LPV system with state matrices that are arbitrary functions of the parameter. The input/output behavior of the uncertainty is described by integral quadratic constraints (IQCs). The robust synthesis problem leads to a non-convex optimization. The proposed algorithm is a coordinate-wise descent similar to the well-known DK iteration for μ synthesis. It alternates between an LPV synthesis step and an IQC analysis step. Both steps can be efficiently solved as semidefinite programs. The derivation of the synthesis algorithm is less obvious for LPV systems as compared to its LTI counterpart due to the lack of a valid frequency response interpretation. The main contribution is the construction of the iterative synthesis algorithm using time domain dissipation inequalities and a scaled system analogous to that appearing in μ synthesis. It is shown that the proposed algorithm ensures that the robust performance is non-increasing at each iteration step. The effectiveness of the proposed method is demonstrated on a simple numerical example.

1 INTRODUCTION

This paper considers the robust synthesis problem for a class of uncertain linear parameter varying (LPV) systems. The uncertain system is described as an interconnection of a nominal (not-uncertain) LPV system and a block structured uncertainty. The state matrices of the nominal system have an arbitrary dependence on parameters, i.e. it is a "gridded" LPV system. Such models arise naturally in many applications via linearization of a nonlinear model around parameterized operating (trim) points (Moreno et al., 2012; Bobanac et al., 2010). The input/output behavior of the uncertainty is described by integral quadratic constraints (IQCs) (Megretski and Rantzer, 1997). The use of IQCs is sufficiently general to describe "uncertain" components that include nonlinearities, in addition to (parametric or dynamic) uncertainty.

The robust synthesis problem, formulated in Section 3.1, is to synthesize a controller that minimizes a closed-loop robust performance metric. This leads to a non-convex optimization that involves a search for both the controller state matrices and the IQC analysis variables. The proposed algorithm, given in Section 3.2, consists of a coordinate-wise descent similar to the well-known DK-iteration (Zhou et al., 1996; Balas et al., 2007) for μ synthesis. Specifically, the proposed algorithm alternates between an LPV synthesis step and an IQC analysis step. The synthesis step essentially relies on existing results for nominal LPV systems in Wu et al. (1996). The analysis step is performed using a matrix inequality condition to bound the robust performance of the closed-loop uncertain LPV system (Section 4.1). Both steps can be efficiently solved as semidefinite programs (SDPs). The effectiveness of the proposed method is demonstrated on a numerical example in Section 5.

There are two main technical challenges. First, the nominal LPV system does not have a valid frequency response interpretation and hence the analysis requires a time domain approach. Section 4.1 develops a matrix inequality robustness analysis condition (Theorem 2) using (time domain) dissipation inequality techniques. This analysis condition is an extension of the worst-case gain condition in Pfifer and Seiler (2014). An alternative to the dissipation inequality based approach for IQCs in the time domain is given in Cantoni et al. (2013). It is purely based on operator theory and uses homotopy arguments to proof stability. This alternative approach can potentially be used to develop synthesis algorithms complementary to the one developed here or provide an alternative proof for the presented algorithm. The second

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technical challenge is that an appropriate scaled system must be constructed to link the analysis and synthesis steps. This construction, described in Section 4.2, is such that the next synthesis step on the scaled plant yields a controller that improves the closed-loop robust performance. These technical results are used to show the following main result in Section 4.3: the robust performance metric is non-increasing at each iteration step and hence the algorithm converges.

This paper builds on known results for both LPV systems and IQCs. A brief review is provided in Section 2. In addition, there are several related robust synthesis results for LPV systems whose state matrices have a rational dependence on the parameters (Apkarian and Adams, 1998; Veenman and Scherer, 2010, 2014; Scherer and Kose, 2012). This rational dependence leads to finite-dimensional matrix inequalities in the algorithm. In contrast, the algorithm in this paper is developed for the case where the state matrices have an arbitrary dependence on the parameters. This leads to parameter-dependent matrix inequalities for both the synthesis and analysis steps. As a result, parameter gridding is required to obtained finite-dimensional matrix inequality conditions. Finally, this paper builds on Wang et al. (2014) which only considered LTI uncertainty. This paper extends the algorithm to uncertainties described by a general class of IQCs.

2 BACKGROUND

2.1 Linear Parameter Varying (LPV) Systems

LPV systems are a class of systems whose state-space matrices are continuous functions of a time-varying parameter $\rho : \mathbb{R}^+ \to \mathbb{R}^{n_\rho}$. The set of admissible parameter trajectories is defined as $\mathcal{T} := \{\rho : \mathbb{R}_+ \to \mathbb{R}^{n_\rho} \ : \ \rho(t) \in \mathcal{P}\ \forall t \geq 0$ and $\rho(t)$ is continuously differentiable$\}$ where $\mathcal{P} \subset \mathbb{R}^{n_\rho}$ is a known compact set. In some applications, the parameter varying rate $\dot{\rho}$ are assumed to be bounded. However, only the rate unbounded case is considered here for simplicity. All results in this paper generalize, but with extensive notations, to the rate bounded case using existing results in (Wu et al., 1996; Pfifer and Seiler, 2014). An n_G^{th} order LPV system, G_ρ, is defined by

$$\begin{bmatrix} \dot{x} \\ e \end{bmatrix} = \begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} \begin{bmatrix} x \\ d \end{bmatrix} \tag{1}$$

where $A : \mathcal{P} \to \mathbb{R}^{n_G \times n_G}$, $B : \mathcal{P} \to \mathbb{R}^{n_G \times n_d}$, $C : \mathcal{P} \to \mathbb{R}^{n_e \times n_G}$ and $D : \mathcal{P} \to \mathbb{R}^{n_e \times n_d}$. The performance of an LPV system G_ρ is specified by its induced L_2 gain $\|G_\rho\| := \sup_{0 \neq d \in L_2,\ \rho \in \mathcal{T}} \frac{\|e\|}{\|d\|}$. A generalization of the Bounded Real Lemma Wu et al. (1996) provides a sufficient condition to upper bound $\|G_\rho\|$. The next theorem states the condition but simplified for the special case of rate unbounded LPV systems.

Theorem 1. *(Wu et al. (1996)): G_ρ is exponentially stable and $\|G_\rho\| \leq \gamma$ if there exists $P = P^T \geq 0$ such that $\forall \rho \in \mathcal{P}$*

$$\begin{bmatrix} PA(\rho)+A(\rho)^T P & PB(\rho) \\ B^T(\rho)P & -I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C(\rho)^T \\ D(\rho)^T \end{bmatrix} \begin{bmatrix} C(\rho) & D(\rho) \end{bmatrix} < 0 \tag{2}$$

This theorem forms the basis for the induced L_2 norm controller synthesis in Wu et al. (1996). Consider an open loop LPV system G_ρ with inputs $[d^T,\, u^T]^T$ and outputs $[e^T,\, y^T]^T$. The objective is to synthesize a controller K_ρ:

$$\begin{bmatrix} \dot{x}_K \\ u \end{bmatrix} = \begin{bmatrix} A_K(\rho) & B_K(\rho) \\ C_K(\rho) & D_K(\rho) \end{bmatrix} \begin{bmatrix} x_K \\ y \end{bmatrix} \tag{3}$$

such that the closed-loop interconnection of G_ρ and K_ρ, which is given by the lower linear fractional transformation (LFT), denoted $\mathcal{F}_l(G_\rho, K_\rho)$, has the minimal induced L_2 gain: $\min_{K_\rho} \|\mathcal{F}_l(G_\rho, K_\rho)\|$. This LPV synthesis problem can be solved via parameterized LMI conditions. Details on the solution can be found in Wu et al. (1996). It should be noted that both the analysis and synthesis problems involve an infinite collection of LMI constraints parameterized by $\rho \in \mathcal{P}$. A remedy to this issue, which works in many practical examples, is to approximate the set $\mathcal{P}$ by a finite gridding set $\mathcal{P}_{grid} \in \mathcal{P}$.

2.2 Integral Quadratic Constraints (IQCs)

IQCs (Megretski and Rantzer, 1997) provide a framework for robustness analysis building on work by Yakubovich (1971). The IQC specifies a constraint on the input/output signals of the perturbation. The following definitions characterize the constraint in the frequency and time domain.

Definition 1. *Let $\Pi \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ be given. Two signals $v \in L_2^{n_v}[0,\infty)$ and $w \in L_2^{n_w}[0,\infty)$ satisfy the frequency domain IQC defined by the multiplier Π if*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{V}(j\omega) \\ \hat{W}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{V}(j\omega) \\ \hat{W}(j\omega) \end{bmatrix} d\omega \geq 0 \tag{4}$$

where $\hat{V}$ and $\hat{W}$ are Fourier transforms of v and w. A bounded, causal operator $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ satisfies the frequency domain IQC defined by Π if Eq. 4 holds for all $v \in L_2^{n_v}[0,\infty)$ and $w = \Delta(v)$.

