An Iterative Algorithm to Estimate Invariant Sets for Uncertain Systems

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***Abstract*— This paper develops an iterative algorithm to estimate an invariant set for uncertain systems. The uncertain system is given as a connection of a nominal linear time-invariant system and a perturbation. The input/output behavior of the perturbation is described by integral quadratic constraints (IQCs). The proposed approach incorporates IQCs into a dissipation inequality formulation. One issue is that it is often useful to specify the IQC in the frequency domain or, equivalently, in the time-domain as a "soft" infinite-horizon constraint. However, the dissipation inequality formulation requires constraints that are valid over all finite time horizons. The main technical result is a finite-horizon bound on soft IQCs constructed using a state-feedback transformation. This forms the basis for the proposed iterative algorithm to estimate invariant sets. A simple example is provided to demonstrate the proposed approach.**

I. Introduction

This paper proposes an algorithm to estimate invariant sets for uncertain systems. The uncertain system is given as a connection of a linear, time invariant (LTI) system and a perturbation. Integral quadratic constraints (IQCs) [7] are used to model the perturbation. A library of IQCs has been developed for various types of uncertainties and nonlinearities as summarized in [7], [13]. An IQC input/output stability theorem was formulated in [7] with frequency domain conditions and was proved using a homotopy method.

The algorithm proposed in this paper uses a time-domain dissipation inequality [15], [16]. This introduces technical issues associated with the use of IQCs. In particular, many existing IQCs are conveniently derived in the frequency domain. These can be equivalently expressed in the time-domain as infinite horizon "soft" constraints. However, the dissipation inequality approach requires constraints that are valid over all finite horizons, often referred to as "hard" IQCs. In general a time-domain IQC need not be hard, i.e. it need not specify a valid finite-horizon integral constraint. Thus additional theory is required to exploit the most general IQC parameterizations for invariant set calculation. A brief review of frequency and (soft/hard) time domain IQCs is given in Section II.

The main contribution of this paper is a new method to compute finite-horizon bounds for soft IQCs. Lemma 3 in Section III-B introduces an additional freedom involving a state feedback. This builds upon prior work in [12], [3]. It was shown in [12] that soft IQCs can be bounded on finite horizons by a min/max game. The cost of this game can be explicitly computed via an algebraic Riccati equation (ARE). However, this ARE introduces a non-convex constraint between variables in the optimization to estimate an invariant set. This min/max result was subsequently used to construct a new soft IQC bound in [3]. The bound in [3] is specified as a (convex) linear matrix inequality (LMI). The drawback is that the bound in [3] is weaker than the one derived in [12].

The state feedback introduced in Lemma 3 improves upon the soft IQC bound in [3]. An iterative invariant set algorithm is proposed in Section III-C to efficiently exploit the additional freedom introduced by the state feedback. The first step of the iterative algorithm is essentially identical to the one given in [3]. It is shown that the iteration costs are monotonically non-increasing. Hence the iterative algorithm can provide significantly improved bounds on the invariant set. A numerical example is provided in Section IV.

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II. Integral Quadratic Constraints

A. Notation

Most notation is from [17]. $\mathbb{R}$ and $\mathbb{R}^+$ denote real and non-negative real numbers, respectively. $\mathbb{C}_0$ and $\mathbb{C}_0^\infty := \mathbb{C}_0 \cup \{\infty\}$ denote the imaginary and extended imaginary axes, respectively. The para-Hermitian conjugate of $G \in \mathbb{RL}_\infty^{m\times n}$, denoted as $G^\sim$, is defined by $G^\sim(s) := G(-s)^T$. Next, $ARE(A,B,C,D,M)$ denotes the following Algebraic Riccati Equation (ARE)

$$A^TX + XA - (XB+S)R^{-1}(XB+S)^T + Q = 0 \quad (1)$$

where $Q := C^TMC$, $R := D^TMD$, and $S := C^TMD$. The stabilizing solution $X = X^T$, if it exists, yields a gain $K_s = R^{-1}(XB+S)^T$ such that $A - BK_s$ is Hurwitz. $KYP(A,B,C,D,M)$ denotes the constraint on $Y = Y^T$:

$$\begin{bmatrix} A^TY + YA & YB \\ B^TY & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} < 0 \quad (2)$$

Finally, for $H = H^T \geq 0$ and $\eta \geq 0$ define the ellipsoidal set $\mathcal{E}(H,\eta) := \{x \in \mathbb{R}^n \, : \, x^THx \leq \eta\}$.

