

Analysis of Approximate Stochastic Gradient Using Quadratic Constraints and Sequential Semidefinite Programs

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Abstract

We present convergence rate analysis for the approximate stochastic gradient method, where individual gradient updates are corrupted by computation errors. We develop stochastic quadratic constraints to formulate a small linear matrix inequality (LMI) whose feasible set characterizes convergence properties of the approximate stochastic gradient. Based on this LMI condition, we develop a sequential minimization approach to analyze the intricate trade-offs that couple step-size selection, convergence rate, optimization accuracy, and robustness to gradient inaccuracy. We also analytically solve this LMI condition and obtain theoretical formulas that quantify the convergence properties of the approximate stochastic gradient under various assumptions on the loss functions.

1 Introduction

Empirical risk minimization (ERM) is prevalent topic in machine learning research [7, 32]. Ridge regression, ℓ_2 -regularized logistic regression, and support vector machines (SVM) can all be formulated as the following standard ERM problem

$$\min_{x \in \mathbb{R}^p} g(x) = \frac{1}{n} \sum_{i=1}^n f_i(x), \quad (1.1)$$

where $g : \mathbb{R}^p \rightarrow \mathbb{R}$ is the objective function. The stochastic gradient (SG) method [3, 5, 26] has been widely used for ERM to exploit redundancy in the training data. The SG method applies the update rule

$$x_{k+1} = x_k - \alpha_k w_k, \quad (1.2)$$

where $w_k = \nabla f_{i_k}(x_k)$ and the index i_k is uniformly sampled from $\{1, 2, \dots, n\}$ in an independent and identically distributed (IID) manner. The convergence properties of the SG method are well understood. SG with a diminishing stepsize converges sublinearly, while SG with a constant stepsize

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converges linearly to a ball around the optimal solution [13, 21, 23, 24]. In the latter case, epochs can be used to balance convergence rate and optimization accuracy. Some recently-developed stochastic methods such as SAG [27, 28], SAGA [10], Finito [11], SDCA [30], and SVRG [17] converge linearly with low iteration cost when applied to (1.1), though SG is still popular because of its simple iteration form and low memory footprint. SG is also commonly used as an initialization for other algorithms [27, 28].

In this paper, we present a general analysis for *approximate SG*. This is a version of SG where the gradient updates $\nabla f_{i_k}(x_k)$ are corrupted by additive as well as multiplicative noise. In practice, such errors can be introduced by sources such as: inaccurate numerical solvers, quantization, or sparsification. The approximate SG update equation is given by

$$x_{k+1} = x_k - \alpha_k(w_k + e_k). \quad (1.3)$$

Here, $w_k = \nabla f_{i_k}(x_k)$ is the individual gradient update and e_k is an error term. We consider the following error model, which unifies the error models in [2]:

$$\|e_k\|^2 \leq \delta^2 \|w_k\|^2 + c^2, \quad (1.4)$$

where $\delta \geq 0$ and $c \geq 0$ bound the relative error and the absolute error in the oracle computation, respectively. If $\delta = c = 0$, then $e_k = 0$ and we recover the standard SG setup. The model (1.4) unifies the error models in [2] since:

1. If $c = 0$, then (1.4) reduces to a relative error model, i.e.

$$\|e_k\| \leq \delta \|w_k\| \quad (1.5)$$

2. If $\delta = 0$, then e_k is a bounded absolute error, i.e.

$$\|e_k\| \leq c \quad (1.6)$$

We assume that both δ and c are known in advance. We make no assumptions about how e_k is generated, just that it satisfies (1.4). Thus, we will seek a worst-case bound that holds regardless of whether e_k is random, set in advance, or chosen adversarially.

Suppose the cost function g admits a unique minimizer x_* . For standard SG (without computation error), w_k is an unbiased estimator of $\nabla g(x_k)$. Hence under many circumstances, one can control the final optimization error $\|x_k - x_*\|$ by decreasing the stepsize α_k . Specifically, suppose g is m -strongly convex. Under various assumptions on f_i , one can prove the following typical bound for standard SG with a constant stepsize α [4, 22, 24]:

$$\mathbb{E}\|x_k - x_*\|^2 \leq \rho^{2k} \mathbb{E}\|x_0 - x_*\|^2 + H_* \quad (1.7)$$

where $\rho^2 = 1 - 2m\alpha + O(\alpha^2)$ and $H_* = O(\alpha)$. By decreasing stepsize α , one can control the final optimization error H_* at the price of slowing down the convergence rate ρ . The convergence behavior of the approximate SG method is different. Since the error term e_k can be chosen adversarially, the sum $(w_k + e_k)$ may no longer be an unbiased estimator of $\nabla g(x_k)$. The error term e_k may introduce a bias which cannot be overcome by decreasing stepsize α . Hence the final optimization error in the approximate SG method heavily depends on the error model of e_k . This paper quantifies the convergence properties of the approximate SG iteration (1.3) with the error model (1.4).

Main contribution. The main novelty of this paper is that we present a unified analysis approach to simultaneously address the relative error and the absolute error in the gradient computation. We formulate a linear matrix inequality (LMI) that characterizes the convergence properties of the approximate SG method and couples the relationship between δ , c , α_k and the assumptions on f_i . This convex program can be solved both numerically and analytically to obtain various convergence bounds for the approximate SG method. Based on this LMI, we develop a sequential minimization approach that can analyze the approximate SG method with an arbitrary time-varying stepsize. We also obtain analytical rate bounds in the form of (1.7) for the approximate SG method with constant stepsize. However, our bound requires $\rho^2 = 1 - \frac{m^2 - \delta^2 \tilde{M}}{m} \alpha + O(\alpha^2)$ ¹ and $H_\star = \frac{c^2 + 2\delta^2 G^2}{m^2 - \delta^2 \tilde{M}} + O(\alpha)$ where \tilde{M} and G^2 are some prescribed constants determined by the assumptions on f_i . Based on this result, there is no way to shrink H_\star to 0. This is consistent with our intuition since the gradient estimator as well as the final optimization result can be biased. We show that this “uncontrollable” biased optimization error is $\frac{c^2 + 2\delta^2 G^2}{m^2 - \delta^2 \tilde{M}}$. The resultant analytical rate bounds highlight the design trade-offs for the approximate SG method.

The work in this paper complements the ongoing research on stochastic optimization methods, which mainly focuses on the case where the oracle computation is exact. Notice there is typically no need to optimize below the so-called estimation error for machine learning problems [3, 6] so stepsize selection in approximate SG must also address the trade-offs between speed, accuracy, and inexactness in the oracle computations. Our approach addresses such trade-offs and can be used to provide guidelines for stepsize selection of approximate SG with inexact gradient computation. It is also worth mentioning that the robustness of full gradient methods with respect to gradient inexactness has been extensively studied [9, 12, 29]. The existing analysis for inexact full gradient methods may be directly extended to handle the approximate SG with only bounded absolute error, i.e. $\delta = 0$ and $c > 0$. However, addressing a general error model that combines the absolute error and the relative error is non-trivial. Our proposed approach complements the existing analysis methods in [9, 12, 29] by providing a unified treatment of the general error model (1.4).

The approach taken in this paper can be viewed as a stochastic generalization of the work in [19, 25] that analyzes deterministic optimization methods (gradient descent, Nesterov’s method, ADMM, etc.) using a dynamical systems perspective and semidefinite programs. Our generalization is non-trivial since the convergence properties of SG are significantly different from its deterministic counterparts and new stochastic notions of quadratic constraints are required in the analysis. The trade-offs involving ρ^2 and H_\star in (1.7) are unique to SG because $H_\star = 0$ in the deterministic case. In addition, the analysis for (deterministic) approximate gradient descent in [19] is numerical. In this paper, we derive analytical formulas quantifying the convergence properties of the approximate SG. Another related work [16] combines jump system theory with quadratic constraints to analyze SAGA, Finito, and SDCA. The analysis in [16] also does not involve the trade-offs between the convergence speed and the optimization error, and cannot be easily tailored to the SG method and its approximate variant.

The rest of the paper is organized as follows. In Section 2, we develop a stochastic quadratic constraint approach to formulate LMI testing conditions for convergence analysis of the approximate SG method. The resultant LMIs are then solved sequentially, yielding recursive convergence bounds for the approximate SG method. In Section 3, we simplify the analytical solutions of the resultant

¹When $\delta = c = 0$, this rate bound does not reduce to $\rho^2 = 1 - 2m\alpha + O(\alpha^2)$. This is due to the inherent differences between the analyses of the approximate SG and the standard SG. See Remark 4 for a detailed explanation.

sequential LMIs and derive analytical rate bounds in the form of (1.7) for the approximate SG method with a constant stepsize. Our results highlight various design trade-offs for the approximate SG method. Finally, we show how existing results on standard SG (without gradient computation error) can be recovered using our proposed LMI approach, and discuss how a time-varying stepsize can potentially impact the convergence behaviors of the approximate SG method (Section 4).

1.1 Notation

The $p \times p$ identity matrix and the $p \times p$ zero matrix are denoted as I_p and 0_p , respectively. The subscript p is occasionally omitted when the dimensions are clear by the context. The Kronecker product of two matrices A and B is denoted $A \otimes B$. We will make use of the properties $(A \otimes B)^\top = A^\top \otimes B^\top$ and $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ when the matrices have the compatible dimensions.

Definition 1 (Smooth functions). *A differentiable function $f : \mathbb{R}^p \rightarrow \mathbb{R}$ is L -smooth for some $L > 0$ if the following inequality is satisfied:*

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\| \quad \text{for all } x, y \in \mathbb{R}^p.$$

Definition 2 (Convex functions). *Let $\mathcal{F}(m, L)$ for $0 \leq m \leq L \leq \infty$ denote the set of differentiable functions $f : \mathbb{R}^p \rightarrow \mathbb{R}$ satisfying the following inequality:*

$$\begin{bmatrix} x - y \\ \nabla f(x) - \nabla f(y) \end{bmatrix}^\top \begin{bmatrix} -2mI_p & (1 + \frac{m}{L})I_p \\ (1 + \frac{m}{L})I_p & -\frac{2}{L}I_p \end{bmatrix} \begin{bmatrix} x - y \\ \nabla f(x) - \nabla f(y) \end{bmatrix} \geq 0 \quad \text{for all } x, y \in \mathbb{R}^p. \quad (1.8)$$

Note that $\mathcal{F}(0, \infty)$ is the set of all convex functions, $\mathcal{F}(0, L)$ is the set of all convex L -smooth functions, $\mathcal{F}(m, \infty)$ with $m > 0$ is the set of all m -strongly convex functions, and $\mathcal{F}(m, L)$ with $m > 0$ is the set of all m -strongly convex and L -smooth functions. If $f \in \mathcal{F}(m, L)$ with $m > 0$, then f has a unique global minimizer.