Definition 2. *Let Ψ be a stable LTI system, i.e. $\Psi \in \mathbb{RH}_\infty^{n_z\times(n_v+n_w)}$, and $M = M^T \in \mathbb{R}^{n_z \times n_z}$. Two signals $v \in L_{2e}^{n_v}[0,\infty)$ and $w \in L_{2e}^{n_w}[0,\infty)$ satisfy the time domain IQC defined by the multiplier Ψ and matrix M if the following inequality holds for all $T \geq 0$*

$$\int_0^T z^T(t) M z(t)\, dt \geq 0 \tag{5}$$

where z is the output of Ψ driven by inputs (v, w) with zero initial conditions. A bounded, causal operator $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ satisfies the time domain IQC *defined by* (Ψ, M) *if Eq. 5 holds for all* $v \in L_{2e}^{n_v}[0,\infty)$, $w = \Delta(v)$ *and* $T \geq 0$.

IQCs can be used to model a variety of nonlinearities and uncertainties, e.g. saturation and norm bounded uncertainty (Megretski and Rantzer, 1997). Fig. 1 provides a graphical interpretation for the time domain IQC. If Δ satisfies the time domain IQC defined by Ψ then the filtered signal z satisfies the constraint in Eq. 5 for any finite-horizon $T \geq 0$.

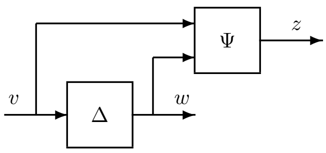

Fig. 1. Graphical interpretation of the IQC

A precise connection between the frequency and time domain IQC formulations is important for the robust synthesis algorithm described in this paper. Specifically, if Δ satisfies the time domain IQC defined by (Ψ, M) then it satisfies the frequency domain IQC defined by $\Pi = \Psi^{\sim} M \Psi$. However, the reverse implication fails to hold in general Seiler (2015). A time domain IQC as in Definition 2 is referred to as a hard IQC in Megretski and Rantzer (1997). In contrast, factorizations for which the time domain constraint holds only for $T = \infty$ are called soft IQCs. Lemma 4 in Seiler (2015) provide a specific "hard" factorization (Ψ, M) (called a J-spectral factorization of Π) that can be constructed under additional assumptions on the frequency domain multiplier Π. The distinction between hard and soft IQCs is important because LPV systems do not have a valid frequency response interpretation. Hence existing conditions for robust analysis of gridded LPV systems (Pfifer and Seiler, 2014) rely on the use of valid time domain (hard) IQCs. Section 4.1 generalizes these existing results to handle factorizations (Ψ, M) that are not necessarily hard.

3 Robust Synthesis

3.1 Problem Formulation

The robust synthesis problem involves an uncertain system (Fig. 2) described by the interconnection of an LPV system G_ρ, a perturbation Δ, and an LPV controller K_ρ. The state-space realization for G_ρ is given by:

$$\begin{bmatrix} \dot{x}_G \\ v \\ e \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_w(\rho) & B_d(\rho) & B_u(\rho) \\ C_v(\rho) & D_{vw}(\rho) & D_{vd}(\rho) & D_{vu}(\rho) \\ C_e(\rho) & D_{ew}(\rho) & D_{ed}(\rho) & D_{eu}(\rho) \\ C_y(\rho) & D_{yw}(\rho) & D_{yd}(\rho) & D_{yu}(\rho) \end{bmatrix} \begin{bmatrix} x_G \\ w \\ d \\ u \end{bmatrix} \tag{6}$$

where $x_G \in \mathbb{R}^{n_G}$, $w \in \mathbb{R}^{n_w}$, $d \in \mathbb{R}^{n_d}$, $u \in \mathbb{R}^{n_u}$, $v \in \mathbb{R}^{n_v}$, $e \in \mathbb{R}^{n_e}$ and $y \in \mathbb{R}^{n_y}$. The following assumptions are made regarding G_ρ and Δ:

Assumption 1. *G_ρ is quadratically stabilizable from u and quadratically detectable from y as defined in Chapter 1 of Wu (1995).*
Assumption 2. *The perturbation is a bounded, causal operator $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ that satisfies a collection of frequency domain IQCs defined by $\{\Pi_k\}_{k=1}^N \subset \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$, denoted $\Delta \in \mathbf{\Delta}(\Pi_1, \ldots, \Pi_N)$.*
Assumption 3. *Partition the frequency domain multipliers $\{\Pi_k\}_{k=1}^N$ as $\begin{bmatrix} \Pi_{k,11} & \Pi_{k,12} \\ \Pi_{k,12}^{\sim} & \Pi_{k,22} \end{bmatrix}$ where $\Pi_{k,11}$ is $n_v \times n_v$. Each frequency domain multiplier satisfies $\Pi_{k,11}(j\omega) \geq 0$ and $\Pi_{k,22}(j\omega) \leq 0 \; \forall \omega \in \mathbb{R} \cup \{\infty\}$.*
Assumption 4. *The perturbation has been normalized to satisfy $\|\Delta\| \leq 1$ and the first IQC is defined by the multiplier $\Pi := diag(I_{n_v}, -I_{n_w})$.*

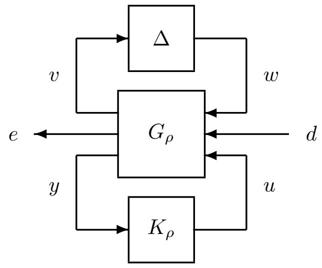

Fig. 2. Interconnection for LPV Robust Synthesis

Assumption 1 ensures that there is a controller K_ρ that stabilizes $\mathcal{F}_l(G_\rho, K_\rho)$. The IQCs in Assumption 2 are used to bound the input/output behavior of the perturbation Δ. The term "uncertainty" is used for simplicity when referring to Δ. Assumptions 3 and 4 are used to simplify the algorithm. Assumption 3 only requires the non-strict definiteness conditions $\Pi_{k,11} \geq 0$ and $\Pi_{k,22} \leq 0$. This is sufficiently general to cover most typical frequency domain multipliers used in IQC analysis (Megretski and Rantzer, 1997). However, Assumptions 3 and 4 are used to ensure that a "combined" multiplier $\Pi_\lambda := \sum_{k=1}^N \lambda_k \Pi_k$ that appears in the proposed robust synthesis algorithm satisfies the strict definiteness conditions by forcing $\lambda_1 > 0$ and $\lambda_k \geq 0$ for $k = 2, \ldots N$. Therefore, Π_λ is a hard IQC and has a J-spectral factorization (Theorem 4 in Seiler (2015)).

To simplify notation, define $H_\rho := \mathcal{F}_l(G_\rho, K_\rho)$. The uncertain LPV system in Fig. 2 can therefore be expressed as an upper LFT, denoted $\mathcal{F}_u(H_\rho, \Delta)$. A natural performance metric for the uncertain LPV system is the worst-case gain: $\sup_{\Delta \in \mathbf{\Delta}(\Pi_1,\ldots,\Pi_N)} \|\mathcal{F}_u(H_\rho, \Delta)\|$. This is the largest induced L_2 gain of the uncertain LPV system over all uncertainties consistent with the specified IQCs. This metric has been widely used for robustness analysis (Pfifer and Seiler, 2014; Turner and Kerr, 2012). However, it is inconvenient for robust synthesis as it requires an initial controller that achieves robust stability with respect to $\mathbf{\Delta}(\Pi_1, \ldots, \Pi_N)$. Thus it is standard, e.g. in D-K synthesis, to instead use a robust performance metric that simultaneously scales both the uncertainty level and the system gain. The definition of robust performance requires the notion of a scaled uncertainty set. Specifically, define S_b as the scaling matrix $diag(bI_{n_v}, I_{n_w})$. Let $\mathbf{\Delta}_b(\Pi_1, \ldots, \Pi_N)$ denote the set of bounded, causal operators Δ that satisfy the frequency domain IQCs defined by $S_b \Pi_k S_b$ for $k = 1, \ldots, N$. For the scaled set, if $b_2 \geq b_1$ then $\mathbf{\Delta}_{b_2} \supseteq \mathbf{\Delta}_{b_1}$. The definition of robust performance uses this scaled uncertainty set.