B. Frequency and Time Domain IQCs

IQCs [7] describe the behavior of a system Δ using quadratic constraints on its inputs and outputs. In particular, an IQC can be defined in the frequency domain as follows:

Definition 1. *Let $\Pi = \Pi^\sim \in \mathbb{RL}_\infty^{(n_v+n_w)\times(n_v+n_w)}$ be given. A bounded, causal operator $\Delta : L_{2e}^{n_v} \to L_{2e}^{n_w}$ satisfies the frequency domain IQC defined by the multiplier Π, if the following inequality holds for all $v \in L_2^{n_v}$ and $w = \Delta(v)$*

$$\int_{-\infty}^{\infty} \begin{bmatrix} V(j\omega) \\ W(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} V(j\omega) \\ W(j\omega) \end{bmatrix} d\omega \geq 0 \quad (3)$$

where V and W are Fourier transforms of v and w.

IQCs can also be defined in the time domain based on the graphical interpretation in Figure 1. The input and output signals of Δ are filtered through an LTI system Ψ with zero initial condition $\psi(0) = 0$. In particular, the dynamics of Ψ are given as follows:

$$\begin{aligned} \dot{\psi}(t) &= A_\psi \psi(t) + B_{\psi 1}\, v(t) + B_{\psi 2}\, w(t) \\ z(t) &= C_\psi \psi(t) + D_{\psi 1}\, v(t) + D_{\psi 2}\, w(t) \end{aligned} \quad (4)$$

where $\psi \in \mathbb{R}^{n_\psi}$ is the state and $(A_\psi, B_\psi, C_\psi, D_\psi)$ denote the state matrices of Ψ. Moreover $B_\psi := [B_{\psi 1},\, B_{\psi 2}]$ and $D_\psi := [D_{\psi 1},\, D_{\psi 2}]$ are partitioned conformably with the dimensions of v and w. A time domain IQC is an inequality enforced on the output z over infinite (soft IQC) or finite (hard IQC) horizons. The formal definitions for time domain IQCs are provided next.

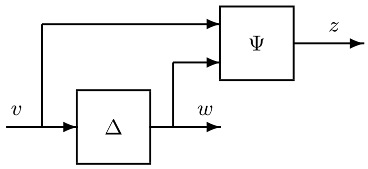

Fig. 1. Graphical interpretation for time domain IQCs

Definition 2. *Let $\Psi \in \mathbb{RH}_\infty^{n_z \times (n_v+n_w)}$ and $M = M^T \in \mathbb{R}^{n_z \times n_z}$ be given.*

(a) *A bounded, causal operator $\Delta : L_{2e}^{n_v} \to L_{2e}^{n_w}$ satisfies the time domain soft IQC defined by (Ψ, M) if the following inequality holds for all $v \in L_2^{n_v}$ and $w = \Delta(v)$*

$$\int_0^\infty z(t)^T M z(t)\, dt \geq 0 \tag{5}$$

where z is the output of Ψ driven by inputs (v, w) with zero initial conditions.

(b) *A bounded, causal operator $\Delta : L_{2e}^{n_v} \to L_{2e}^{n_w}$ satisfies the time domain hard IQC defined by (Ψ, M) if the following inequality holds for all $v \in L_{2e}^{n_v}$, $w = \Delta(v)$ and for all $T \geq 0$*

$$\int_0^T z(t)^T M z(t)\, dt \geq 0 \tag{6}$$

where z is the output of Ψ driven by inputs (v, w) with zero initial conditions.

The notation $\Delta \in \text{IQC}(\Pi)$, $\Delta \in \text{SoftIQC}(\Psi, M)$ and $\Delta \in \text{HardIQC}(\Psi, M)$ is used when Δ satisfies the corresponding frequency, soft, or hard IQC. A library of IQCs exists for various types of uncertainties and nonlinearities as summarized in [7], [13]. This library includes IQCs for LTI dynamic and real parametric uncertainty analogous to the D and G scalings used in structured singular value μ analysis [8]. It also includes IQCs for static and time-varying nonlinearities, e.g. the classical Popov and Zames-Falb multipliers. Many of these existing IQCs are conveniently derived in the frequency domain. However, Lyapunov and dissipation inequality approaches require the use of time domain IQCs. Thus it is useful to connect frequency and time domain (soft/hard) IQCs.