Definition 3. *Let $\mathcal{S}(m, L)$ for $0 \leq m \leq L \leq \infty$ denote the set of differentiable functions $g : \mathbb{R}^p \rightarrow \mathbb{R}$ having some global minimizer $x_\star \in \mathbb{R}^p$ and satisfying the following inequality:*

$$\begin{bmatrix} x - x_\star \\ \nabla g(x) \end{bmatrix}^\top \begin{bmatrix} -2mI_p & (1 + \frac{m}{L})I_p \\ (1 + \frac{m}{L})I_p & -\frac{2}{L}I_p \end{bmatrix} \begin{bmatrix} x - x_\star \\ \nabla g(x) \end{bmatrix} \geq 0 \quad \text{for all } x \in \mathbb{R}^p. \quad (1.9)$$

If $g \in \mathcal{S}(m, L)$ with $m > 0$, then x_\star is also the unique stationary point of g . It is worth noting that $\mathcal{F}(m, L) \subset \mathcal{S}(m, L)$. In general, a function $g \in \mathcal{S}(m, L)$ may not be convex. If $g \in \mathcal{S}(m, \infty)$, then g may not be smooth. The condition (1.9) is similar to the notion of *one-point convexity* [1, 8, 31] and *star-convexity* [18].

1.2 Assumptions

Referring to the problem setup (1.1), we will adopt the general assumption that $g \in \mathcal{S}(m, \infty)$ with $m > 0$. So in general, g may not be convex. We will analyze four different cases, characterized by different assumptions on the individual f_i :

- I. Bounded shifted gradients: $\|\nabla f_i(x) - mx\| \leq \beta$ for all $x \in \mathbb{R}^p$.²

²This case is a variant of the common assumption $\frac{1}{n} \sum_{i=1}^n \|\nabla f_i(x)\|^2 \leq \beta$. One can check that this case holds for several ℓ_2 -regularized problems including SVM and logistic regression.

II. f_i is L -smooth.

III. $f_i \in \mathcal{F}(0, L)$.

IV. $f_i \in \mathcal{F}(m, L)$.

Assumption I is a natural assumption for SVM³ and logistic regression while Assumptions II, III, or IV can be used for ridge regression, logistic regression, and smooth SVM. The m assumed in cases I and IV is the same as the m used in the assumption on $g \in \mathcal{S}(m, \infty)$.

2 Analysis framework

2.1 LMI conditions for the analysis of approximate SG

To analyze the convergence properties of approximate SG, we can formulate LMI testing conditions using stochastic quadratic constraints. This is formalized by the next lemma.

Lemma 1. *Consider the approximate SG iteration (1.3). Suppose a fixed $x_\star \in \mathbb{R}^p$ and matrices $X^{(j)} = (X^{(j)})^\top \in \mathbb{R}^{3 \times 3}$ and a scalar $\gamma^{(j)} \geq 0$ for $j = 1, \dots, J$ are given such that the following quadratic constraints hold for all k :*

$$\mathbb{E} \left(\begin{bmatrix} x_k - x_\star \\ w_k \\ e_k \end{bmatrix}^\top (X^{(j)} \otimes I_p) \begin{bmatrix} x_k - x_\star \\ w_k \\ e_k \end{bmatrix} \right) \geq -\gamma^{(j)}, \quad j = 1, \dots, J. \quad (2.1)$$

If there exist non-negative scalars $z_k^{(j)} \geq 0$ for $j = 1, \dots, J$ and $\rho_k \geq 0$ such that

$$\begin{bmatrix} 1 - \rho_k^2 & -\alpha_k & -\alpha_k \\ -\alpha_k & \alpha_k^2 & \alpha_k^2 \\ -\alpha_k & \alpha_k^2 & \alpha_k^2 \end{bmatrix} + \sum_{j=1}^J z_k^{(j)} X^{(j)} \leq 0, \quad (2.2)$$

then the approximate SG iterates satisfy

$$\mathbb{E} \|x_{k+1} - x_\star\|^2 \leq \rho_k^2 \mathbb{E} \|x_k - x_\star\|^2 + \sum_{j=1}^J z_k^{(j)} \gamma^{(j)}. \quad (2.3)$$

Proof. The proof relies on the following algebraic relation

$$\begin{aligned} & \|x_{k+1} - x_\star\|^2 - \rho_k^2 \|x_k - x_\star\|^2 \\ &= \|x_k - x_\star - \alpha_k w_k - \alpha_k e_k\|^2 - \rho_k^2 \|x_k - x_\star\|^2 \\ &= \begin{bmatrix} x_k - x_\star \\ w_k \\ e_k \end{bmatrix}^\top \begin{bmatrix} (1 - \rho_k^2)I_p & -\alpha_k I_p & -\alpha_k I_p \\ -\alpha_k I_p & \alpha_k^2 I_p & \alpha_k^2 I_p \\ -\alpha_k I_p & \alpha_k^2 I_p & \alpha_k^2 I_p \end{bmatrix} \begin{bmatrix} x_k - x_\star \\ w_k \\ e_k \end{bmatrix} \end{aligned} \quad (2.4)$$

³The loss functions for SVM are non-smooth, and w_k is actually updated using the subgradient information. For simplicity, we abuse our notation and use ∇f_i to denote the subgradient of f_i for SVM problems.

Since (2.2) holds, we have

$$\left(\begin{bmatrix} 1 - \rho_k^2 & -\alpha_k & -\alpha_k \\ -\alpha_k & \alpha_k^2 & \alpha_k^2 \\ -\alpha_k & \alpha_k^2 & \alpha_k^2 \end{bmatrix} + \sum_{j=1}^J z_k^{(j)} X^{(j)} \right) \otimes I_p \leq 0$$

Left and right multiply the above inequality by $[(x_k - x_\star)^\top \ w_k^\top \ e_k^\top]$ and $[(x_k - x_\star)^\top \ w_k^\top \ e_k^\top]^\top$ respectively, take full expectation, and apply (2.4) to get

$$\mathbb{E}\|x_{k+1} - x_\star\|^2 - \rho_k^2 \mathbb{E}\|x_k - x_\star\|^2 + \sum_{j=1}^J z_k^{(j)} \mathbb{E} \left(\begin{bmatrix} x_k - x_\star \\ w_k \\ e_k \end{bmatrix}^\top (X^{(j)} \otimes I_p) \begin{bmatrix} x_k - x_\star \\ w_k \\ e_k \end{bmatrix} \right) \leq 0$$

Applying the quadratic constraints (2.1) to the above inequality, we obtain (2.3), as required. \square

When $X^{(j)}$ and α_k are given, the matrix in (2.2) is linear in ρ_k^2 and $z_k^{(j)}$, so (2.2) is a *linear matrix inequality* (LMI) whose feasible set is convex and can be efficiently searched using interior point methods. Since the goal is to obtain the best possible bound, it is desirable to make both ρ_k^2 and $\sum_{j=1}^J z_k^{(j)} \gamma^{(j)}$ small. These are competing objectives, and we can trade them off by minimizing an objective of the form $\rho_k^2 t_k + \sum_{j=1}^J z_k^{(j)} \gamma^{(j)}$.

Remark 1. Notice (2.3) can be used to prove various types of convergence results. For example, when a constant stepsize is used, i.e. $\alpha_k = \alpha$ for all k , a naive analysis can be performed by setting $t_k = t$ for all k . Then, $\rho_k = \rho$ and $z_k^{(j)} = z^{(j)}$ for all k and both (2.2) and (2.3) become independent of k . We can rewrite (2.3) as

$$\mathbb{E}\|x_{k+1} - x_\star\|^2 \leq \rho^2 \mathbb{E}\|x_k - x_\star\|^2 + \sum_{j=1}^J z^{(j)} \gamma^{(j)}. \quad (2.5)$$

If $\rho < 1$, then we may recurse (2.5) to obtain the following convergence result:

$$\begin{aligned} \mathbb{E}\|x_k - x_\star\|^2 &\leq \rho^{2k} \mathbb{E}\|x_0 - x_\star\|^2 + \left(\sum_{i=0}^{k-1} \rho^{2i} \right) \left(\sum_{j=1}^J z^{(j)} \gamma^{(j)} \right) \\ &\leq \rho^{2k} \mathbb{E}\|x_0 - x_\star\|^2 + \frac{\sum_{j=1}^J z^{(j)} \gamma^{(j)}}{1 - \rho^2}. \end{aligned} \quad (2.6)$$

The inequality (2.6) is an error bound of the familiar form (1.7). Nevertheless, this bound may be conservative even in the constant stepsize case. To minimize the right-hand side of (2.3), the best choice for t_k in the objective function is $\mathbb{E}\|x_k - x_\star\|^2$, which changes with k . Consequently, setting t_k to be a constant may introduce conservatism even in the constant stepsize case. To overcome this issue, we will later introduce a sequential minimization approach.

Lemma 1 presents a general connection between the quadratic constraints (2.1) and convergence properties of approximate SG. The quadratic constraints (2.1) couple x_k , w_k , and e_k . In the following lemma, we will collect specific instances of such quadratic constraints. In particular, the properties of g and f_i imply a coupling between x_k and w_k , while the error model for e_k implies a coupling between w_k and e_k .

Case	Desired assumption on each f_i	Value of M	Value of G
I	$(f_i(x) - \frac{m}{2}\ x\ ^2)$ have bounded gradients; $\ \nabla f_i(x) - mx\ \leq \beta$.	$M = \begin{bmatrix} -m^2 & m \\ m & -1 \end{bmatrix}$	$G^2 = \beta^2 + m^2 \ x_\star\ ^2$
II	The f_i are L -smooth, but are not necessarily convex.	$M = \begin{bmatrix} 2L^2 & 0 \\ 0 & -1 \end{bmatrix}$	$G^2 = \frac{1}{n} \sum_{i=1}^n \ \nabla f_i(x_\star)\ ^2$
III	The f_i are convex and L -smooth; $f_i \in \mathcal{F}(0, L)$.	$M = \begin{bmatrix} 0 & L \\ L & -1 \end{bmatrix}$	$G^2 = \frac{1}{n} \sum_{i=1}^n \ \nabla f_i(x_\star)\ ^2$
IV	The f_i are m -strongly convex and L -smooth; $f_i \in \mathcal{F}(m, L)$	$M = \begin{bmatrix} -2mL & L + m \\ L + m & -1 \end{bmatrix}$	$G^2 = \frac{1}{n} \sum_{i=1}^n \ \nabla f_i(x_\star)\ ^2$

Table 1: Given that $g \in \mathcal{S}(m, \infty)$, this table shows different possible assumptions about the f_i and their corresponding values of M and G such that (2.8) holds.