Definition 3. *The system H_ρ achieves robust performance of level γ with respect to the uncertainty described by $\{\Pi_k\}_{k=1}^N$ if*

$$\sup_{\Delta \in \mathbf{\Delta}_{1/\gamma}(\Pi_1, \ldots, \Pi_N)} \|\mathcal{F}_u(H_\rho, \Delta)\| \leq \gamma \tag{7}$$

Let $r_{\mathbf{\Delta}(\Pi_1, \ldots, \Pi_N)}[H_\rho]$ denote the smallest level of robust performance achievable by H_ρ.

H_ρ achieves robust performance of level γ if the worst-case gain is $\leq \gamma$ over all uncertainties in the scaled set $\mathbf{\Delta}_{1/\gamma}(\Pi_1, \ldots, \Pi_N)$. For decreasing levels of robust performance, the gain decreases and the bound on the tolerable uncertainty increases. The robust synthesis problem is to synthesize a controller K_ρ that stabilizes G_ρ and minimizes the closed-loop robust performance, i.e.:

$$\inf_{K_\rho \text{ stabilizing}} r_{\mathbf{\Delta(\Pi_1, \ldots, \Pi_N)}} \left[\mathcal{F}_l(G_\rho, K_\rho)\right] \tag{8}$$

3.2 Algorithm

This section gives a high-level overview of the proposed LPV robust synthesis algorithm. Detailed steps are described in Algorithm 1. Technical results regarding the algorithm are given in Section 4. As in DK synthesis, the robust LPV synthesis is non-convex. In particular, Theorem 2 in Section 4.1 provides a linear matrix inequality (LMI) formulation for robust performance. Applying this result for synthesis leads to a matrix inequality that is bilinear in the state matrices for K_ρ and the analysis variables consisting of a storage matrix $P \geq 0$ and IQC coefficients $\{\lambda_k\}_{k=1}^N$. A coordinate-wise approach is used to decouple the design into a nominal controller synthesis step (for K_ρ) and a robust performance analysis step (for P and λ). The technical results in Section 4 are used to link these steps. The proposed algorithm will not, in general, converge to a local (nor global) optima. However, it is a pragmatic approach that decouples the synthesis and analysis steps into convex optimizations. The main technical result (Theorem 3 in Section 4.3) is that the algorithm iteration is well posed at each step and the robust performance is non-increasing. This property is similar to DK synthesis.

Algorithm 1 Robust Synthesis for LPV Systems

1: **Given:** LPV system G_ρ and multipliers $\{\Pi_k\}_{k=1}^N$ satisfying Assumptions 1-4; Stopping tolerance parameters $i_{max} \in \mathbb{N}$ and $\epsilon_{tol} > 0$.
2: **Initialization:** Initialize the iteration count to $i = 0$. Set $\lambda(0) = [1, 0, \ldots, 0] \in \mathbb{R}^N_{\geq 0}$ and $\gamma(0) = +\infty$. Factorize each Π_k as (Ψ_k, M_k) with $\Psi_k \in \mathbb{RH}_\infty^{n_z \times (n_v + n_w)}$ according to Section 7.3 of Francis (1987).
3: **if** $i < i_{max}$ **then**
4: **Iteration Count:** Increment count $i := i + 1$.
5: **Performance Scaling:** If $i > 1$ then define the scaling matrix $S(i-1) := diag(\gamma^{-1}(i-1) I_{n_v}, I_{n_w})$, otherwise $S(0) := I_{n_v + n_w}$.
6: **Combined Multiplier:** Construct $\Pi_\lambda := \sum_{k=1}^N \lambda_k(i-1) S(i-1) \Pi_k S(i-1)$. Compute a J-spectral factorization $(\Psi_\lambda, M_\lambda)$ of Π_λ according to Lemma 4 in Seiler (2015).
7: **Scaled System Construction:** Invert the w/w_λ channels of Ψ_λ (Eq. 22) to construct $\Psi_\lambda^\dagger$ with (Eq. 23). Form the (open-loop) scaled system G_ρ^{scl} by interconnecting G_ρ and $\Psi_\lambda^\dagger$ as shown in Fig. 3.
8: **Synthesis Step:** Use results in Wu (1995); Wu et al. (1996) to solve the synthesis problem: $\min_{K_\rho} \left\|\mathcal{F}_l(G_\rho^{scl}, K_\rho)\right\|$. This minimizes the (upper bound) on the closed-loop induced gain from (w_λ, d) to (v_λ, e). The result is a bound on closed-loop induced gain, denoted $\nu(i)$, and controller $K_\rho(i)$.
9: **Analysis Step:** Use Theorem 2 in Section 4.1 to compute the best upper bound on the robust performance of the closed-loop $H_\rho := \mathcal{F}_l(G_\rho, K_\rho(i))$ with respect to $\mathbf{\Delta}(\Pi_1, \ldots, \Pi_N)$. The result is the robust performance bound $\gamma(i)$, scalars $\{\lambda_k(i)\}_{k=1}^N$, and storage function matrix $P(i) = P(i)^T$.
10: **Termination Condition:** If $\gamma(i) - \gamma(i-1) \leq \epsilon_{tol}$ then stop the iteration.
11: **end if**
12: **Return:** Final controller $K_\rho(i)$ and robust performance upper bound $\gamma(i)$.

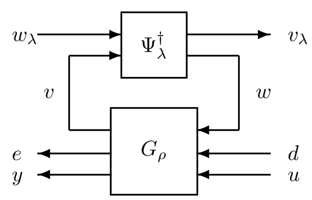

Fig. 3. LFT Interconnection of Scaled System, G_ρ^{scl}

4 Technical Results

4.1 Robust Performance Condition

This section derives a matrix inequality condition to bound the robust performance for an uncertain LPV system $\mathcal{F}_u(H_\rho, \Delta)$. The nominal LPV system H_ρ has the following state-space realization:

$$\begin{bmatrix} \dot{x}_H \\ v \\ e \end{bmatrix} = \begin{bmatrix} A(\rho) & B_w(\rho) & B_d(\rho) \\ C_v(\rho) & D_{vw}(\rho) & D_{vd}(\rho) \\ C_e(\rho) & D_{ew}(\rho) & D_{ed}(\rho) \end{bmatrix} \begin{bmatrix} x_H \\ w \\ d \end{bmatrix} \quad (9)$$

where $x_H \in \mathbb{R}^{n_H}$, $w \in \mathbb{R}^{n_w}$, $d \in \mathbb{R}^{n_d}$, $v \in \mathbb{R}^{n_v}$ and $e \in \mathbb{R}^{n_e}$. The uncertainty Δ is assumed to satisfy Assumptions 2-4 in Section 3.1. Construct a factorization for each Π_k as (Ψ_k, M_k) where Ψ_k is stable, e.g. using the basic method described in Section 7.3 of Francis (1987). It is emphasized that the factorization (Ψ_k, M_k) need not specify a valid time domain IQC as given by Definition 2. Robust performance is defined with the scaled uncertainty set $\mathbf{\Delta}_{1/\gamma}(\Pi_1, \ldots, \Pi_N)$ corresponding to scaled multipliers $S_{1/\gamma}\Pi_k S_{1/\gamma}$ $(k = 1, \ldots, N)$. A factorization for each scaled multiplier is given by $(\Psi_k S_{1/\gamma}, M_k)$. Let z_k denote the output of the scaled system $\Psi_k S_{1/\gamma}$ driven by the input/output signals (v, w) of Δ. Then $\{\Psi_k S_{1/\gamma}\}_{k=1}^N$ can be aggregated into a single system denoted $\Psi_{1/\gamma}$ with the following (minimal) state-space realization:

$$\begin{bmatrix} \dot{x}_\psi(t) \\ z_k(t) \end{bmatrix} = \begin{bmatrix} \tilde{A} & \gamma^{-1}\tilde{B}_v & \tilde{B}_w \\ \tilde{C}_{z_k} & \gamma^{-1}\tilde{D}_{z_k v} & \tilde{D}_{z_k w} \end{bmatrix} \begin{bmatrix} x_\psi(t) \\ v(t) \\ w(t) \end{bmatrix} \quad (k = 1, \ldots, N) \quad (10)$$

Eq. 10 uses an abbreviated notation to denote the outputs of $\Psi_{1/\gamma}$ are $[z_1^T, \ldots z_N^T]^T$. The robust performance analysis is based on Fig. 4 with $\Delta \in \mathbf{\Delta}_{1/\gamma}(\Pi_1, \ldots, \Pi_N)$. This interconnection is described by $w = \Delta(v)$ and the extended system of H_ρ and $\Psi_{1/\gamma}$:

$$\begin{bmatrix} \dot{x} \\ z_k \\ e \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\rho) & \mathcal{B}_w(\rho) & \mathcal{B}_d(\rho) \\ \mathcal{C}_{z_k}(\rho) & \mathcal{D}_{z_k w}(\rho) & \mathcal{D}_{z_k d}(\rho) \\ \mathcal{C}_e(\rho) & \mathcal{D}_{ew}(\rho) & \mathcal{D}_{ed}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ d \end{bmatrix} \quad (k = 1, \ldots, N) \quad (11)$$

where the state vector is $x = [x_H; x_\psi] \in \mathbb{R}^{n_H + n_\psi}$. The extended system can be expressed in terms of the state matrices for H_ρ (Eq. 9) and $\Psi_{1/\gamma}$ (Eq. 10).