Note that any time domain IQC yields a valid frequency domain IQC. Specifically, if Δ satisfies the (soft or hard) IQC defined by (Ψ, M) then, by Parseval's theorem [17], Δ also satisfies the frequency domain IQC defined by $\Pi = \Psi^\sim M \Psi$. Conversely, any frequency domain multiplier can be factorized (non-uniquely) as $\Pi = \Psi^\sim M \Psi$ with Ψ stable [10]. It follows, again by Parseval's theorem, that $\Delta \in \text{IQC}(\Pi)$ implies $\Delta \in \text{SoftIQC}(\Psi, M)$ for any such factorization. Thus there is an equivalence between frequency domain and time domain soft IQCs. In special cases, such factorizations also yield the stronger condition $\Delta \in \text{HardIQC}(\Psi, M)$. However, $\Delta \in \text{IQC}(\Pi)$ does not, in general, imply $\Delta \in \text{HardIQC}(\Psi, M)$. In fact, some factorizations of Π may yield hard IQCs while others do not. Thus the hard/soft property is not inherent to the multiplier Π but depends on the factorization (Ψ, M).

C. Game Theoretic Bound for Soft IQCs

As noted above, the dissipation inequality framework requires the use of time domain IQCs. In particular, it requires the IQC to be specified as a finite horizon constraint. This is typically done using hard IQCs where $\int_0^T z(t)^T M z(t)\, dt \geq 0$ for all $T \geq 0$. It is often useful (for numerical implementations) to instead search over parameterized soft IQCs specified as infinite horizon constraints. The use of soft IQCs within the dissipation inequality framework requires a lower bound on $\int_0^T z(t)^T M z(t)\, dt$ valid for all finite horizons $T \geq 0$. This section reviews one such bound from [12].

Assume $\Delta : L_{2e}^{n_v}[0,\infty) \to L_{2e}^{n_w}[0,\infty)$ is a causal, bounded operator satisfying the soft IQC defined by (Ψ, M). Define the following min/max game using (Ψ, M):

$$J_{\Psi,M}(\psi_0) := \inf_{v \in L_2^{n_v}[0,\infty)} \sup_{w \in L_2^{n_w}[0,\infty)} \int_0^\infty z(t)^T M z(t)\, dt \quad \text{subject to } z = \Psi \begin{bmatrix} v \\ w \end{bmatrix} \text{ (Eq. 4) and } \psi(0) = \psi_0 \tag{7}$$

It follows from Lemma 2 in [12] that for all $T \geq 0$, $v \in L_{2e}^{n_v}[0,\infty)$ and $w = \Delta(v)$, the soft IQC is bounded by:

$$\int_0^T z(t)^T M z(t)\, dt \geq -J_{\Psi,M}(\psi(T)) \tag{8}$$

where $\psi(T)$ is the state of Ψ at time T when driven by inputs (v, w) with initial condition $\psi(0) = 0$. The proof involves simple manipulations of the soft IQC and uses the causality of Δ. Tighter bounds on $\int_0^T z(t)^T M z(t)\, dt$ can be derived with additional assumptions on Δ, e.g. the bounds derived in [1] for the case where Δ is LTI and norm bounded.

The next lemma gives an explicit expression for $J_{\Psi,M}(\psi(T))$ under additional assumptions on the multiplier.

Lemma 1 ([12])**.** *Let $\Psi \in \mathbb{RH}_\infty^{n_z \times (n_v+n_w)}$ and $M = M^T \in \mathbb{R}^{n_z \times n_z}$ be given and define $\Pi := \Psi^\sim M \Psi$. If $\Pi_{11} > 0$ and $\Pi_{22} < 0$ on $\mathbb{C}_0^\infty$ then*[1]