Lemma 2. Consider the approximate SG iteration (1.3) with $g \in \mathcal{S}(m, \infty)$ for some $m > 0$, and let x_\star be the unique global minimizer of g .

1. The following inequality holds.

$$\mathbb{E} \begin{bmatrix} x_k - x_\star \\ w_k \end{bmatrix}^\top \left(\begin{bmatrix} -2m & 1 \\ 1 & 0 \end{bmatrix} \otimes I_p \right) \begin{bmatrix} x_k - x_\star \\ w_k \end{bmatrix} \geq 0. \quad (2.7)$$

2. For each of the four conditions on f_i and the corresponding M and G in Table 1, the following inequality holds.

$$\mathbb{E} \begin{bmatrix} x_k - x_\star \\ w_k \end{bmatrix}^\top (M \otimes I_p) \begin{bmatrix} x_k - x_\star \\ w_k \end{bmatrix} \geq -2G^2. \quad (2.8)$$

3. If $\|e_k\|^2 \leq \delta^2 \|w_k\|^2 + c^2$ for every k , then

$$\mathbb{E} \begin{bmatrix} w_k \\ e_k \end{bmatrix}^\top \left(\begin{bmatrix} \delta^2 & 0 \\ 0 & -1 \end{bmatrix} \otimes I_p \right) \begin{bmatrix} w_k \\ e_k \end{bmatrix} \geq -c^2. \quad (2.9)$$

Proof. See Appendix A. □

Statements 1 and 2 in Lemma 2 present quadratic constraints coupling $(x_k - x_\star)$ and w_k , while Statement 3 presents a quadratic constraint coupling w_k and e_k . All these quadratic constraints can be trivially lifted into the general form (2.1). Specifically, we define

$$\begin{aligned} X^{(1)} &= \begin{bmatrix} -2m & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & X^{(2)} &= \begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, & X^{(3)} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta^2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ \gamma^{(1)} &= 0, & \gamma^{(2)} &= 2G^2, & \gamma^{(3)} &= c^2. \end{aligned} \quad (2.10)$$

Then $j = 1$, $j = 2$, and $j = 3$ correspond to (2.7), (2.8), and (2.9), respectively. Therefore, we have quadratic constraints of the form (2.1) with $(X^{(j)}, \gamma^{(j)})$ defined by (2.10). It is worth mentioning

that our constraints (2.1) can be viewed as stochastic counterparts of the so-called biased local quadratic constraints for deterministic dynamical systems [20]. Now we are ready to present a small linear matrix inequality (LMI) whose feasible set characterizes the convergence properties of the approximate SG (1.3) with the error model (1.4).

Theorem 1 (Main Theorem). *Consider the approximate SG iteration (1.3) with $g \in \mathcal{S}(m, \infty)$ for some $m > 0$, and let x_* be the unique global minimizer of g . Given one of the four conditions on f_i and the corresponding $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ and G from Table 1, if the following holds for some choice of nonnegative $\lambda_k, \nu_k, \mu_k, \rho_k$,*

$$\begin{bmatrix} -\rho_k^2 - 2\nu_k m + \lambda_k M_{11} & \nu_k + \lambda_k M_{12} & -1 & 0 \\ \nu_k + \lambda_k M_{21} & \mu_k \delta^2 + \lambda_k M_{22} & \alpha_k & 0 \\ -1 & \alpha_k & -1 & \alpha_k \\ 0 & 0 & \alpha_k & -\mu_k \end{bmatrix} \leq 0 \quad (2.11)$$

where the inequality is taken in the semidefinite sense, then the approximate SG iterates satisfy

$$\mathbb{E}\|x_{k+1} - x_*\|^2 \leq \rho_k^2 \mathbb{E}\|x_k - x_*\|^2 + (2\lambda_k G^2 + \mu_k c^2) \quad (2.12)$$

Proof. Let x_k be generated by the approximate SG iteration (1.3) under the error model (1.4). By Lemma 2, the quadratic constraints (2.1) hold with $(X^{(j)}, \gamma^{(j)})$ defined by (2.10). Taking the Schur complement of (2.11) with respect to the (3, 3) entry, (2.11) is equivalent to

$$\begin{bmatrix} 1 - \rho_k^2 & -\alpha_k & -\alpha_k \\ -\alpha_k & \alpha_k^2 & \alpha_k^2 \\ -\alpha_k & \alpha_k^2 & \alpha_k^2 \end{bmatrix} + \nu_k \begin{bmatrix} -2m & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda_k \begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu_k \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta^2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \leq 0 \quad (2.13)$$

which is exactly LMI (2.2) if we set $z_k^{(1)} = \nu_k$, $z_k^{(2)} = \lambda_k$, and $z_k^{(3)} = \mu_k$. Now the desired conclusion follows as a consequence of Lemma 1. \square

Applying Theorem 1. For a fixed δ , the matrix in (2.11) is linear in $(\rho_k^2, \nu_k, \mu_k, \lambda_k, \alpha_k)$, so potentially the LMI (2.11) can be used to adaptively select stepsizes. The feasibility of (2.11) can be numerically checked using standard semidefinite program solvers. All simulations in this paper were implemented using CVX, a package for specifying and solving convex programs [14, 15]. One may also obtain analytical formulas for certain feasibility points of the LMI (2.11) due to its relatively simple form. Our analytical bounds for approximate SG are based on the following result.

Corollary 1. *Choose one of the four conditions on f_i and the corresponding $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ and G from Table 1. Also define $\tilde{M} = M_{11} + 2mM_{12}$. Consider the approximate SG iteration (1.3) with $g \in \mathcal{S}(m, \infty)$ for some $m > 0$, and let x_* be the unique global minimizer of g . Suppose the stepsize satisfies $0 < M_{21}\alpha_k \leq 1$. Then the approximate SG method (1.3) with the error model (1.4) satisfies the bound (2.12) with the following nonnegative parameters*

$$\mu_k = \alpha_k^2(1 + \zeta_k^{-1}) \quad (2.14a)$$

$$\lambda_k = \alpha_k^2(1 + \zeta_k)(1 + \delta^2\zeta_k^{-1}) \quad (2.14b)$$

$$\rho_k^2 = (1 + \zeta_k)(1 - 2m\alpha_k + \tilde{M}\alpha_k^2(1 + \delta^2\zeta_k^{-1})) \quad (2.14c)$$

where ζ_k is a parameter that satisfies $\zeta_k > 0$ and $\zeta_k \geq \frac{\alpha_k M_{21} \delta^2}{1 - \alpha_k M_{21}}$. Each choice of ζ_k yields a different bound in (2.12).

Proof. We further define

$$\nu_k = \alpha_k(1 + \zeta_k)(1 - \alpha_k M_{21}(1 + \delta^2 \zeta_k^{-1})) \quad (2.15)$$

We will show that (2.14) and (2.15) are a feasible solution for (2.11). We begin with (2.11) and take the Schur complement with respect to the (3,3) entry of the matrix, leading to

$$\begin{bmatrix} 1 - \rho_k^2 - 2\nu_k m + \lambda_k M_{11} & \nu_k + \lambda_k M_{12} - \alpha_k & -\alpha_k \\ \nu_k + \lambda_k M_{21} - \alpha_k & \mu_k \delta^2 + \lambda_k M_{22} + \alpha_k^2 & \alpha_k^2 \\ -\alpha_k & \alpha_k^2 & \alpha_k^2 - \mu_k \end{bmatrix} \leq 0 \quad (2.16)$$

Examining the (3,3) entry, we deduce that $\mu_k > \alpha_k^2$, for if we had equality instead, the rest of the third row and column would be zero, forcing $\alpha_k = 0$. Substituting $\mu_k = \alpha_k^2(1 + \zeta_k^{-1})$ for some $\zeta_k > 0$ and taking the Schur complement with respect to the (3,3) entry, we see (2.16) is equivalent to

$$\begin{bmatrix} 1 - \rho_k^2 - 2\nu_k m + \lambda_k M_{11} + \zeta_k & \nu_k + \lambda_k M_{12} - \alpha_k(1 + \zeta_k) \\ \nu_k + \lambda_k M_{21} - \alpha_k(1 + \zeta_k) & \lambda_k M_{22} + \alpha_k^2(1 + \zeta_k)(1 + \delta^2 \zeta_k^{-1}) \end{bmatrix} \leq 0 \quad (2.17)$$

In (2.17), $\zeta_k > 0$ is a parameter that we are free to choose, and each choice yields a different set of feasible tuples $(\rho_k^2, \lambda_k, \mu_k, \nu_k)$. One way to obtain a feasible tuple is to set the left side of (2.17) equal to the zero matrix. This shows (2.11) is feasible with the following parameter choices.

$$\mu_k = \alpha_k^2(1 + \zeta_k^{-1}) \quad (2.18a)$$

$$\lambda_k = -\alpha_k^2(1 + \zeta_k)(1 + \delta^2 \zeta_k^{-1})M_{22}^{-1} \quad (2.18b)$$

$$\nu_k = \alpha_k(1 + \zeta_k) - \lambda_k M_{21} \quad (2.18c)$$

$$\rho_k^2 = 1 - 2\nu_k m + \lambda_k M_{11} + \zeta_k \quad (2.18d)$$

Since we always have $M_{22} = -1$ in Table 1, it is straightforward to verify that (2.18) is equivalent to (2.14) and (2.15). Notice that we directly have $\mu_k \geq 0$ and $\lambda_k \geq 0$ because $\zeta_k > 0$. In order to ensure $\rho_k^2 \geq 0$ and $\nu_k \geq 0$, we must have $1 - 2m\alpha_k + \tilde{M}\alpha_k^2(1 + \delta^2 \zeta_k^{-1}) \geq 0$ and $\alpha_k M_{21}(1 + \delta^2 \zeta_k^{-1}) \leq 1$, respectively. The first inequality always holds because $\tilde{M} \geq m^2$ and we have $1 - 2m\alpha_k + \tilde{M}\alpha_k^2(1 + \delta^2 \zeta_k^{-1}) \geq 1 - 2m\alpha_k + m^2\alpha_k^2 \geq (1 - m\alpha_k)^2 \geq 0$. Based on the conditions $0 \leq \alpha_k M_{21} < 1$ and $\zeta_k \geq \frac{\alpha_k M_{21} \delta^2}{1 - \alpha_k M_{21}}$, we conclude that the second inequality always holds as well. Since we have constructed a feasible solution to the LMI (2.11), the bound (2.12) follows from Theorem 1. \square

Given α_k , Corollary 1 provides a one-dimensional family of solutions to the LMI (2.11). These solutions are given by (2.14) and (2.15) and are parameterized by the auxiliary variable ζ_k .