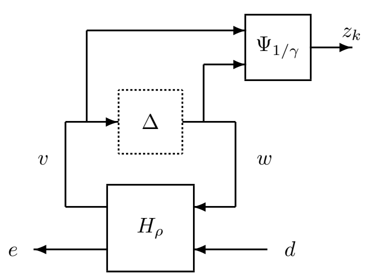

Fig. 4. Uncertain LPV system extended to include filter $\Psi_{1/\gamma}$

The robust performance analysis condition relies on a connection between $\Psi_{1/\gamma}$ and a combined multiplier $\Pi_\lambda := \sum_{k=1}^N \lambda_k S_{1/\gamma}\Pi_k S_{1/\gamma}$ where $\lambda_1 \in \mathbb{R}_{>0}$ and $\lambda_k \in \mathbb{R}_{\geq 0}$ $(k = 2, \ldots, N)$. Π_λ can be expressed in terms of the state-space realization of $\Psi_{1/\gamma}$ (Eq. 10) as:

$$\Pi_\lambda = \begin{bmatrix} (-sI - \tilde{A})^{-1}\tilde{B} \\ I \end{bmatrix}^T \begin{bmatrix} \tilde{Q}_\lambda & \tilde{S}_\lambda \\ \tilde{S}_\lambda^T & \tilde{R}_\lambda \end{bmatrix} \begin{bmatrix} (sI - \tilde{A})^{-1}\tilde{B} \\ I \end{bmatrix} \quad (12)$$

where $\tilde{B} := \begin{bmatrix} \gamma^{-1}\tilde{B}_v & \tilde{B}_w \end{bmatrix}$ and

$$\begin{bmatrix} \tilde{Q}_\lambda & \tilde{S}_\lambda \\ \tilde{S}_\lambda^T & \tilde{R}_\lambda \end{bmatrix} := \sum_{k=1}^N \lambda_k \begin{bmatrix} \tilde{C}_{z_k}^T \\ \gamma^{-1}\tilde{D}_{z_k v}^T \\ \tilde{D}_{z_k w}^T \end{bmatrix} M_k \begin{bmatrix} \tilde{C}_{z_k} & \gamma^{-1}\tilde{D}_{z_k v} & \tilde{D}_{z_k w} \end{bmatrix} \quad (13)$$

The conditions on $\{\lambda_k\}_{k=1}^N$ along with Assumptions 3 and 4 imply $(\Pi_\lambda)_{11}(j\omega) > 0$ and $(\Pi_\lambda)_{22}(j\omega) < 0$ $\forall \omega \in \mathbb{R} \cup \{+\infty\}$. Thus Π_λ has a J-spectral factorization (Lemma 4 in Seiler (2015)). This factorization is constructed from the stabilizing solution X to the ARE in Eq. A.2 with $(\tilde{A}, \tilde{B}, \tilde{Q}_\lambda, \tilde{S}_\lambda, \tilde{R}_\lambda)$. Let (Ψ, M) be a J-spectral factorization of Π_λ with $M := diag(I, -I)$. Then $(\Psi_\lambda, M_\lambda) := (S_\gamma \Psi, S_{1/\gamma} M S_{1/\gamma})$ is another factorization of Π_λ. This rescaled factorization has $M_\lambda := diag(\gamma^{-2}I, -I)$. A state-space realization for the rescaled filter Ψ_λ is:

$$\begin{bmatrix} \dot{x}_\psi(t) \\ z_\lambda(t) \end{bmatrix} = \begin{bmatrix} \tilde{A} & \gamma^{-1}\tilde{B}_v & \tilde{B}_w \\ \tilde{C}_{z_\lambda} & \tilde{D}_{z_\lambda v} & \tilde{D}_{z_\lambda w} \end{bmatrix} \begin{bmatrix} x_\psi(t) \\ v(t) \\ w(t) \end{bmatrix} \quad (14)$$

$(\Psi_\lambda, M_\lambda)$ is a valid time domain IQC for Δ (Seiler, 2015). Finally, an extended system of H_ρ and Ψ_λ can be formed:

$$\begin{bmatrix} \dot{x} \\ z_\lambda \\ e \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\rho) & \mathcal{B}_w(\rho) & \mathcal{B}_d(\rho) \\ \mathcal{C}_{z_\lambda}(\rho) & \mathcal{D}_{z_\lambda w}(\rho) & \mathcal{D}_{z_\lambda d}(\rho) \\ \mathcal{C}_e(\rho) & \mathcal{D}_{ew}(\rho) & \mathcal{D}_{ed}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ d \end{bmatrix} \quad (15)$$

This extended system can be expressed in terms of the state matrices for H_ρ (Eq. 9) and Ψ_λ (Eq. 14).

Two extended systems have been presented thus far. The extended system of H_ρ and $\Psi_{1/\gamma}$ (Eq. 11) can be used to define the following parameterized matrix inequality involving multiple IQCs (neglecting dependence on ρ):

$$\begin{bmatrix} P\mathcal{A} + \mathcal{A}^T P & P\mathcal{B}_w & P\mathcal{B}_d \\ \mathcal{B}_w^T P & 0 & 0 \\ \mathcal{B}_d^T P & 0 & -I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} \mathcal{C}_e^T \\ \mathcal{D}_{ew}^T \\ \mathcal{D}_{ed}^T \end{bmatrix} \begin{bmatrix} \mathcal{C}_e & \mathcal{D}_{ew} & \mathcal{D}_{ed} \end{bmatrix} + \sum_{k=1}^N \lambda_k \begin{bmatrix} \mathcal{C}_{z_k}^T \\ \mathcal{D}_{z_k w}^T \\ \mathcal{D}_{z_k d}^T \end{bmatrix} M_k \begin{bmatrix} \mathcal{C}_{z_k} & \mathcal{D}_{z_k w} & \mathcal{D}_{z_k d} \end{bmatrix} < 0 \quad (16)$$

Similarly, the extended system of H_ρ and Ψ_λ (Eq. 15) defines an inequality with the single, rescaled IQC:

$$\begin{bmatrix} \tilde{P}\mathcal{A} + \mathcal{A}^T \tilde{P} & \tilde{P}\mathcal{B}_w & \tilde{P}\mathcal{B}_d \\ \mathcal{B}_w^T \tilde{P} & 0 & 0 \\ \mathcal{B}_d^T \tilde{P} & 0 & -I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} \mathcal{C}_e^T \\ \mathcal{D}_{ew}^T \\ \mathcal{D}_{ed}^T \end{bmatrix} \begin{bmatrix} \mathcal{C}_e & \mathcal{D}_{ew} & \mathcal{D}_{ed} \end{bmatrix} + \begin{bmatrix} \mathcal{C}_{z_\lambda}^T \\ \mathcal{D}_{z_\lambda w}^T \\ \mathcal{D}_{z_\lambda d}^T \end{bmatrix} M_\lambda \begin{bmatrix} \mathcal{C}_{z_\lambda} & \mathcal{D}_{z_\lambda w} & \mathcal{D}_{z_\lambda d} \end{bmatrix} < 0 \quad (17)$$

The technical result regarding these two matrix inequalities is formally stated in the next lemma.

Lemma 1. *Let $\{\Pi_k\}_{k=1}^N \subset \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$, $\gamma > 0$, and $\{\lambda_k\}_{k=1}^N$ be given where $\{\Pi_k\}_{k=1}^N$ satisfies Assumptions 3 and 4, $\lambda_1 \in \mathbb{R}_{>0}$ and $\lambda_k \in \mathbb{R}_{\geq 0}$ $(k = 2, \ldots, N)$. Let each Π_k have a factorization (Ψ_k, M_k) where Ψ_k is stable. Define $\Pi_\lambda := \sum_{k=1}^N \lambda_k S_{1/\gamma} \Pi_k S_{1/\gamma}$. Let X denote the corresponding stabilizing solution to the ARE in Eq. A.2 with $(\tilde{A}, \tilde{B}, \tilde{Q}_\lambda, \tilde{S}_\lambda, \tilde{R}_\lambda)$. Finally, assume H_ρ is stable. Then the symmetric matrix $P = P^T$ satisfies Eq. 16 for all $\rho \in \mathcal{P}$ if and only if $\tilde{P} := P - \left[\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right] \geq 0$ satisfies Eq. 17 for all $\rho \in \mathcal{P}$.*

Proof. See Appendix A. □

Lemma 1 is used to prove the following result.