- *$D_\psi^T M D_\psi$ is nonsingular and there exists a unique, real, stabilizing solution $X = X^T$ to $ARE(A_\psi, B_\psi, C_\psi, D_\psi, M)$.*
- *If $\Delta \in SoftIQC(\Psi, M)$ then for all $T \geq 0$, $v \in L_{2e}^{n_v}[0,\infty)$ and $w = \Delta(v)$,*

$$\int_0^T z(t)^T M z(t)\, dt \geq -\psi(T)^T X \psi(T) \tag{9}$$

Proof. The first conclusion corresponds to Lemma 4 in [12]. Next, $J_{\Psi,M}(\psi(T)) = \psi(T)^T X \psi(T)$ follows from Lemma 5 in [12]. Combining this with the soft IQC min/max bound in Equation 8 yields the second conclusion. □

Lemma 1 is valid for multipliers that satisfy the strict conditions $\Pi_{11} > 0$ and $\Pi_{22} < 0$. Multipliers satisfying the non-strict conditions $\Pi_{11} \geq 0$ and $\Pi_{22} \leq 0$ can be handled via a perturbation argument [12]. Most multipliers used in IQC analysis satisfy the non-strict conditions [7], [13]. These conditions have the following interpretations [6], [4]. $\Pi_{11} \geq 0$ is necessary and sufficient for $0 \in \text{IQC}(\Pi)$. $\Pi_{22} < 0$ implies any $\Delta \in \text{IQC}(\Pi)$ maps zero input to zero output. Bounded gain operators automatically have this zero input-zero output property. Moreover, $\Pi_{22} \leq 0$ further implies that the set of all $\Delta \in \text{IQC}(\Pi)$ is a convex set.

The bound $-\psi(T)^T X \psi(T)$ can be interpreted in a dissipation inequality as the energy stored in the IQC. An issue with this soft IQC bound is that the multiplier (Ψ, M) and bound X are related by an ARE. Typically the IQCs are used in optimizations that search over parameterized multipliers. This is done by selecting Ψ and optimizing over parameterizations for M. Section IV gives an example of this numerical procedure. In such situations, the ARE is a non-convex constraint on M and X.

[1]The notation Π_{11} and Π_{22} refers to the partitioning $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\sim & \Pi_{22} \end{bmatrix}$ conformably with the dimensions of v and w. Thus $\Pi_{11} = \Psi_1^\sim M \Psi_1$ and $\Pi_{22} = \Psi_2^\sim M \Psi_2$ where $\Psi = [\Psi_1, \Psi_2]$. Moreover, $\Pi_{11} < 0$ on $\mathbb{C}_0^\infty$ denotes $\Pi_{11}(j\omega) < 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

D. Convex Bound for Soft IQCs

This section reviews a soft IQC bound from [3] specified as a convex constraint. First set $v = 0$ in the min/max game (Equation 7) to obtain the following quadratic optimization:

$$\begin{aligned} \tilde{J}_{\Psi,M}(\psi_0) := & \sup_{w\in L_2^{n_w}[0,\infty)} \int_0^\infty z(t)^T M z(t)\, dt \\ & \text{subject to } z = \Psi \left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right] \text{ (Eq. 4) and } \psi(0) = \psi_0 \end{aligned} \tag{10}$$

Selecting $v = 0$ in the min/max game (Equation 7) can only increase the cost, i.e. $\tilde{J}_{\Psi,M}(\psi_0) \geq J_{\Psi,M}(\psi_0)$. This yields the following soft IQC bound (accounting for the sign in Equation 8):

$$\int_0^T z(t)^T M z(t)\, dt \geq -\tilde{J}_{\Psi,M}(\psi(T)) \tag{11}$$

where $\psi(T)$ is the state of Ψ at time T when driven by inputs $(0, w)$ with initial condition $\psi(0) = 0$. Note that Equation 10 is a standard LQ problem and can be explicitly rewritten in the form

$$\begin{aligned} \tilde{J}_{\Psi,M}(\psi_0) := & \sup_{w\in L_2^{n_w}[0,\infty)} \int_0^\infty z(t)^T M z(t)\, dt \\ & \text{subject to:} \\ & \dot{\psi}(t) = A_\psi \psi(t) + B_{\psi 2} w(t),\ \psi(0) = \psi_0 \\ & z(t) = C_\psi \psi(t) + D_{\psi 2} w(t) \end{aligned} \tag{12}$$

The next lemma gives an explicit expression for the soft IQC bound under additional assumptions on the multiplier.