Remark 2. Depending on the case being considered, the condition $\alpha_k M_{21} \leq 1$ imposes the following strict upper bound on α_k :

Case	I	II	III	IV
M_{21}	m	0	L	$L + m$
$\tilde{M} = M_{11} + 2mM_{12}$	m^2	$2L^2$	$2mL$	$2m^2$
α_k bound	$\frac{1}{m}$	∞	$\frac{1}{L}$	$\frac{1}{L+m}$

Corollary 1 does not require $\rho_k \leq 1$. Hence it actually does not impose any upper bound on α_k in Case II. Later we will impose refined upper bounds on α_k such that the bound (2.12) can be transformed into a useful bound in the form of (1.7).

2.2 Sequential minimization approach for approximate SG

We will quantify the convergence behaviors of the approximate SG method by providing upper bounds for $\mathbb{E}\|x_k - x_\star\|^2$. To do so, we will recursively make use of the bound (2.12). Suppose δ , c , and G are constant. Define $\mathcal{T}_k \subseteq \mathbb{R}_+^4$ to be the set of tuples $(\rho_k, \lambda_k, \mu_k, \nu_k)$ that are feasible points for the LMI (2.11). Also define the real number sequence $\{U_k\}_{k \geq 0}$ via the recursion:

$$U_0 \geq \mathbb{E}\|x_0 - x_\star\|^2 \quad \text{and} \quad U_{k+1} = \rho_k^2 U_k + 2\lambda_k G^2 + \mu_k c^2 \quad (2.19)$$

where $(\rho_k, \lambda_k, \mu_k, \nu_k) \in \mathcal{T}_k$. By induction, we can show that U_k provides an upper bound for the error at timestep k . Indeed, if $\mathbb{E}\|x_k - x_\star\|^2 \leq U_k$, then by Theorem 1,

$$\begin{aligned} \mathbb{E}\|x_{k+1} - x_\star\|^2 &\leq \rho_k^2 \mathbb{E}\|x_k - x_\star\|^2 + 2\lambda_k G^2 + \mu_k c^2 \\ &\leq \rho_k^2 U_k + 2\lambda_k G^2 + \mu_k c^2 \\ &= U_{k+1} \end{aligned}$$

A key issue in computing a useful upper bound U_k is which tuple $(\rho_k, \lambda_k, \mu_k, \nu_k) \in \mathcal{T}_k$ to choose. If the stepsize is constant ($\alpha_k = \alpha$), then \mathcal{T}_k is independent of k . Thus we may choose the same particular solution $(\rho, \lambda, \mu, \nu)$ for each k . Then, based on (2.6), if $\rho < 1$ we can obtain a bound of the following form for the approximate SG method:

$$\mathbb{E}\|x_k - x_\star\|^2 \leq \rho^{2k} U_0 + \frac{2\lambda G^2 + \mu c^2}{1 - \rho^2}. \quad (2.20)$$

As discussed in Remark 1, the above bound may be unnecessarily conservative. Because of the recursive definition (2.19), the bound U_k depends solely on U_0 and the parameters $\{\rho_t, \lambda_t, \mu_t\}_{t=0}^{k-1}$. So we can seek the smallest possible upper bound by solving the optimization problem:

$$\begin{aligned} U_{k+1}^{\text{opt}} &= \underset{\{\rho_t, \lambda_t, \mu_t, \nu_t\}_{t=0}^k}{\text{minimize}} && U_{k+1} \\ &\text{subject to} && U_{t+1} = \rho_t^2 U_t + 2\lambda_t G^2 + \mu_t c^2 \quad t = 0, \dots, k \\ &&& (\rho_t, \lambda_t, \mu_t, \nu_t) \in \mathcal{T}_t \quad t = 0, \dots, k \end{aligned}$$

This optimization problem is similar to a dynamic programming and a recursive solution reminiscent of the Bellman equation can be derived for the optimal bound U_k .

$$\begin{aligned} U_{k+1}^{\text{opt}} &= \underset{(\rho, \lambda, \mu, \nu) \in \mathcal{T}_k}{\text{minimize}} \left\{ \begin{array}{l} \underset{\{\rho_t, \lambda_t, \mu_t, \nu_t\}_{t=0}^{k-1}}{\text{minimize}} \quad U_{k+1} \\ \text{subject to} \quad U_{t+1} = \rho_t^2 U_t + 2\lambda_t G^2 + \mu_t c^2 \quad t = 0, \dots, k \\ (\rho_t, \lambda_t, \mu_t, \nu_t) \in \mathcal{T}_t \quad t = 0, \dots, k-1 \\ (\rho, \lambda, \mu, \nu) = (\rho_k, \lambda_k, \mu_k, \nu_k) \end{array} \right\} \\ &= \underset{(\rho, \lambda, \mu, \nu) \in \mathcal{T}_k}{\text{minimize}} \left\{ \begin{array}{l} \underset{\{\rho_t, \lambda_t, \mu_t, \nu_t\}_{t=0}^{k-1}}{\text{minimize}} \quad \rho^2 U_k + 2\lambda G^2 + \mu c^2 \\ \text{subject to} \quad U_{t+1} = \rho_t^2 U_t + 2\lambda_t G^2 + \mu_t c^2 \quad t = 0, \dots, k-1 \\ (\rho_t, \lambda_t, \mu_t, \nu_t) \in \mathcal{T}_t \quad t = 0, \dots, k-1 \end{array} \right\} \\ &= \underset{(\rho, \lambda, \mu, \nu) \in \mathcal{T}_k}{\text{minimize}} \quad \rho^2 U_k^{\text{opt}} + 2\lambda G^2 + \mu c^2 \quad (2.21) \end{aligned}$$

Where the final equality in (2.21) relies on the fact that $\rho^2 \geq 0$. Thus, a greedy approach where U_{t+1} is optimized in terms of U_t recursively for $t = 0, \dots, k-1$ yields a bound U_k that is in fact globally optimal over all possible choices of parameters $\{\rho_t, \lambda_t, \mu_t, \nu_t\}_{t=0}^k$.

Obtaining an explicit analytical formula for U_k^{opt} is not straightforward, since it involves solving a sequence of semidefinite programs. However, we can make use of Corollary 1 to further upper-bound U_k^{opt} . This works because Corollary 1 gives an analytical parameterization of a *subset* of \mathcal{T}_k . Denote this new upper bound by \hat{U}_k . By Corollary 1, we have:

$$\begin{aligned} \hat{U}_{k+1} = \underset{\zeta > 0}{\text{minimize}} \quad & \rho^2 \hat{U}_k + 2\lambda G^2 + \mu c^2 \\ \text{subject to} \quad & \mu = \alpha_k^2 (1 + \zeta^{-1}) \\ & \lambda = \alpha_k^2 (1 + \zeta) (1 + \delta^2 \zeta^{-1}) \\ & \rho^2 = (1 + \zeta) (1 - 2m\alpha_k + \tilde{M}\alpha_k^2 (1 + \delta^2 \zeta^{-1})) \\ & \zeta \geq \frac{\alpha_k M_{21} \delta^2}{1 - \alpha_k M_{21}} \end{aligned} \tag{2.22}$$

Note that Corollary 1 also places bounds on α_k , which we assume are being satisfied here. The optimization problem (2.22) is a single-variable smooth constrained problem. It is straightforward to verify that μ , λ , and ρ^2 are convex functions of ζ when $\zeta > 0$. Moreover, the inequality constraint on ζ is linear, so we deduce that (2.22) is a convex optimization problem.

Thus, we have reduced the problem of recursively solving semidefinite programs (finding U_k^{opt}) to recursively solving single-variable convex optimization problems (finding \hat{U}_k). Ultimately, we obtain an upper bound on the expected error of approximate SG that is easy to compute:

$$\mathbb{E} \|x_k - x_\star\|^2 \leq U_k^{\text{opt}} \leq \hat{U}_k \tag{2.23}$$

Preliminary numerical simulations suggest that \hat{U}_k seems to be equal to U_k^{opt} under the four sets of assumptions in this paper. However, we are unable to show $\hat{U}_k = U_k^{\text{opt}}$ analytically. In the subsequent sections, we will solve the recursion for \hat{U}_k analytically and thereby derive bounds on the performance of approximate SG.

2.3 Analytical recursive bounds for approximate SG

We showed in the previous section that $\mathbb{E} \|x_k - x_\star\|^2 \leq \hat{U}_k$ for the approximate SG method, where \hat{U}_k is the solution to (2.22). We now derive an analytical recursive formula for \hat{U}_k . Let us simplify the optimization problem (2.22). Eliminating ρ , λ , μ , we obtain

$$\begin{aligned} \hat{U}_{k+1} = \underset{\zeta > 0}{\text{minimize}} \quad & a_k (1 + \zeta^{-1}) + b_k (1 + \zeta) \\ \text{subject to} \quad & a_k = \alpha_k^2 (c^2 + 2\delta^2 G^2 + \tilde{M}\delta^2 \hat{U}_k) \\ & b_k = (1 - 2m\alpha_k + \tilde{M}\alpha_k^2) \hat{U}_k + 2\alpha_k^2 G^2 \\ & \zeta \geq \frac{\alpha_k M_{21} \delta^2}{1 - \alpha_k M_{21}} \end{aligned} \tag{2.24}$$

The assumptions on α_k from Corollary 1 also imply that $a_k \geq 0$ and $b_k \geq 0$. We may now solve this problem explicitly and we summarize the solution to (2.24) in the following lemma.