Theorem 2. *Assume $\mathcal{F}_u(H_\rho, \Delta)$ is well posed for all $\Delta \in \mathbf{\Delta}_{1/\gamma}(\Pi_1, \ldots, \Pi_N)$. Then H_ρ achieves robust performance of level γ if there exists a matrix $P = P^T \in \mathbb{R}^{(n_H+n_\psi)\times(n_H+n_\psi)}$ and scalars $\{\lambda_k\}_{k=1}^N$ such that (P, λ, γ) satisfy the parameterized matrix inequality in Eq. 16 for all $\rho \in \mathcal{P}$, $\lambda_1 \in \mathbb{R}_{>0}$ and $\lambda_k \in \mathbb{R}_{\geq 0}$ $(k = 2, \ldots, N)$.*

Proof. As described above, Π_λ has a rescaled J-spectral factorization $(\Psi_\lambda, M_\lambda)$. Define $\tilde{P} := P - \left[\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right] \geq 0$ where X is the stabilizing ARE solution used to construct this factorization. By Lemma 1, $\tilde{P}$ satisfies Eq. 17. Define the storage function $V : R^{n_H+n_\psi} \to \mathbb{R}_+$ as $V(x) := x^T \tilde{P} x$. Left and right multiply Eq. 17 by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ to show that V satisfies:

$$\dot{V}(t) + z_\lambda^T(t) M_\lambda z_\lambda(t) \leq d(t)^T d(t) - \gamma^{-2} e(t)^T e(t) \quad (18)$$

Append Ψ_λ to the (v, w) channels of the uncertain system $F_u(H_\rho, \Delta)$. This corresponds to the interconnection shown in Fig. 4 except with Ψ_λ replacing $\Psi_{1/\gamma}$. Let (x, w, d, z, e) be the solution of this interconnection for some $\Delta \in \mathbf{\Delta}_{1/\gamma}(\Pi_1, \ldots, \Pi_N)$, disturbance $d \in L_2^{n_d}$, admissible trajectory $\rho \in \mathcal{T}$, and zero initial conditions. Integrating Eq. 18 along this solution from $t = 0$ to $t = T$ yields: $V(x(T)) + \int_0^T z_\lambda(t)^T M_\lambda z_\lambda(t)\, dt + \frac{1}{\gamma^2} \int_0^T e(t)^T e(t)\, dt \leq \int_0^T d(t)^T d(t)\, dt$. It follows from $\lambda_k \geq 0$ that $\Delta \in \mathbf{\Delta}(\Pi_\lambda)$. In addition $(\Psi_\lambda, M_\lambda)$ is a valid time domain IQC for Δ. Apply this time domain IQC along with $\tilde{P} \geq 0$ to conclude that $\|e\| \leq \gamma \|d\|$. Hence H_ρ achieves robust performance of level γ. □

The parameterized matrix inequality (Eq. 16) involves N IQCs. Note that left/right multiplying Eq. 16 by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ does not yield a true dissipation inequality for two reasons. First, (Ψ_k, M_k) does not need to be a hard factorization and hence is not a valid time domain IQC. Second, the matrix P need not be positive definite and thus does not necessarily define a valid storage function. Lemma 1 addresses both issues. It converts the original problem to an alternative form (Eq. 17) involving only a single, valid time domain IQC. The alternative form involves $\tilde{P} = P - \left[\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right] \geq 0$ which defines a valid storage function. The term X can be interpreted as additional stored energy.

The conditions in Theorem 2 are sufficient to prove that $\|\mathcal{F}_u(H_\rho, \Delta)\| \leq \gamma$ for all $\Delta \in \mathbf{\Delta}_{1/\gamma}$. This is a (finite-gain) input-output stability result that appears frequently in literature, e.g. Schaft and A.J.Schaft (1999). The conditions are also sufficient to prove asymptotic stability of the system from any initial condition and for any disturbance input $d \in L_2^{n_d}$ using the arguments in Jönsson (1996) (Proposition 1.2 and its proof).

4.2 Scaled System

This section constructs a scaled system that is used to link the analysis and synthesis steps in our robust synthesis algorithm. Consider the uncertain system $\mathcal{F}_u(H_\rho, \Delta)$. Theorem 2 provides a sufficient condition to upper bound the robust performance of H_ρ. Recall that $M_\lambda := diag(\gamma^{-2} I, -I)$. Thus partitioning $z_\lambda := \left[\begin{smallmatrix} v_\lambda \\ w_\lambda \end{smallmatrix}\right]$ simplifies the dissipation inequality (Eq. 18) to $\dot{V} \leq \left(d^T d - \gamma^{-2} e^T e\right) + \left(w_\lambda^T w_\lambda - \gamma^{-2} v_\lambda^T v_\lambda\right)$. This has the form of a dissipation inequality used to prove a (nominal) LPV system from inputs (w_λ, d) to outputs (v_λ, e) has L_2 gain $\leq \gamma$. A scaled system is constructed based on this insight. First, rewrite the extended system of H_ρ and Ψ_λ (Eq. 15) by partitioning $z_\lambda := \left[\begin{smallmatrix} v_\lambda \\ w_\lambda \end{smallmatrix}\right]$:

$$\begin{bmatrix} \dot{x} \\ v_\lambda \\ w_\lambda \\ e \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\rho) & \mathcal{B}_w(\rho) & \mathcal{B}_d(\rho) \\ \mathcal{C}_{v_\lambda}(\rho) & \mathcal{D}_{v_\lambda w}(\rho) & \mathcal{D}_{v_\lambda d}(\rho) \\ \mathcal{C}_{w_\lambda}(\rho) & \mathcal{D}_{w_\lambda w}(\rho) & \mathcal{D}_{w_\lambda d}(\rho) \\ \mathcal{C}_e(\rho) & \mathcal{D}_{ew}(\rho) & \mathcal{D}_{ed}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ d \end{bmatrix} \quad (19)$$

Assume that $\mathcal{D}_{w_\lambda w}$ is nonsingular $\forall \rho \in \mathcal{P}$. Then the output equation for w_λ can be rewritten as: $w = \mathcal{D}_{w_\lambda w}^{-1}(w_\lambda - \mathcal{C}_{w_\lambda} x - \mathcal{D}_{w_\lambda d} d)$. Use this relation to substitute for w in Eq. 19. This gives the following "scaled" system:

$$\begin{bmatrix} \dot{x} \\ v_\lambda \\ e \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\rho) & \mathcal{B}_w(\rho) & \mathcal{B}_d(\rho) \\ \mathcal{C}_{v_\lambda}(\rho) & \mathcal{D}_{v_\lambda w}(\rho) & \mathcal{D}_{v_\lambda d}(\rho) \\ \mathcal{C}_e(\rho) & \mathcal{D}_{ew}(\rho) & \mathcal{D}_{ed}(\rho) \end{bmatrix} T(\rho) \begin{bmatrix} x \\ w_\lambda \\ d \end{bmatrix} \quad (20)$$

where $T(\rho)$ is defined as:

$$T(\rho) := \begin{bmatrix} I & 0 & 0 \\ -\mathcal{D}_{w_\lambda w}^{-1}(\rho)\mathcal{C}_{w_\lambda}(\rho) & \mathcal{D}_{w_\lambda w}^{-1}(\rho) & -\mathcal{D}_{w_\lambda w}^{-1}(\rho)\mathcal{D}_{w_\lambda d}(\rho) \\ 0 & 0 & I \end{bmatrix}$$

The next lemma gives a formal statement connecting robust performance of the extended system to nominal performance of this scaled system.