Lemma 2 ([3]). *Let $\Psi \in \mathbb{RH}_\infty^{n_z \times (n_v+n_w)}$ and $M = M^T \in \mathbb{R}^{n_z \times n_z}$ be given and define $\Pi := \Psi^\sim M \Psi$. If $\Pi_{22} < 0$ on $\mathbb{C}_0^\infty$ then*

- *$D_{\psi,2}^T M D_{\psi,2} < 0$ and there exists a solution $Y_{22} = Y_{22}^T$ to $KYP(A_\psi, B_{\psi,2}, C_\psi, D_{\psi,2}, M)$.*
- *If $\Delta \in SoftIQC(\Psi, M)$ then for all $T \geq 0$, $v \in L_{2e}^{n_v}[0,\infty)$ and $w = \Delta(v)$,*
$$\int_0^T z(t)^T M z(t)\, dt \geq -\psi(T)^T Y_{22} \psi(T) \tag{13}$$
for any Y_{22} satisfying $KYP(A_\psi, B_{\psi,2}, C_\psi, D_{\psi,2}, M)$.

Proof. The assumption $\Pi_{22} < 0$ implies that $\Pi_{22}(\infty) := D_{\psi,2}^T M D_{\psi,2} < 0$. By the KYP Lemma [9], $\Pi_{22} < 0$ also ensures there exists a $Y_{22} = Y_{22}^T$ satisfying the KYP constraint. Hence the first conclusion holds. Next note that $\Pi_{22} < 0$ ensures there exists a unique stabilizing solution X_{22} to $ARE(A_\psi, B_{\psi,2}, C_\psi, D_{\psi,2}, M)$ such that $\tilde{J}_{\Psi,M}(\psi(T)) = \psi(T)^T X_{22} \psi(T)$ [17], [14]. Moreover the ARE solution is minimal: $X_{22} < Y_{22}$ for any feasible solution to the KYP constraint [5], [11]. Combine this with the soft IQC bound in Equation 11 to yield the second conclusion. □

Lemma 2 relates the multiplier (Ψ, M) and the bound Y_{22} via a KYP constraint. As noted above, it is common to search over classes of IQC by selecting Ψ and optimizing over M. In this case, the KYP constraint is a convex, linear matrix inequality on M and Y_{22}. This is the key benefit of the bound in Lemma 2 as compared to the one in Lemma 1. The drawback is that Lemma 2 is a weaker lower bound. Specifically, the choice $v = 0$ yields a smaller soft IQC bound $-J_{\Psi,M}(\psi_0) \geq -\tilde{J}_{\Psi,M}(\psi_0)$.

Another difference between the two bounds is that Lemma 2 only requires $\Pi_{22} < 0$. Hence it applies to a larger class of multipliers than the bound in Lemma 1 which, in addition, requires $\Pi_{11} > 0$. However, as mentioned previously, most multipliers satisfy $\Pi_{11} > 0$ (or $\Pi_{11} \geq 0$ using a perturbation argument).

III. Iterative Algorithm for Robust Invariant Sets

This section considers the problem of computing invariant sets for uncertain systems. Subsection III-A formulates the problem and presents an existing (convex) condition from [3] for computing invariant sets. The remaining two subsections present the new results to improve upon the previous result in [3]. These include a technical lemma (Subsection III-B) and an iterative algorithm to compute invariant sets (Subsection III-C).

A. Robust Invariant Sets

Consider the uncertain system $F_u(G, \Delta)$ shown in Figure 2. This uncertain system is described by the interconnection of a nominal continuous-time LTI system G and a perturbation Δ. The dynamics are governed by $w = \Delta(v)$ and the following LTI, state-space dynamics for G:

$$\begin{aligned} \dot{x}_G(t) &= A_G\, x_G(t) + B_{G1}\, w(t) + B_{G2}\, d(t) \\ v(t) &= C_G\, x_G(t) + D_{G1}\, w(t) + D_{G2}\, d(t) \end{aligned} \tag{14}$$

$x_G \in \mathbb{R}^{n_G}$ is the state of G. The inputs to G are $w \in \mathbb{R}^{n_w}$ and $d \in \mathbb{R}^{n_d}$ while $v \in \mathbb{R}^{n_v}$ is the output. The state matrices of G have dimensions compatible with these signals, e.g. $A_G \in \mathbb{R}^{n_G \times n_G}$.