Lemma 3. Consider the approximate SG iteration (1.3) with $g \in \mathcal{S}(m, \infty)$ for some $m > 0$, and let x_* be the unique global minimizer of g . Given one of the four conditions on f_i and the corresponding $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ and G from Table 1, further assume α_k is strictly positive and satisfies $M_{21}\alpha_k \leq 1$. Then the error bound \hat{U}_k defined in (2.24) can be computed recursively as follows.

$$\hat{U}_{k+1} = \begin{cases} (\sqrt{a_k} + \sqrt{b_k})^2 & \sqrt{\frac{a_k}{b_k}} \geq \frac{\alpha_k M_{21} \delta^2}{1 - \alpha_k M_{21}} \\ a_k + b_k + a_k \frac{1 - \alpha_k M_{21}}{\alpha_k M_{21} \delta^2} + b_k \frac{\alpha_k M_{21} \delta^2}{1 - \alpha_k M_{21}} & \text{otherwise} \end{cases} \quad (2.25)$$

where a_k and b_k are defined in (2.24). We may initialize the recursion at any $\hat{U}_0 \geq \mathbb{E} \|x_0 - x_*\|^2$.

Proof. In Case II, we have $M_{21} = 0$ so the constraint on ζ is vacuously true. Therefore, the only constraint on ζ in (2.24) is $\zeta > 0$ and we can solve the problem by setting the derivative of the objective function with respect to ζ equal to zero. The result is $\zeta_k = \sqrt{\frac{a_k}{b_k}}$. In Cases I, III, and IV, we have $M_{21} > 0$. By convexity, the optimal ζ_k is either the unconstrained optimum (if it is feasible) or the boundary point (otherwise). Hence (2.25) holds as desired. Note that if $\delta = c = 0$, then $a_k = 0$. This corresponds to the pathological case where the objective reduces to $b_k(1 + \zeta)$. Here, the optimum is achieved as $\zeta \rightarrow 0$, which corresponds to $\mu \rightarrow \infty$ in (2.22). This does not cause a problem because $c = 0$ so μ does not appear in the objective function. The recursion (2.25) then simplifies to $\hat{U}_{k+1} = b_k$. \square

Remark 3. If $M_{21} = 0$ (Case II in Table 1) or if $\delta = 0$ (no multiplicative noise), the optimization problem (2.24) reduces to an unconstrained optimization problem whose solution is

$$\begin{aligned} \hat{U}_{k+1} &= (\sqrt{a_k} + \sqrt{b_k})^2 \\ &= \left(\alpha_k \sqrt{c^2 + 2\delta^2 G^2 + \tilde{M} \delta^2 \hat{U}_k} + \sqrt{(1 - 2m\alpha_k + \tilde{M} \alpha_k^2) \hat{U}_k + 2G^2 \alpha_k^2} \right)^2 \end{aligned} \quad (2.26)$$

3 Analytical rate bounds for the constant stepsize case

In this section, we present non-recursive error bounds for approximate SG with constant stepsize. Specifically, we assume $\alpha_k = \alpha$ for all k and we either apply Lemma 3 or carefully choose a constant ζ in order to obtain a tractable bound for \hat{U}_k . The bounds derived in this section highlight the trade-offs inherent in the design of the approximate SG method.

3.1 Linearization of the nonlinear recursion

This first result applies to the case where $\delta = 0$ or $M_{21} = 0$ (Case II) and leverages Remark 3 to obtain a bound for approximate SG.

Corollary 2. Consider the approximate SG iteration (1.3) with $g \in \mathcal{S}(m, \infty)$ for some $m > 0$, and let x_* be the unique global minimizer of g . Given one of the four conditions on f_i and the corresponding $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ and G from Table 1, further assume that $\alpha_k = \alpha > 0$ (constant stepsize), $M_{21}\alpha \leq 1$, and either $\delta = 0$ or $M_{21} = 0$. Define $p, q, r, s \geq 0$ as follows.

$$p = \tilde{M} \delta^2 \alpha^2, \quad q = (c^2 + 2G^2 \delta^2) \alpha^2, \quad r = 1 - 2m\alpha + \tilde{M} \alpha^2, \quad s = 2G^2 \alpha^2. \quad (3.1)$$

Where $\tilde{M} = M_{11} + 2mM_{12}$. If $\sqrt{p} + \sqrt{r} < 1$ then we have the following iterate error bound:

$$\mathbb{E} \|x_k - x_\star\|^2 \leq \left(\frac{p\sqrt{\hat{U}_\star}}{\sqrt{p\hat{U}_\star + q}} + \frac{r\sqrt{\hat{U}_\star}}{\sqrt{r\hat{U}_\star + s}} \right)^k \mathbb{E} \|x_0 - x_\star\|^2 + \hat{U}_\star, \quad (3.2)$$

where the fixed point \hat{U}_\star is given by

$$\hat{U}_\star = \frac{(p-r)(s-q) + q + s + 2\sqrt{ps^2 + q^2r + qs(1-p-r)}}{(p-r)^2 - 2(p+r) + 1}. \quad (3.3)$$

Proof. By Remark 3, we have the nonlinear recursion (2.26) for \hat{U}_k . This recursion is of the form

$$\hat{U}_{k+1} = \left(\sqrt{p\hat{U}_k + q} + \sqrt{r\hat{U}_k + s} \right)^2, \quad (3.4)$$

where $p, q, r, s > 0$ are given in (3.1). It is straightforward to verify that the right-hand side of (3.4) is a monotonically increasing concave function of \hat{U}_k and its asymptote is a line of slope $(\sqrt{p} + \sqrt{r})^2$. Thus, (3.4) will have a unique fixed point when $\sqrt{p} + \sqrt{r} < 1$. We will return to this condition shortly. When a fixed point exists, it is found by setting $\hat{U}_k = \hat{U}_{k+1} = \hat{U}_\star$ in (3.4) and yields U_\star given by (3.3). The concavity property further guarantees that any first-order Taylor expansion of the right-hand side of (3.4) yields an upper bound to \hat{U}_{k+1} . Expanding about \hat{U}_\star , we obtain:

$$\hat{U}_{k+1} - \hat{U}_\star \leq \left(\frac{p\sqrt{\hat{U}_\star}}{\sqrt{p\hat{U}_\star + q}} + \frac{r\sqrt{\hat{U}_\star}}{\sqrt{r\hat{U}_\star + s}} \right) (\hat{U}_k - \hat{U}_\star) \quad (3.5)$$

which leads to the following non-recursive bound for the approximate SG method.

$$\begin{aligned} \mathbb{E} \|x_k - x_\star\|^2 \leq \hat{U}_k &\leq \left(\frac{p\sqrt{\hat{U}_\star}}{\sqrt{p\hat{U}_\star + q}} + \frac{r\sqrt{\hat{U}_\star}}{\sqrt{r\hat{U}_\star + s}} \right)^k (\hat{U}_0 - \hat{U}_\star) + \hat{U}_\star \\ &\leq \left(\frac{p\sqrt{\hat{U}_\star}}{\sqrt{p\hat{U}_\star + q}} + \frac{r\sqrt{\hat{U}_\star}}{\sqrt{r\hat{U}_\star + s}} \right)^k \hat{U}_0 + \hat{U}_\star \end{aligned} \quad (3.6)$$

Since this bound holds for any $\hat{U}_0 \geq \mathbb{E} \|x_0 - x_\star\|^2$, it holds in particular when we have equality, and thus we obtain (3.2) as required. \square

The condition that $\sqrt{p} + \sqrt{r} < 1$ from Corollary 2, which is necessary for the existence of a fixed-point of (3.4), is equivalent to an upper bound on α . After manipulation, it amounts to:

$$\alpha < \frac{2(m - \delta\sqrt{\tilde{M}})}{\tilde{M}(1 - \delta^2)} \quad (3.7)$$

Therefore, we can ensure that $\sqrt{p} + \sqrt{r} < 1$ when $\delta < m/\sqrt{\tilde{M}}$, and α is sufficiently small. If $\delta = 0$, the stepsize bound (3.7) is only relevant in Case II. For Cases I, III, and IV, the bound $M_{21}\alpha \leq 1$ imposes a stronger restriction on α (see Remark 2). If $\delta \neq 0$, we only consider Case II ($M_{21} = 0$) and the resultant bound for α is $\frac{m - \sqrt{2}L\delta}{L^2(1 - \delta^2)}$. The condition $\delta < m/\sqrt{\tilde{M}}$ becomes $\delta < m/(\sqrt{2}L)$.

To see the trade-offs in the design of the approximate SG method, we can take Taylor expansions of several key quantities about $\alpha = 0$ to see how changes in α affect convergence:

$$\hat{U}_\star \approx \frac{c^2 + 2\delta^2 G^2}{m^2 - \delta^2 \tilde{M}} + \frac{m \left(c^2(\tilde{M} - m^2) + 2(1 - \delta^2) G^2 m^2 \right)}{(m^2 - \delta^2 \tilde{M})^2} \alpha + O(\alpha^2) \quad (3.8a)$$

$$\left(\frac{p\sqrt{\hat{U}_\star}}{\sqrt{p\hat{U}_\star + q}} + \frac{r\sqrt{\hat{U}_\star}}{\sqrt{r\hat{U}_\star + s}} \right) \approx 1 - \frac{(m^2 - \delta^2 \tilde{M})}{m} \alpha + O(\alpha^2) \quad (3.8b)$$

We conclude that when $\delta < m/\sqrt{\tilde{M}}$, the approximate SG method converges linearly to a ball whose radius is roughly $\hat{U}_\star \geq 0$. One can decrease the stepsize α to control the final error \hat{U}_\star . However, due to the errors in the individual gradient updates, one cannot guarantee the final error $\mathbb{E} \|x_k - x_\star\|^2$ smaller than $\frac{c^2 + 2\delta^2 G^2}{m^2 - \delta^2 \tilde{M}}$. This is consistent with our intuition; one could inject noise in an adversarial manner to shift the optimum point away from x_\star so there is no way to guarantee that $\{x_k\}$ converges to x_\star just by decreasing the stepsize α .