Lemma 2. *Let $\tilde{P} \geq 0$ and $\gamma > 0$ be given. The following statements are equivalent:*

(1) $(\tilde{P}, \gamma)$ satisfy the robust performance LMI (Eq. 17).
(2) $\mathcal{D}_{w_\lambda w}$ is nonsingular . Let $(\mathcal{A}_{scl}, \mathcal{B}_{scl}, \mathcal{C}_{scl}, \mathcal{D}_{scl})$ denote the state-space representation of the scaled system formed from H_ρ and Ψ_λ (Eq. 20). $(\tilde{P}, \gamma)$ satisfy the induced L_2 gain LMI (Eq. 2) associated with the scaled system $\forall \rho \in \mathcal{P}$:

$$\begin{bmatrix} \tilde{P}\mathcal{A}_{scl} + \mathcal{A}_{scl}^T \tilde{P} & \tilde{P}\mathcal{B}_{scl} \\ \mathcal{B}_{scl}^T \tilde{P} & -I \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} \mathcal{C}_{scl}^T \\ \mathcal{D}_{scl}^T \end{bmatrix} \begin{bmatrix} \mathcal{C}_{scl} & \mathcal{D}_{scl} \end{bmatrix} < 0 \quad (21)$$

Proof. ($1 \Rightarrow 2$) Assume statement 1 holds. The (2,2) block of Eq. 17 implies $\mathcal{D}_{w_\lambda w}^T \mathcal{D}_{w_\lambda w} > \gamma^{-2}(\mathcal{D}_{v_\lambda w}^T \mathcal{D}_{v_\lambda w} + \mathcal{D}_{ew}^T \mathcal{D}_{ew}) \geq 0$ and hence $\mathcal{D}_{w_\lambda w}$ is nonsingular. Next, note that T is nonsingular $\forall \rho \in \mathcal{P}$. Multiply Eq. 17 on the left/right by T^T/T to demonstrate that Eq. 21 holds. ($2 \Rightarrow 1$) follows by the inverse transformation. □

The lemma states that H_ρ satisfies the robust performance condition if and only if the scaled system satisfies the nominal performance condition. This scaled system is a complicated function of the state matrices of H_ρ and Ψ_λ. This is an issue because the robust synthesis algorithm requires the use of this result with the closed-loop, $H_\rho := \mathcal{F}_l(G_\rho, K_\rho)$. In fact, the scaled system has a simpler construction. It is formed by inverting the input/output channel associated with w to w_λ. This channel only involves the filter Ψ_λ which can be expressed as:

$$\begin{bmatrix} \dot{x}_\psi \\ v_\lambda \\ w_\lambda \end{bmatrix} = \begin{bmatrix} \tilde{A} & \gamma^{-1}\tilde{B}_v & \tilde{B}_w \\ \tilde{C}_{v_\lambda} & \tilde{D}_{v_\lambda v} & \tilde{D}_{v_\lambda w} \\ \tilde{C}_{w_\lambda} & \tilde{D}_{w_\lambda v} & \tilde{D}_{w_\lambda w} \end{bmatrix} \begin{bmatrix} x_\psi \\ v \\ w \end{bmatrix} \quad (22)$$

The condition $(\Pi_\lambda)_{22}(j\omega) < 0 \; \forall \omega \in \mathbb{R} \cup \{+\infty\}$ is sufficient to ensure that $\tilde{D}_{w_\lambda w}$ is nonsingular. Then w can be solved as: $w = \tilde{D}_{w_\lambda w}^{-1}(w_\lambda - \tilde{C}_{w_\lambda} x_\psi - \tilde{D}_{w_\lambda v} v)$. In this case, let $\Psi_\lambda^\dagger$ denote the filter from (v, w_λ) to (v_λ, w) obtained by inverting the w to w_λ channel of Ψ_λ:

$$\begin{bmatrix} \dot{x}_\psi \\ v_\lambda \\ w \end{bmatrix} = \begin{bmatrix} \tilde{A}(\rho) & \gamma^{-1}\tilde{B}_v(\rho) & \tilde{B}_w(\rho) \\ \tilde{C}_{v_\lambda}(\rho) & \tilde{D}_{v_\lambda v}(\rho) & \tilde{D}_{v_\lambda w}(\rho) \\ 0 & 0 & I \end{bmatrix} \tilde{T}(\rho) \begin{bmatrix} x_\psi \\ w_\lambda \\ v \end{bmatrix} \quad (23)$$

where $\tilde{T}(\rho)$ is defined as:

$$\tilde{T}(\rho) := \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ -\tilde{D}_{w_\lambda w}^{-1}(\rho)\tilde{C}_{w_\lambda}(\rho) & \tilde{D}_{w_\lambda w}^{-1}(\rho) & -\tilde{D}_{w_\lambda w}^{-1}(\rho)\tilde{D}_{w_\lambda v}(\rho) \end{bmatrix}$$

The next lemma provides an alternative, but equivalent, construction for the scaled system.

Lemma 3. *Assume $\tilde{D}_{w_\lambda w}$ is nonsingular so that $\Psi_\lambda^\dagger$ as defined in Eq. 23 is well-defined. Moreover, assume $\mathcal{D}_{w_\lambda w}$ is nonsingular $\forall \rho \in \mathcal{P}$ so that the scaled system formed from H_ρ and Ψ_λ (Eq. 20) is well-posed. Then the scaled system is equivalent to the LFT interconnection of H_ρ and $\Psi_\lambda^\dagger$ as shown in Fig. 5.*

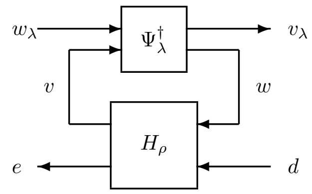

Fig. 5. LFT Interconnection of H_ρ and $\Psi_\lambda^\dagger$

Proof. Only algebra is involved to verify the equivalence of the two state-space realizations (Eqs. 20 and 22). □

4.3 Main Theorem

The main technical result for the proposed algorithm is that the iteration is well posed at each step and the robust performance is non-increasing at each iteration. Thus the robust performance will converge and the iteration will terminate. As with DK synthesis, there are no guarantees that the iteration will lead to a local optima let alone a global optima. However, it is a useful heuristic that enables robust synthesis to extend from LTI to LPV systems. This result is now stated.

Theorem 3. *The iteration is well-posed at each step and the iteration is non-increasing, i.e. $\gamma(i) \leq \gamma(i-1)$ for $i = 1, 2, \ldots$.*

Proof. The initial iteration $i = 1$ differs slightly from the subsequent ones. Specifically, $\lambda(0) = [1, 0, \ldots, 0]$ yields $\Pi_{\lambda(0)} = \Pi_1$ in Step 6. The definition of Π_1 implies that it has a J-factorization with $\Psi_1 := I_{n_v + n_w}$ and $M_1 := diag(I_{n_v}, -I_{n_w})$ in Step 7. Since no rescaling is used on the first iteration, the scaled system in Step 7 is simply $G_\rho^{scl} = G_\rho$. The synthesis step 8 is then performed with no modifications and yields a controller $K_\rho(1)$ that stabilizes G_ρ and achieves a closed-loop gain $\nu(1) < \infty$. The analysis step then achieves a robust performance $\gamma(1) < \infty$ because H_ρ is stable. Thus the first iteration is well-posed and achieves $\gamma(1) < \gamma(0) = +\infty$.

Subsequent iterations ($i > 1$) begin with the iteration count update (Step 4) and performance scaling definition (Step 5). Next the combined multiplier Π_λ is constructed. It has a J-spectral factorization. In addition, $(\Pi_\lambda)_{22}(+\infty) < 0$ implies that $\tilde{D}_{w_\lambda w}$ must be nonsingular. Hence by Lemma 3, the construction of $\Psi_\lambda^\dagger$ in Step 7 is well-defined. The analysis step from the previous iteration shows that there exists $(P(i-1), \lambda(i-1), \gamma(i-1))$ satisfying Eq. 16. By Lemma 1, this implies the existence of $\tilde{P}(i-1) \geq 0$ that, along with $(\lambda(i-1), \gamma(i-1))$, satisfies Eq. 17. Next, Lemma 2 states that feasibility of Eq. 17 implies that the scaled closed-loop of $H_\rho := \mathcal{F}_l(G_\rho, K_\rho(i-1))$ and Ψ_λ is well-posed and has induced gain $\leq \gamma(i-1)$. By Lemma 3, this scaled system can be represented by the LFT of H_ρ and

$\Psi_\lambda^\dagger$ as shown in Fig. 5. Removing the controller yields the scaled open-loop plant. Thus the construction of the scaled system in Step 7 is well-defined. Finally, the synthesis in Step 8 optimizes over all stabilizing controllers. Hence the new controller $K_\rho(i)$ must yield better nominal performance than $K_\rho(i-1)$: $\nu(i) \le \gamma(i-1)$. Thus $K_\rho(i)$ must satisfy the nominal performance LMI in Eq. 21 with the slightly larger cost of $\gamma := \gamma(i-1)$. Lemmas 2 and 1 can be used backward to the analysis condition in Step 9. Specifically, the closed-loop with $K_\rho(i)$ satisfies the analysis condition in Step 9 with $\gamma(i-1)$, $\lambda(i-1)$ and $P(i-1)$. Step 9 involves optimizing over all feasible λ and P. This yields a robust performance cost no greater than the previous step $\gamma(i) \le \gamma(i-1)$. □