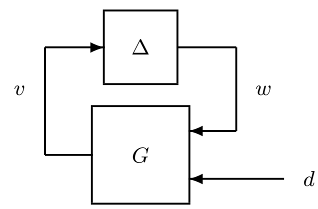

Fig. 2. Interconnection for an Uncertain System $F_u(G, \Delta)$

The perturbation $\Delta : L_{2e}^{n_v} \to L_{2e}^{n_w}$ is a bounded, causal operator whose input/output behavior is specified by IQCs. Δ can have block-structure as is standard in robust control modeling [17]. The operator Δ can include blocks that are hard nonlinearities (e.g. saturations) and infinite dimensional operators (e.g. time delays) in addition to true system uncertainties. The term "uncertainty" is used for simplicity when referring to the perturbation Δ. Well-posedness of the interconnection $F_u(G, \Delta)$ is defined as follows.

Definition 3. *$F_u(G, \Delta)$ is well-posed if for all $x_G(0) \in \mathbb{R}^{n_G}$ and $d \in L_{2e}^{n_d}$ there exists a unique solution $x_G \in L_{2e}^{n_G}$, $v \in L_{2e}^{n_v}$ and $w \in L_{2e}^{n_w}$ with a causal dependence on d.*

The objective is to compute an invariant set for the uncertain system $F_u(G, \Delta)$. The analysis uses a time domain IQC for Δ defined by (Ψ, M) and relies on the interconnection shown in Figure 3. The extended system of G (Equation 14) and Ψ (Equation 4) is governed by the following state space model:

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 d(t) \\ z(t) &= \mathcal{C}x(t) + \mathcal{D}_1 w(t) + \mathcal{D}_2 d(t) \end{aligned} \tag{15}$$

where the extended state vector is $x := \left[\begin{smallmatrix} x_G \\ \psi \end{smallmatrix}\right] \in \mathbb{R}^{n_G + n_\psi}$ and the state matrices are given by

$$\mathcal{A} := \begin{bmatrix} A_G & 0 \\ B_{\psi 1} C_G & A_\psi \end{bmatrix}, \mathcal{B}_1 := \begin{bmatrix} B_{G1} \\ B_{\psi 1} D_{G1} + B_{\psi 2} \end{bmatrix}, \mathcal{B}_2 := \begin{bmatrix} B_{G2} \\ B_{\psi 1} D_{G2} \end{bmatrix} \tag{16}$$

$$\mathcal{C} := \begin{bmatrix} D_{\psi 1} C_G & C_\psi \end{bmatrix}, \mathcal{D}_1 := D_{\psi 1} D_{G1} + D_{\psi 2}, \mathcal{D}_2 := D_{\psi 1} D_{G2} \tag{17}$$

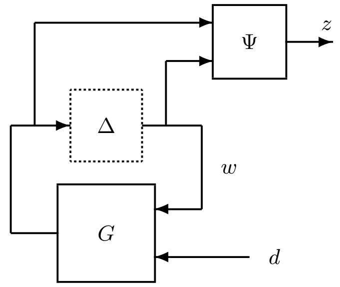

Fig. 3. Extended LTI system of G and filter Ψ

The next theorem provides an analysis condition formulated with this extended system. The proof uses IQCs and a standard dissipation argument. This result is essentially a restatement of Theorem 7 in [3] adapted to the notation used here.

Theorem 1 ([3]). *Let $G \in \mathbb{RH}_\infty^{n_v \times (n_w+n_d)}$ be a stable LTI system defined by (14) and $\Delta : L_{2e}^{n_v} \to L_{2e}^{n_w}$ be a bounded, causal operator. Further assume $F_u(G,\Delta)$ is well-posed and*