Remark 4. *One can check that the left side of (3.8b) is not differentiable at $(c, \alpha) = (0, 0)$. Consequently, taking a Taylor expansion with respect to α and then setting $c = 0$ does not yield the same result as first setting $c = 0$ and then taking a Taylor expansion with respect to α of the resulting expression. This explains why (3.8b) does not reduce to $\rho^2 = 1 - 2m\alpha + O(\alpha^2)$ when $c = \delta = 0$. It is worth noting that the higher order term $O(\alpha^2)$ in (3.8b) depends on c . Indeed, it goes to infinity as $c \rightarrow 0$. Therefore, the rate formula (3.8b) only describes the stepsize design trade-off for a fixed positive c and sufficiently small α .*

The non-recursive bound (3.2) relied on a linearization of the recursive formula (3.4), which involved a time-varying ζ_k . It is emphasized that we assumed that either $\delta = 0$ or $M_{21} = 0$. In the other cases, namely $\delta > 0$ and $M_{21} > 0$ (Case I, III, or IV), we cannot ignore the additional condition $\zeta_k \geq \frac{\alpha M_{21} \delta^2}{1 - \alpha M_{21}}$ and we must use the hybrid recursive formula (2.25). This hybrid formulation is more problematic to solve explicitly. However, if we are mostly interested in the regime where α is small, we can obtain non-recursive bounds similar to (3.2) by carefully choosing a constant ζ for all k . We will develop these bounds in the next section.

3.2 Non-recursive bounds via a fixed ζ parameter

When α is small, we can choose $\zeta = m\alpha$ and we obtain the following result.

Corollary 3. *Consider the approximate SG iteration (1.3) with $g \in \mathcal{S}(m, \infty)$ for some $m > 0$, and let x_\star be the unique global minimizer of g . Given one of the four conditions on f_i and the corresponding $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ and G from Table 1, further assume that $\alpha_k = \alpha > 0$ (constant stepsize), and $M_{21} \left(\alpha + \frac{\delta^2}{m} \right) \leq 1$.⁴ Finally, assume that*

$$0 < \tilde{\rho}^2 < 1 \quad \text{where } \tilde{\rho}^2 = 1 - \frac{m^2 - \tilde{M}\delta^2}{m} \alpha + (\tilde{M}(1 + \delta^2) - 2m^2) \alpha^2 + \tilde{M}m\alpha^3. \quad (3.9)$$

⁴When $M_{21} = 0$, this condition always holds. When $\delta = 0$, this condition is equivalent to $M_{21}\alpha \leq 1$. Hence the above corollary can be directly applied if $M_{21} = 0$ or $\delta = 0$. If $M_{21} > 0$ and $\delta > 0$, the condition $M_{21} \left(\alpha + \frac{\delta^2}{m} \right) \leq 1$ can be rewritten as a condition on α in a case-by-case manner.

Note (3.9) holds for α sufficiently small. Then, we have the following error bound for the iterates

$$\mathbb{E}\|x_k - x_\star\|^2 \leq \tilde{\rho}^{2k} \mathbb{E}\|x_0 - x_\star\|^2 + \tilde{U}_\star \quad (3.10)$$

where \tilde{U}_\star is given by

$$\tilde{U}_\star = \frac{2\delta^2 G^2 + c^2 + m(c^2 + 2G^2(1 + \delta^2))\alpha + 2m^2 G^2 \alpha^2}{(m^2 - \tilde{M}\delta^2) - m(\tilde{M}(1 + \delta^2) - 2m^2)\alpha - \tilde{M}m^2 \alpha^2} \quad (3.11)$$

Proof. Set $\zeta = m\alpha$ in the optimization problem (2.24). This defines a new recursion for a quantity \tilde{U}_k that upper-bounds \hat{U}_k since we are choosing a possibly sub-optimal ζ . Our assumption $M_{21} \left(\alpha + \frac{\delta^2}{m}\right) \leq 1$ guarantees that $\zeta \geq \frac{\alpha M_{21} \delta^2}{1 - \alpha M_{21}}$ when $\zeta = m\alpha$. Hence our choice of ζ is a feasible choice for (2.24). This leads to:

$$\begin{aligned} \tilde{U}_{k+1} &= a_k \left(1 + \frac{1}{m\alpha}\right) + b_k (1 + m\alpha) \\ &= \tilde{\rho}^2 \tilde{U}_k + \left(\alpha^2 (c^2 + 2\delta^2 G^2) \left(1 + \frac{1}{m\alpha}\right) + 2\alpha^2 G^2 (1 + m\alpha)\right) \end{aligned}$$

This is a simple linear recursion that we can solve explicitly in a similar way to the recursion in Remark 1. After simplifications, we obtain (3.10) and (3.11). \square

The linear rate of convergence in (3.10) is of the same order as the one obtained in Corollary 2 and (3.8b). Namely,

$$\tilde{\rho}^2 \approx 1 - \frac{(m^2 - \tilde{M}\delta^2)}{m} \alpha + O(\alpha^2) \quad (3.12)$$

Likewise, the limiting error \tilde{U}_\star from (3.11) can be expanded as a series in α and we obtain a result that matches the small- α limit of \hat{U}_\star from (3.8a) up to linear terms. Namely,

$$\tilde{U}_\star \approx \frac{c^2 + 2\delta^2 G^2}{m^2 - \tilde{M}\delta^2} + \frac{m \left(c^2(\tilde{M} - m^2) + 2(1 - \delta^2)G^2 m^2\right)}{(m^2 - \delta^2 \tilde{M})^2} \alpha + O(\alpha^2) \quad (3.13)$$

Therefore, (3.10) can give a reasonable non-recursive bound for the approximate SG method with small α even for the cases where $M_{21} > 0$ and $\delta > 0$.

Now we discuss the acceptable relative noise level under various assumptions on f_i . Based on (3.12), we need $m^2 - \tilde{M}\delta^2 > 0$ to ensure $\tilde{\rho}^2 < 1$ for sufficiently small α . The other constraint $M_{21} \left(\alpha + \frac{\delta^2}{m}\right) \leq 1$ enforces $M_{21}\delta^2 < m$. Depending on which case we are dealing with, the conditions $\delta < m/\sqrt{\tilde{M}}$ and $M_{21}\delta^2 < m$ impose an upper bound on admissible values of δ . See Table 2.

Case	I	II	III	IV
$\tilde{M} = M_{11} + 2mM_{12}$	m^2	$2L^2$	$2mL$	$2m^2$
δ bound	1	$\frac{m}{\sqrt{2}L}$	$\sqrt{\frac{m}{2}L}$	$\sqrt{\frac{m}{L+m}}$

Table 2: Upper bound on δ for the four different cases described in Table 1.

We can clearly see that for ℓ_2 -regularized logistic regression and support vector machines which admit the assumption in Case I, the approximate SG method is robust to the relative noise. Given

the condition $\delta < 1$, the iterates of the approximate SG method will stay in some ball, although the size of the ball could be quite large. Comparing the bound for Cases II, III, and IV, we can see the allowable relative noise level increases as the assumptions on f_i become stronger.

As previously mentioned, the bound of Corollary 3 requires a sufficiently small α . Specifically, the stepsize α must satisfy $M_{21} \left(\alpha + \frac{\delta^2}{m} \right) \leq 1$ and (3.9), which can be solved to obtain an explicit upper bound on α . Details are omitted.

4 Further discussion

4.1 Connections to existing SG results

In this section, we relate the results of Theorem 1 and its corollaries to existing results on standard SG. We also discuss the effect of replacing our error model (1.4) with IID noise.

If there is no noise at all, $c = \delta = 0$ and none of the approximations of Section 3 are required to obtain an analytical bound on the iteration error. Returning to Theorem 1 and Corollary 1, the objective to be minimized no longer depends on μ_k . Examining (2.14), we conclude that optimality occurs as $\zeta \rightarrow 0$ ($\mu \rightarrow \infty$). This leads directly to the bound

$$\mathbb{E} \|x_{k+1} - x_\star\|^2 \leq (1 - 2m\alpha_k + \tilde{M}\alpha_k^2) \mathbb{E} \|x_k - x_\star\|^2 + 2G^2\alpha_k^2, \quad (4.1)$$

where α_k is constrained such that $M_{21}\alpha_k \leq 1$. The bound (4.1) directly leads to existing convergence results for standard SG. For example, we can apply the argument in Remark 1 to obtain the following bound for standard SG with a constant stepsize $\alpha_k = \alpha$

$$\mathbb{E} \|x_k - x_\star\|^2 \leq \left(1 - 2m\alpha + \tilde{M}\alpha^2\right)^k \mathbb{E} \|x_0 - x_\star\|^2 + \frac{2G^2\alpha}{2m - \tilde{M}\alpha}, \quad (4.2)$$

where α is further required to satisfy $1 - 2m\alpha + \tilde{M}\alpha^2 \leq 1$. For Cases I, III, and IV, the condition $M_{21}\alpha \leq 1$ dominates, and the valid values of α are documented in Remark 2. For Case II, the condition $\alpha \leq 2m/\tilde{M}$ dominates and the upper bound on α is m/L^2 .

The bound recovers existing results that describe the design trade-off of standard (noiseless) SG under a variety of conditions [21, 23, 24]. Case I is a slight variant of the well-known result [23, Prop. 3.4]. The extra factor of 2 in the rate and errors terms are due to the fact that [23, Prop. 3.4] poses slightly different conditions on g and f_i . Cases II and III are also well-known [13, 21, 24].

Remark 5. *If the error term e_k is IID noise with zero mean and bounded variance, then a slight modification to our analysis yields the bound*

$$\mathbb{E} \|x_{k+1} - x_\star\|^2 \leq (1 - 2m\alpha_k + \tilde{M}\alpha_k^2) \mathbb{E} \|x_k - x_\star\|^2 + (2G^2 + \sigma^2)\alpha_k^2, \quad (4.3)$$

where $\sigma^2 \geq \mathbb{E} \|e_k\|^2$. To see this, first notice that e_k is independent from w_k and x_k . We can modify the proof of Lemma 2 to show

$$\begin{aligned} \mathbb{E} \begin{bmatrix} x_k - x_\star \\ w_k + e_k \end{bmatrix}^\top \left(\begin{bmatrix} -2m & 1 \\ 1 & 0 \end{bmatrix} \otimes I_p \right) \begin{bmatrix} x_k - x_\star \\ w_k + e_k \end{bmatrix} &\geq 0, \\ \mathbb{E} \begin{bmatrix} x_k - x_\star \\ w_k + e_k \end{bmatrix}^\top (M \otimes I_p) \begin{bmatrix} x_k - x_\star \\ w_k + e_k \end{bmatrix} &\geq -2G^2 - \sigma^2. \end{aligned} \quad (4.4)$$

Next, we notice that if there exist non-negative ρ_k^2 , ν_k , and λ_k such that

$$\begin{bmatrix} 1 - \rho_k^2 & -\alpha_k \\ -\alpha_k & \alpha_k^2 \end{bmatrix} + \nu_k \begin{bmatrix} -2m & 1 \\ 1 & 0 \end{bmatrix} + \lambda_k \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \leq 0, \quad (4.5)$$

then (1.3) with an IID error e_k satisfies $\mathbb{E} \|x_{k+1} - x_\star\|^2 \leq \rho_k^2 \mathbb{E} \|x_k - x_\star\|^2 + \lambda_k(2G^2 + \sigma^2)$. This can be proved using the argument in Lemma 1. Specifically, we can left and right multiply (4.5) by $[(x_k - x_\star)^\top \quad w_k^\top + e_k^\top]$ and $[(x_k - x_\star)^\top \quad w_k^\top + e_k^\top]^\top$, take full expectation, and apply (4.4) to finish the argument. Finally, we can choose $\lambda_k = \alpha_k^2$, $\nu_k = \alpha_k - M_{21}\alpha_k^2$, and $\rho_k^2 = 1 - 2m\alpha_k + \tilde{M}\alpha_k^2$ to show (4.3). We can recurse (4.3) to obtain a bound in the form of (1.7) with $\rho^2 = 1 - 2m\alpha + O(\alpha^2)$ and $H_\star = O(\alpha)$. Therefore, the zero-mean noise does not introduce bias into the gradient estimator, and the iterates behave similarly to standard SG.