5 Numerical Example

An example is used to demonstrate the applicability of the proposed robust synthesis algorithm. The example is based on an example that appears in Veenman and Scherer (2014) to test an alternative IQC synthesis algorithm for LTI systems. Here the example is extended to include plant dynamics described by an LPV system. As shown in Fig. 6, the nominal plant dynamics are given by the following LPV system F_ρ:

$$\dot{x}(t) = \left(-\frac{1}{71+2\rho} I_2\right) x(t) + \left(\frac{1}{71+2\rho} I_2\right) u(t) \quad (24)$$

$$y(t) = \begin{bmatrix} 87+0.2\rho^2 & -87.2+0.2\rho^2 \\ 107.4+0.2\rho^2 & -110.4+0.2\rho^2 \end{bmatrix} x(t) \quad (25)$$

F_ρ depends on a single scheduling parameter $\rho \in [1,3]$. The objective is to synthesize a robust controller K_{rob} that offers good tracking performance at low frequencies while penalizing control input at high frequencies. These objectives are specified via the weights $W_e = \frac{0.3(s+0.1)}{2s+10^{-5}} I_2$ and $W_u = \frac{s+10}{s+100} I_2$ on the error e and control input u, respectively. The controller should also be robust to the uncertainty Δ which is a block diagonal nonlinear perturbation, i.e. $\Delta := diag(\Delta_1, \Delta_2)$. Each block of Δ is a (scalar) dead zone operator $w_i = \Delta_i(v_i)$ defined by:

$$w_i = \Delta_i(v_i) := \begin{cases} v_i - b_i, & v_i > b_i \\ 0, & v_i \in [-b_i, b_i] \\ v_i + b_i, & v_i < -b_i \end{cases} \quad (26)$$

where $b_i = 0.05$ $(i = 1, 2)$. The uncertainty weight is defined as $W_d := diag(0.6, 0.3)$.

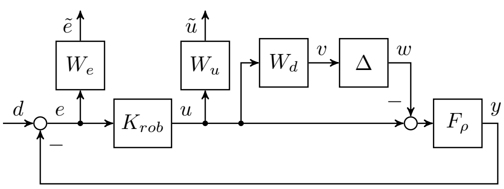

Fig. 6. Synthesis interconnection

Three IQCs are chosen to describe each dead zone Δ_i. The first is $\Pi_a = diag(1, -1)$. The second one $\Pi_b = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ is used to model the $[0,1]$ sector bound (Megretski and Rantzer, 1997) on the dead zone. The last IQC $\Pi_c = \begin{bmatrix} 0 & 1+H(s) \\ 1+H^\sim(s) & -(2+H(s)+H^\sim(s)) \end{bmatrix}$ with $H(s) = \frac{1}{s+1}$ corresponds to a Zames-Falb multiplier. This is used to model the monotonic odd nonlinearity (Megretski and Rantzer, 1997). These multipliers can be combined to obtain the following multiplier for Δ:

$$(\Pi_i, \Pi_j) := \begin{bmatrix} (\Pi_i)_{11} & 0 & (\Pi_i)_{12} & 0 \\ 0 & (\Pi_j)_{11} & 0 & (\Pi_j)_{12} \\ (\Pi_i)_{21} & 0 & (\Pi_i)_{22} & 0 \\ 0 & (\Pi_j)_{21} & 0 & (\Pi_j)_{22} \end{bmatrix} \quad (27)$$

Five extended IQCs (Eq. 27) are constructed to model Δ: $\Pi_1 := (\Pi_a, \Pi_a)$, $\Pi_2 := (\Pi_a, \Pi_b)$, $\Pi_3 := (\Pi_b, \Pi_a)$, $\Pi_4 := (\Pi_a, \Pi_c)$ and $\Pi_5 := (\Pi_c, \Pi_a)$. It is easy to check that $\{\Pi_n\}_{n=1}^5$ satisfy Assumptions 2, 3 and 4 in Section 3.1.

To apply the proposed algorithm, F_ρ is approximated with 5 points spaced equally in the parameter range $[1,3]$. After 3 iterations ($46.89\,s$), robust performance with the designed controller K_ρ converges to 0.96 using a stopping criteria $\epsilon_{tol} = 0.05$. As a comparison, a nominal LPV controller K_{nom} is designed for the system without uncertainty ($\Delta = 0$). The induced L_2 norm of the nominal system using K_{nom} and K_{rob} is given by 0.42 and 0.56, respectively. Next, the robust performance was assessed using the matrix inequality condition in Section 4.1. This yields 3.03 and 0.96 for K_{nom} and K_{rob}, respectively. The gap in robust performance between the two controllers is also illustrated by a time domain step response simulation (Fig. 7). In the simulation, unit step signals are injected into both channels of d simultaneously at $t = 10\,s$ and the parameter trajectory is given by $\rho(t) = \sin(0.05\,t) + 2$. The responses of y_1 and y_2 are shown in Fig. 7. It is seen that K_{nom} performs well (solid blue curve) when there is no uncertainty. However, it degrades dramatically (dash-dot red curve) when the uncertainty is added. In contrast, K_{rob} maintains good tracking and steady state error (dash green curve) even in the presence of the uncertainty.

6 CONCLUSION

This paper described a robust synthesis algorithm for a class of uncertain LPV systems. The proposed algorithm involves a coordinate-wise iteration between an LPV synthesis step and an IQC analysis step. It was shown that the closed-loop robust performance is a nonincreasing function of the iteration number. The effectiveness of this method was shown on a simple numerical example. Future work will consider refinements of the proposed algorithm including a more efficient parameterization of the IQC multipliers. In addition, the algorithm will be applied to design a robust LPV controller for a more realistic system.

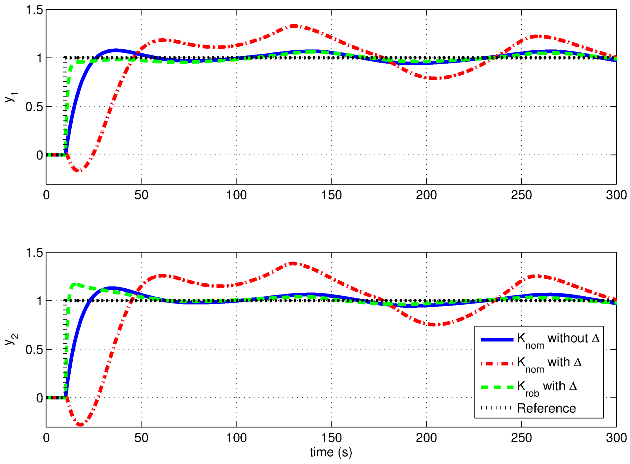

Fig. 7. Step responses

7 ACKNOWLEDGMENTS

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References

Apkarian, P. and Adams, R. (1998). Advanced gain-scheduling techniques for uncertain systems. *IEEE Trans. on Control Systems Technology*, 6(1), 21–32.

Balas, G., Chiang, R., Packard, A., and Safonov, M. (2007). Robust control toolbox 3 user's guide. Technical report, The Math Works, Inc.

Bobanac, V., Jelavić, M., and Perić, N. (2010). Linear parameter varying approach to wind turbine control. In *14th International Power Electronics and Motion Control Conference*, T12–60–T12–67.

Cantoni, M., Jönsson, U., and Khong, S. (2013). Robust stability analysis for feedback interconnections of time-varying linear systems. *SIAM J. of Control Optim.*, 51(1), 353–379.

Francis, B. (1987). *A Course in H_∞ Control Theory*. Springer-Verlag.

Jönsson, U. (1996). *Robustness analysis of uncertain and nonlinear systems*. Ph.D. thesis, Lund Institute of Technology.

Megretski, A. and Rantzer, A. (1997). System analysis via integral quadratic constraints. *IEEE Trans. on Automatic Control*, 42, 819–830.

Moreno, C., Seiler, P., and Balas, G. (2012). Linear parameter varying model reduction for aeroservoelastic systems. In *AIAA Atmospheric Flight Mechanics Conference*, Paper No. AIAA–2012–4859. doi: http://dx.doi.org/10.2514/6.2012-4859.

Pfifer, H. and Seiler, P. (2014). Robustness analysis of linear parameter varying systems using integral quadratic constraints. *accepted to the International Journal of Robust and Nonlinear Control*.

Schaft, A. and A.J.Schaft (1999). *L_2-gain and passivity in nonlinear control*. Springer-Verlag New York, Inc.