1) *$\Delta \in SoftIQC(\Psi, M)$*
2) *$\Pi := \Psi^\sim M \Psi$ satisfies $\Pi_{22} < 0$ on $\mathbb{C}_0^\infty$*
3) *$\exists\, Y_{22} = Y_{22}^T$ satisfying $KYP(A_\psi, B_{\psi,2}, C_\psi, D_{\psi,2}, M)$*
4) *$\exists\, P = P^T$ satisfying*

$$\begin{bmatrix} \mathcal{A}^T P + P\mathcal{A} & P\mathcal{B}_1 & P\mathcal{B}_2 \\ \mathcal{B}_1^T P & 0 & 0 \\ \mathcal{B}_2^T P & 0 & -I \end{bmatrix} + \begin{bmatrix} \mathcal{C}^T \\ \mathcal{D}_1^T \\ \mathcal{D}_2^T \end{bmatrix} M \begin{bmatrix} \mathcal{C}^T \\ \mathcal{D}_1^T \\ \mathcal{D}_2^T \end{bmatrix}^T < 0 \tag{18}$$

5) *P and Y_{22} satisfy the following LMI for some $H = H^T$,*

$$\begin{bmatrix} H & I & 0 \\ I & P_{11} & P_{12} \\ 0 & P_{12}^T & P_{22} - Y_{22} \end{bmatrix} > 0 \tag{19}$$

Then for any $d \in L_2^{n_d}[0,\infty)$ with $\|d\| \le \alpha$, the trajectories of $F_u(G,\Delta)$ starting from $x(0)=0$ satisfy $x_G(t) \in \mathcal{E}(H^{-1}, \alpha^2)$ for all $t \ge 0$.

Proof. The full proof is given in [3] and a sketch is provided here for completeness. Define a storage function $V : \mathbb{R}^{n_G+n_\psi} \to \mathbb{R}^+$ by $V(x) = x^T P x$. Let $d \in L_2^{n_d}[0,\infty)$ be any input signal. From well-posedness, the extended system has a solution (x_G, w, z). Multiply Equation 18 on the left and right by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$ to show that V satisfies the dissipation inequality:

$$\dot{V}(t) + z(t)^T M z(t) \le d(t)^T d(t) \tag{20}$$

The dissipation inequality (20) can be integrated from $t=0$ to $t=T$ with the initial condition $x(0)=0$ to yield:

$$V(x(T)) + \int_0^T z(t)^T M z(t)\, dt \le \int_0^T d(t)^T d(t)\, dt \tag{21}$$

Apply the soft IQC bound in Lemma 2 and $\|d\|_2 \le \alpha$ to conclude that for all $T \ge 0$:

$$\begin{bmatrix} x_G(T) \\ \psi(T) \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} - Y_{22} \end{bmatrix} \begin{bmatrix} x_G(T) \\ \psi(T) \end{bmatrix} \le \alpha^2 \tag{22}$$

Finally, the Schur complement of Equation 19 yields

$$\begin{bmatrix} I \\ 0 \end{bmatrix} H^{-1} \begin{bmatrix} I & 0 \end{bmatrix} < \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} - Y_{22} \end{bmatrix} \tag{23}$$

Combine Equations 22 and 23 to conclude that $x_G(T) \in \mathcal{E}(H^{-1}, \alpha^2)$ □

Note that the second assumption $\Pi_{22} < 0$ implies the third assumption. A "tight" bound on the invariant set can be obtained by minimizing some metric related to the ellipsoid volume, e.g. $trace(H)$. In this optimization, the soft IQC can be parameterized by selecting (fixed) $(A_\psi, B_\psi, C_\psi, D_\psi)$ while M is constrained to lie in a set described by LMI constraints. Moreover, conditions 2-4 in Theorem 1 are LMI constraints on Y_{22}, P, H, and M. Minimizing $trace(H)$ subject to these LMI constraints is a semidefinite program (SDP) in variables Y_{22}, P, H, and the variables used to parameterize M. Hence it can be efficient solved. Also note that this theorem can be modified to compute bounds on individual signals rather than on the state x_G as in Corollary 8 of [3]. Other measures of ellipsoid size, e.g. ellipsoid volume $\log\det(H)$, also lead to a convex optimization.