4.2 Adaptive stepsize via sequential minimization

In Section 3, we fixed $\alpha_k = \alpha$ and derived bounds on the worst-case performance of approximate SG. In this section, we discuss the potential impacts of adopting time-varying stepsizes. First, we refine the bounds by optimizing over α_k as well. What makes this approach tractable is that in Theorem 1, the LMI (2.11) is also linear in α_k . Therefore, we can easily include α_k as one of our optimization variables.

In fact, the development of Section 2.2 carries through if we augment the set \mathcal{T}_k to be the set of tuples $(\rho_k, \lambda_k, \mu_k, \nu_k, \alpha_k)$ that makes the LMI (2.11) feasible. We then obtain a Bellman-like equation analogous to (2.21) that holds when we also optimize over α at every step. The net result is an optimization problem similar to (2.24) but that now includes α as a variable:

$$\begin{aligned} V_{k+1} &= \underset{\alpha > 0, \zeta > 0}{\text{minimize}} && a_k(1 + \zeta^{-1}) + b_k(1 + \zeta) \\ &\text{subject to} && a_k = \alpha^2 \left(c^2 + 2\delta^2 G^2 + \tilde{M}\delta^2 V_k \right) \\ &&& b_k = \left(1 - 2m\alpha + \tilde{M}\alpha^2 \right) V_k + 2\alpha^2 G^2 \\ &&& \alpha M_{21}(1 + \delta^2 \zeta^{-1}) \leq 1 \end{aligned} \quad (4.6)$$

As we did in Section 2.2, we can show that $\mathbb{E} \|x_k - x_\star\|^2 \leq V_k$ for any iterates of the approximate SG method. We would like to learn two things from (4.6): how the optimal α changes as a function of k in order to produce the fastest possible convergence rate, and whether this optimized rate is different from the rate we obtained when assuming α was constant in Section 3.

To simplify the analysis, we will restrict our attention to Case II, where $M_{21} = 0$ and $\tilde{M} = 2L^2$. In this case, the inequality constraint in (4.6) is satisfied for any $\alpha > 0$ and $\zeta > 0$, so it may be removed. Observe that the objective in (4.6) is a quadratic function of α .

$$\begin{aligned} &a_k(1 + \zeta^{-1}) + b_k(1 + \zeta) \\ &= (1 + \zeta)V_k - 2m(1 + \zeta)V_k\alpha + (1 + \zeta^{-1})(c^2 + 2G^2\delta^2 + \tilde{M}V_k\delta^2 + 2G^2\zeta + \tilde{M}V_k\zeta)\alpha^2 \end{aligned} \quad (4.7)$$

This quadratic is always positive definite, and the optimal α is given by:

$$\alpha_k^{\text{opt}} = \frac{mV_k\zeta}{(c^2 + 2\delta^2 G^2 + \delta^2 \tilde{M}V_k) + (2G^2 + \tilde{M}V_k)\zeta} \quad (4.8)$$

Substituting (4.8) into (4.6) to eliminate α , we obtain the optimization problem:

$$V_{k+1} = \underset{\zeta > 0}{\text{minimize}} \quad \frac{(\zeta + 1)V_k(c^2 + (2G^2 + \tilde{M}V_k)(\delta^2 + \zeta) - m^2V_k\zeta)}{c^2 + (2G^2 + \tilde{M}V_k)(\delta^2 + \zeta)} \quad (4.9)$$

By taking the second derivative with respect to ζ of the objective function in (4.9), one can check that we will have convexity as long as $(2G^2 + \tilde{M}V_k)(1 - \delta^2) \geq c^2$. In other words, as long as the noise parameters c and δ are not too large. If this bound holds for $V_k = 0$, then it will hold for any $V_k > 0$. So it suffices to ensure that $2G^2(1 - \delta^2) \geq c^2$.

Upon careful analysis of the objective function, we note that when $\zeta = 0$, we obtain $V_{k+1} = V_k$. In order to obtain a decrease for some $\zeta > 0$, we require a negative derivative at $\zeta = 0$. This amounts to the condition: $c^2 + (2G^2 + \tilde{M}V_k)\delta^2 < m^2V_k$. As V_k gets smaller, this condition will eventually be violated. Specifically, the condition holds whenever $m^2 - \tilde{M}\delta^2 > 0$ and

$$V_k > \frac{c^2 + 2\delta^2G^2}{m^2 - \tilde{M}\delta^2}$$

Note that this is the same limit as was observed in the constant- α limits \hat{U}_* and \tilde{U}_* when $\alpha \rightarrow 0$ in (3.8a) and (3.13), respectively. This is to be expected; choosing a time-varying stepsize cannot outperform a constant stepsize in terms of final optimization error. Notice that we have not ruled out the possibility that V_k suddenly jumps below $\frac{c^2 + 2\delta^2G^2}{m^2 - \tilde{M}\delta^2}$ at some k and then stays unchanged after that. We will make a formal argument to rule out this possibility in the next lemma. Moreover, the question remains as to whether this minimal error can be achieved *faster* by varying α_k in an optimal manner. We describe the final nonlinear recursion in the next lemma.

Lemma 4. *Consider the approximate SG iteration (1.3) with $g \in \mathcal{S}(m, \infty)$ for some $m > 0$, and let x_* be the unique global minimizer of g . Suppose Case II holds and (M, G) are the associated values from Table 1. Further assume $2G^2(1 - \delta^2) \geq c^2$ and $V_0 > \frac{c^2 + 2\delta^2G^2}{m^2 - \tilde{M}\delta^2} = V_*$.*

1. *The sequential optimization problem (4.9) can be solved using the following nonlinear recursion*

$$V_{k+1} = \frac{V_k}{(2G^2 + \tilde{M}V_k)^2} \left(\sqrt{(2G^2 + (\tilde{M} - m^2)V_k)((2G^2 + \tilde{M}V_k)(1 - \delta^2) - c^2)} + \sqrt{m^2V_k(c^2 + \delta^2(2G^2 + \tilde{M}V_k))} \right)^2 \quad (4.10)$$

and V_k satisfies $V_k > V_*$ for all k .

2. *Suppose $\hat{U}_0 = V_0 \geq \mathbb{E} \|x_0 - x_*\|^2$ (all recurrences are initialized the same way), then $\{V_k\}_{k \geq 0}$ provides an upper bound to the iterate error satisfying $\mathbb{E} \|x_k - x_*\|^2 \leq V_k \leq \hat{U}_k$.*
3. *The sequence $\{V_k\}_{k \geq 0}$ converges to V_* :*

$$\lim_{k \rightarrow \infty} V_k = V_* = \frac{c^2 + 2\delta^2G^2}{m^2 - \tilde{M}\delta^2}$$

Proof. See Appendix B. □

To learn more about the rate of convergence, we can once again use a Taylor series approximation. Specializing to Case II (where $\tilde{M} > 0$), we can consider two cases. When V_k is large, perform a Taylor expansion of (4.10) about $V_k = \infty$ and obtain:

$$V_{k+1} \approx \left(\frac{m\delta + \sqrt{(\tilde{M} - m^2)(1 - \delta^2)}}{\tilde{M}} \right) V_k + O(1)$$

In other words, we obtain linear convergence. When V_k is close to V_* , the behavior changes. To see this, perform a Taylor expansion of (4.10) about $V_k = V_*$ and obtain:

$$V_{k+1} \approx V_k - \frac{(m^2 - \tilde{M}\delta^2)^3}{4m^2(c^2(\tilde{M} - m^2) + 2G^2m^2(1 - \delta^2))} (V_k - V_*)^2 + O((V_k - V_*)^3) \quad (4.11)$$

We will ignore the higher-order terms, and apply the next lemma to show that the above recursion roughly converges at a $O(1/k)$ rate.

Lemma 5. *Consider the recurrence relation*

$$v_{k+1} = v_k - \eta v_k^2 \quad \text{for } k = 0, 1, \dots \quad (4.12)$$

where $v_0 > 0$ and $0 < \eta < v_0^{-1}$. Then the iterates satisfy the following bound for all $k \geq 0$.

$$v_k \leq \frac{1}{\eta k + v_0^{-1}} \quad (4.13)$$

Proof. The recurrence (4.12) is equivalent to $\eta v_{k+1} = \eta v_k - (\eta v_k)^2$ with $0 < \eta v_0 < 1$. Clearly, the sequence $\{\eta v_k\}_{k \geq 0}$ is monotonically decreasing to zero. To bound the iterates, invert the recurrence:

$$\frac{1}{\eta v_{k+1}} = \frac{1}{\eta v_k - (\eta v_k)^2} = \frac{1}{\eta v_k} + \frac{1}{1 - \eta v_k} \geq \frac{1}{\eta v_k} + 1$$

Recurring the above inequality, we obtain: $\frac{1}{\eta v_k} \geq \frac{1}{\eta v_0} + k$. Inverting this inequality yields (4.13), as required. \square

Applying Lemma 5 to the sequence $v_k = V_k - V_*$ defined in (4.11), we deduce that when V_k is close to its optimal value of V_* , we have:

$$V_k \sim V_* + \frac{1}{\eta k + (V_0 - V_*)^{-1}} \quad \text{with: } \eta = \frac{(m^2 - \tilde{M}\delta^2)^3}{4m^2(c^2(\tilde{M} - m^2) + 2G^2m^2(1 - \delta^2))} \quad (4.14)$$

We can also examine how α_k changes in this optimal recursive scheme by taking (4.8) and substituting the optimal ζ found in the optimization of Lemma 4. The result is messy, but a Taylor expansion about $V_k = V_*$ reveals that

$$\alpha_k^{\text{opt}} \approx \frac{(m^2 - \tilde{M}\delta^2)^2}{2m(c^2(\tilde{M} - m^2) + 2G^2m^2(1 - \delta^2))} (V_k - V_*) + O((V_k - V_*)^2).$$

So when V_k is close to V_* , we should be decreasing α_k to zero at a rate of $O(1/k)$ so that it mirrors the rate at which $V_k - V_*$ goes to zero in (4.14).