Scherer, C. and Kose, I. (2012). Gain-scheduled control synthesis using dynamic D-scales. *IEEE Trans. on Automatic Control*, 57, 2219–2234.

Seiler, P. (2015). Stability analysis with dissipation inequalities and integral quadratic constraints. *IEEE Trans. on Automatic Control*, 60(6), 1704–1709.

Turner, M. and Kerr, M. (2012). L_2 gain bounds for systems with sector bounded and slope-restricted nonlinearities. *International Journal of Robust and Nonlinear Control*, 22(13), 1505–1521.

Veenman, J. and Scherer, C. (2010). On robust LPV controller synthesis: A dynamic integral quadratic constraint based approach. In *IEEE Conference on Decision and Control*, 591–596.

Veenman, J. and Scherer, C. (2014). IQC-synthesis with general dynamic multipliers. *International Journal of Robust and Nonlinear Control*, 24, 3027–3056.

Wang, S., Pfifer, H., and Seiler, P. (2014). Robust synthesis for linear parameter varying systems using integral quadratic constraints. In *IEEE Conference on Decision and Control*, 4789–4794.

Willems, J. (1971). Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Trans. on Aut. Control*, 16, 621–634.

Wu, F. (1995). *Control of Linear Parameter Varying Systems*. Ph.D. thesis, University of California, Berkeley.

Wu, F., Yang, X.H., Packard, A., and Becker, G. (1996). Induced $\mathcal{L}_2$ norm control for LPV systems with bounded parameter variation rates. *International Journal of Robust and Nonlinear Control*, 6, 983–998.

Yakubovich, V. (1971). S-procedure in nonlinear control theory. *Vestnik Leningrad Univ.*, 62–77.

Zhou, K., Doyle, J., and Glover, K. (1996). *Robust and Optimal Control*. Prentice-Hall.

A Proof of Lemma 1

Proof. ($\Rightarrow$) Assume $P = P^T$ satisfies Eq. 16. The output z_k from $\Psi_{1/\gamma}$ is a linear function of (x_ψ, v, w) as defined in Eq. 10: $z_k = \left[\tilde{C}_{z_k} \; \gamma^{-1}\tilde{D}_{z_k v} \; \tilde{D}_{z_k w} \right] \left[\begin{smallmatrix} x_\psi \\ v \\ w \end{smallmatrix} \right]$. These variables (x_ψ, v, w) can, in turn, be expressed in terms of the extended system state and inputs (x, w, d) as:

$$\begin{bmatrix} x_\psi \\ v \\ w \end{bmatrix} = L(\rho) \begin{bmatrix} \begin{bmatrix} x_H \\ x_\psi \end{bmatrix} \\ w \\ d \end{bmatrix} \tag{A.1}$$

where $L(\rho)$ is defined as:

$$L(\rho) := \begin{bmatrix} [0, I] & 0 & 0 \\ [C_v(\rho), 0] & D_{vw}(\rho) & D_{vd}(\rho) \\ [0, 0] & I & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x_H \\ x_\psi \end{bmatrix} \\ w \\ d \end{bmatrix}$$

Thus, using the extended system state matrices, the second term of the matrix inequality in Eq. 16 can be rewritten as:

$$\sum_{k=1}^{N} \lambda_k \begin{bmatrix} \mathcal{C}_{z_k}^T \\ \mathcal{D}_{z_k w}^T \\ \mathcal{D}_{z_k d}^T \end{bmatrix} M_k \left[\mathcal{C}_{z_k} \; \mathcal{D}_{z_k w} \; \mathcal{D}_{z_k d} \right] = L(\rho)^T \begin{bmatrix} \tilde{Q}_\lambda & \tilde{S}_\lambda \\ \tilde{S}_\lambda^T & \tilde{R}_\lambda \end{bmatrix} L(\rho)$$

$\tilde{Q}_\lambda$, $\tilde{S}_\lambda$, and $\tilde{R}_\lambda$ are defined in Eq. 13. Substitute for $\tilde{Q}_\lambda$ using the ARE:

$$\tilde{A}^T X + X\tilde{A} - (X\tilde{B} + \tilde{S}_\lambda)\tilde{R}_\lambda^{-1}(X\tilde{B} + \tilde{S}_\lambda)^T + \tilde{Q}_\lambda = 0 \tag{A.2}$$

Rearrange terms in the matrix inequality to show that $\tilde{P} := P - \left[\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right]$ satisfies Eq. 17.

This direction of the proof is completed by showing that $\tilde{P} \geq 0$. Define $V(x_0) := x_0^T \tilde{P} x_0$ and the cost functional $V^*(x_0)$:

$$V^*(x_0) := \sup_{w \in L_2^{n_w}[0,\infty)} \int_0^\infty z_\lambda(t)^T M_\lambda z_\lambda(t)\, dt \tag{A.3}$$

subject to:

$$\dot{x} = \mathcal{A}(\rho)x + \mathcal{B}_w(\rho)w, \qquad x(0) = x_0$$
$$z_\lambda = \mathcal{C}_{z_\lambda}(\rho)x + \mathcal{D}_{z_\lambda w}(\rho)w$$

The disturbance input of the extended system is neglected ($d = 0$) in this linear quadratic optimization. Note that the extended system is stable since H_ρ is stable (by assumption), Ψ_λ is stable (by construction), and Ψ_λ is connected in an open loop fashion to H_ρ. First we show that $V(x_0) \geq V^*(x_0)$ for all $x_0 \in \mathbb{R}^{n_H + n_\psi}$. This follows along the lines of Theorems 2 and 3 in Willems (1971) and hence the proof is only sketched. Let $x(t)$, $z_\lambda(t)$ be the resulting solutions of the extended system of H_ρ and Ψ_λ for a given input $w \in L_2^{n_w}[0,\infty)$, admissible trajectory $\rho \in \mathcal{T}$, and initial condition $x_0 \in \mathbb{R}^{n_H+n_\psi}$ assuming $d = 0$. Multiply the matrix inequality in Eq. 17 on the left/right by $\begin{bmatrix} x(t) \\ w(t) \\ 0 \end{bmatrix}^T$ and $\begin{bmatrix} x(t) \\ w(t) \\ 0 \end{bmatrix}$ to show $\dot{V}(x(t)) + z_\lambda(t)^T M_\lambda z_\lambda(t) \leq 0$. Integrate this inequality from $t = 0$ to $t = T$ to obtain

$$V(x(T)) + \int_0^T z_\lambda(t)^T M_\lambda z_\lambda(t)\, dt \leq V(x_0) \tag{A.4}$$

$\lim_{T\to\infty} x(T) = 0$ for any $w \in L_2^{n_w}[0,\infty)$ because the extended system is stable. Maximizing the left side of Eq. A.4 over $w \in L_2^{n_w}[0,\infty)$ for $T = \infty$ thus yields $V(x_0) \geq V^*(x_0)$.

Next, consider the max/min game defined for the rescaled J-spectral factorization $(\Psi_\lambda, M_\lambda)$:

$$\underline{J}(x_{\psi 0}) := \sup_{w \in L_2^{n_w}[0,\infty)} \inf_{v \in L_2^{n_v}[0,\infty)} \int_0^\infty z_\lambda(t)^T M_\lambda z_\lambda(t)\, dt \tag{A.5}$$

subject to:

$$\dot{x}_\psi = \tilde{A}x_\psi + \tilde{B}\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right], \quad x_\psi(0) = x_{\psi 0}$$
$$z = \tilde{C}_{z_\lambda}x_\psi + \tilde{D}_{z_\lambda}\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$$

where $\tilde{D}_{z_\lambda} := [\tilde{D}_{z_\lambda v}, \, \tilde{D}_{z_\lambda w}]$. This max/min game is connected to the quadratic optimization defined in Eq. A.3. Specifically, restricting v in the max/min game to be the output of H_ρ generated by $w \in L_2$, $d = 0$, and $x_H(0) = x_{H0}$ yields the quadratic optimization in Eq. A.3. This specific choice of v yields a value that is no lower than the infimum over all possible $v \in L_2$. Hence the max/min game yields the bound $\underline{J}(x_{\psi 0}) \leq V^*(x_0)$. By Theorem 4 of Seiler (2015), the cost of this max/min game is $\underline{J}(x_{\psi 0}) = 0$. Putting these results together yields the following inequality $0 = \underline{J}(x_{\psi 0}) \leq V^*(x_0) \leq V(x_0) := x_0^T \tilde{P} x_0$ This holds for any x_0 and thus $\tilde{P} \geq 0$.

($\Leftarrow$) This direction of the proof essentially involves reversing the algebraic rearrangement to go from the matrix inequality in Eq. 17 to the form in Eq. 16. □