In summary, adopting an optimized time-varying stepsize still roughly yields a rate of $O(1/k)$, which is consistent with the sublinear convergence rate of standard SG with diminishing stepsize.

Appendix

A Proof of Lemma 2

To prove (2.7), first notice that since i_k is uniformly distributed on $\{1, \dots, n\}$ and x_k and i_k are independent, we have:

$$\mathbb{E}(w_k | x_k) = \mathbb{E}(\nabla f_{i_k}(x_k) | x_k) = \frac{1}{n} \sum_{i=1}^n \nabla f_i(x_k) = \nabla g(x_k)$$

Consequently, we have:

$$\mathbb{E} \left(\begin{bmatrix} x_k - x_\star \\ w_k \end{bmatrix}^\top \begin{bmatrix} -2mI_p & I_p \\ I_p & 0_p \end{bmatrix} \begin{bmatrix} x_k - x_\star \\ w_k \end{bmatrix} \middle| x_k \right) = \begin{bmatrix} x_k - x_\star \\ \nabla g(x_k) \end{bmatrix}^\top \begin{bmatrix} -2mI_p & I_p \\ I_p & 0_p \end{bmatrix} \begin{bmatrix} x_k - x_\star \\ \nabla g(x_k) \end{bmatrix} \geq 0 \quad (\text{A.1})$$

where the inequality in (A.1) follows from the definition of $g \in \mathcal{S}(m, \infty)$. Taking the expectation of both sides of (A.1) and applying the law of total expectation, we arrive at (2.7).

To prove (2.8), let's start with Case I, the boundedness constraint $\|\nabla f_i(x_k)\| \leq \beta$ implies that $\|w_k\| \leq \beta$ for all k . Rewrite as a quadratic form to obtain:

$$\begin{bmatrix} x_k - x_\star \\ w_k \end{bmatrix}^\top \begin{bmatrix} 0_p & 0_p \\ 0_p & -I_p \end{bmatrix} \begin{bmatrix} x_k - x_\star \\ w_k \end{bmatrix} \geq -\beta^2 \quad (\text{A.2})$$

Taking the expectation of both sides, we obtain the first row of Table 1, as required. We prove Case II in a similar fashion. The boundedness constraint $\|\nabla f_i(x_k) - mx_k\| \leq \beta$ implies that:

$$\begin{aligned} \|w_k - m(x_k - x_\star)\|^2 &\leq \|(w_k - mx_k) + mx_\star\|^2 + \|(w_k - mx_k) - mx_\star\|^2 \\ &= 2\|w_k - mx_k\|^2 + 2m^2\|x_\star\|^2 \\ &\leq 2\beta^2 + 2m^2\|x_\star\|^2 \end{aligned}$$

As in the proof of Case I, rewrite the above inequality as a quadratic form and we obtain the second row of Table 1.

To prove the three remaining cases, we begin by showing that an inequality of the following form holds for each f_i :

$$\begin{bmatrix} x_k - x_\star \\ \nabla f_i(x_k) - \nabla f_i(x_\star) \end{bmatrix}^\top \begin{bmatrix} M_{11}I_p & M_{12}I_p \\ M_{21}I_p & -2I_p \end{bmatrix} \begin{bmatrix} x_k - x_\star \\ \nabla f_i(x_k) - \nabla f_i(x_\star) \end{bmatrix} \geq 0 \quad (\text{A.3})$$

The verification for (A.3) follows directly from the definitions of L -smoothness and convexity. In the smooth case (Definition 1), for example, $\|\nabla f_i(x_k) - \nabla f_i(x_\star)\| \leq L\|x_k - x_\star\|$. So (A.3) holds with $M_{11} = 2L^2$, $M_{12} = M_{21} = 0$. The cases for $\mathcal{F}(0, L)$ and $\mathcal{F}(m, L)$ follow directly from Definition 2. In Table 1, we always have $M_{22} = -1$. Therefore,

$$\begin{aligned} &\mathbb{E} \left(\begin{bmatrix} x_k - x_\star \\ w_k \end{bmatrix}^\top \begin{bmatrix} M_{11}I_p & M_{12}I_p \\ M_{21}I_p & M_{22}I_p \end{bmatrix} \begin{bmatrix} x_k - x_\star \\ w_k \end{bmatrix} \middle| x_k \right) \\ &= \frac{1}{n} \sum_{i=1}^n \begin{bmatrix} x_k - x_\star \\ \nabla f_i(x_k) \end{bmatrix}^\top \begin{bmatrix} M_{11}I_p & M_{12}I_p \\ M_{21}I_p & 0_p \end{bmatrix} \begin{bmatrix} x_k - x_\star \\ \nabla f_i(x_k) \end{bmatrix} - \frac{1}{n} \sum_{i=1}^n \|\nabla f_i(x_k)\|^2 \quad (\text{A.4}) \end{aligned}$$

Since $\frac{1}{n} \sum_{i=1}^n \nabla f_i(x_\star) = \nabla g(x_\star) = 0$, the first term on the right side of (A.4) is equal to

$$\frac{1}{n} \sum_{i=1}^n \begin{bmatrix} x_k - x_\star \\ \nabla f_i(x_k) - \nabla f_i(x_\star) \end{bmatrix} \begin{bmatrix} M_{11}I_p & M_{12}I_p \\ M_{21}I_p & 0_p \end{bmatrix} \begin{bmatrix} x_k - x_\star \\ \nabla f_i(x_k) - \nabla f_i(x_\star) \end{bmatrix}$$

Based on the constraint condition (A.3), we know that the above term is greater than or equal to $\frac{2}{n} \sum_{i=1}^n \|\nabla f_i(x_k) - \nabla f_i(x_\star)\|^2$. Substituting this fact back into (A.4) leads to the inequality:

$$\begin{aligned} \mathbb{E} \left(\begin{bmatrix} x_k - x_\star \\ w_k \end{bmatrix}^\top \begin{bmatrix} M_{11}I_p & M_{12}I_p \\ M_{21}I_p & M_{22}I_p \end{bmatrix} \begin{bmatrix} x_k - x_\star \\ w_k \end{bmatrix} \middle| x_k \right) &\geq \frac{1}{n} \sum_{i=1}^n (2\|\nabla f_i(x_k) - \nabla f_i(x_\star)\|^2 - \|\nabla f_i(x_k)\|^2) \\ &= \frac{1}{n} \sum_{i=1}^n (\|\nabla f_i(x_k) - 2\nabla f_i(x_\star)\|^2 - 2\|\nabla f_i(x_\star)\|^2) \\ &\geq -\frac{2}{n} \sum_{i=1}^n \|\nabla f_i(x_\star)\|^2 \end{aligned} \quad (\text{A.5})$$

Taking the expectation of both sides, we arrive at (2.8), as desired.

Finally, to prove (2.9), we express the error model $\|e_k\|^2 - \delta^2\|w_k\|^2 \leq c^2$ as a quadratic form, and take the expectation to directly obtain (2.9). \square

B Proof of Lemma 4

We use an induction argument to prove Item 1. For simplicity, we denote (4.9) as $V_{k+1} = h(V_k)$. Suppose we have $V_k = h(V_{k-1})$ and $V_{k-1} > V_\star$. We are going to show $V_{k+1} = h(V_k)$ and $V_k > V_\star$. We can rewrite (4.9) as

$$V_{k+1} = \underset{\zeta > 0}{\text{minimize}} \quad A_k(1 + Z_k^{-1}) + B_k(1 + Z_k) \quad (\text{B.1})$$

where A_k , B_k , and Z_k are defined as

$$\begin{aligned} A_k &= \frac{m^2 V_k^2 (c^2 + (2G^2 + \tilde{M}V_k)\delta^2)}{(2G^2 + \tilde{M}V_k)^2} \\ B_k &= \frac{(2G^2 V_k + (\tilde{M} - m^2)V_k^2)((2G^2 + \tilde{M}V_k)(1 - \delta^2) - c^2)}{(2G^2 + \tilde{M}V_k)^2} \\ Z_k &= \frac{(2G^2 + \tilde{M}V_k)(\delta^2 + \zeta_k) + c^2}{(2G^2 + \tilde{M}V_k)(1 - \delta^2) - c^2} \end{aligned}$$

Note that $A_k \geq 0$ and $B_k \geq 0$ due to the condition $2G^2(1 - \delta^2) \geq c^2$. The objective in (B.1) therefore has a form very similar to the objective in (2.24). Applying Lemma 3, we deduce that $V_{k+1} = (\sqrt{A_k} + \sqrt{B_k})^2$, which is the same as (4.10). The associated Z_k^{opt} is $\sqrt{\frac{A_k}{B_k}}$. To ensure this is a feasible choice, it remains to check that the associated $\zeta_k^{\text{opt}} > 0$ as well. Via algebraic manipulations, one can show that $\zeta_k > 0$ is equivalent to $V_k > V_\star$. We can also verify A_k is a monotonically increasing function of V_k , and B_k is a monotonically nondecreasing function of V_k .

Hence h is a monotonically increasing function. Also notice V_\star is a fixed point of (4.10). Therefore, if we assume $V_k = h(V_{k-1})$ and $V_{k-1} > V_\star$, we have $V_k = h(V_{k-1}) > h(V_\star) = V_\star$. Hence we guarantee $\zeta_k > 0$ and $V_{k+1} = h(V_k)$. By similar arguments, one can verify $V_1 = h(V_0)$. And it is assumed that $V_0 > V_\star$. This completes the induction argument.

Item 2 follows from a similar argument to the one used in Section 2.2. Finally, Item 3 can be proven by choosing a sufficiently small constant stepsize α to make \hat{U}_k arbitrarily close to V_\star . Since $V_\star \leq V_k \leq \hat{U}_k$, we conclude that $\lim_{k \rightarrow \infty} V_k = V_\star$, as required. \square

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