B. Soft IQC Bounds via State Feedback

The soft IQC lower bound in Lemma 2 is obtained by setting $v=0$ in the min/max game (Equation 7). This yields a conservative bound, in general, because $v=0$ need not be the minimizing input. The soft IQC bound may be improved by selecting some $v = v_0 \in L_2$ other than $v=0$. The basic intuition for the technical result in this section is to select some (v_0, w_0) and perform a change of variables $\left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right] := \left[\begin{smallmatrix} v_0 \\ w_0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} \delta_v \\ \delta_w \end{smallmatrix}\right]$ in the min/max game. This yields a min/max game involving inf over δ_v and sup over δ_w. Setting $\delta_v = 0$ and maximizing over δ_w yields a soft IQC bound corresponding to some $v = v_0$. If v_0 is properly selected then this will improve upon the bound for (Ψ, M) obtained with $v_0 = 0$.

To derive this new soft IQC bound, consider the inputs $\left[\begin{smallmatrix} v_0 \\ w_0 \end{smallmatrix}\right] = -K\psi$ for some state feedback K. The state feedback $-K\psi$ acting on Ψ yields a new system $\bar{\Psi}_K$ with the following state matrices:

$$(\bar{A}_K, \bar{B}_K, \bar{C}_K, \bar{D}_K) := (A_\psi - B_\psi K, B_\psi, C_\psi - D_\psi K, D_\psi) \tag{24}$$

As before, the inputs to $\bar{\Psi}_K$ are partitioned according to the dimensions of v and w, e.g. $\bar{B}_K := [\bar{B}_{K,1},\ \bar{B}_{K,2}]$. The next lemma provides a soft IQC bound for (Ψ, M) in terms of the related pair $(\bar{\Psi}_K, M)$ obtained after state feedback. It is important to note that (Ψ, M) is the actual multiplier for the uncertainty Δ. Here $(\bar{\Psi}_K, M)$ need not be interpreted as an actual IQC for Δ but is simply used to obtain a new bound for (Ψ, M).

Lemma 3. *Let $\Psi \in \mathbb{RH}_\infty^{n_z \times (n_v+n_w)}$, $M = M^T \in \mathbb{R}^{n_z \times n_z}$, and $K \in \mathbb{R}^{(n_v+n_w)\times n_\psi}$ be given. Assume $A_\psi - B_\psi K$ is Hurwitz and define $\Pi := \Psi^\sim M \Psi$ and $\bar{\Pi}_K := \bar{\Psi}_K^\sim M \bar{\Psi}_K$.*
If $\Pi_{11} > 0$, $\Pi_{22} < 0$, $\bar{\Pi}_{K,11} > 0$, $\bar{\Pi}_{K,22} < 0$ on $\mathbb{C}_0^\infty$ then

- *$\bar{D}_{K,2}^T M \bar{D}_{K,2} < 0$ and there exists a solution $\bar{Y}_{22} = \bar{Y}_{22}^T$ to $KYP(\bar{A}_K, \bar{B}_{K,2}, \bar{C}_K, \bar{D}_{K,2}, M)$.*
- *$D_\psi^T M D_\psi$ is nonsingular and there exists a unique, real, stabilizing solution $X = X^T$ to $ARE(A_\psi, B_\psi, C_\psi, D_\psi, M)$. Moreover $X \le \bar{Y}_{22}$.*
- *If $\Delta \in SoftIQC(\Psi, M)$ then for all $T \ge 0$, $v \in L_{2e}^{n_v}[0,\infty)$ and $w = \Delta(v)$,*

$$\int_0^T z(t)^T M z(t)\, dt \ge -\psi(T)^T \bar{Y}_{22} \psi(T) \tag{25}$$

Proof. $(\bar{\Psi}_K, M)$ satisfies the conditions of Lemma 2 and hence the first conclusion follows immediately. In addition, (Ψ, M) satisfies the conditions of Lemma 1. Hence there exists a unique, real stabilizing solution $X = X^T$ to $ARE(A_\psi, B_\psi, C_\psi, D_\psi, M)$ such that

$\int_0^T z(t)^T M z(t)\,dt \geq -\psi(T)^T X \psi(T)$. The proof is completed by showing $X \leq \bar{Y}_{22}$. First, note that $(\bar{\Psi}_K, M)$ also satisfies the conditions of Lemma 1. Hence there exists a unique, real stabilizing solution $\bar{X} = \bar{X}^T$ to $ARE(\bar{A}_K, \bar{B}_K, \bar{C}_K, \bar{D}_K, M)$. $\bar{X}$ yields the cost of the min/max game $J_{\bar{\Psi}_K,M}$ (Equation 7) while $\bar{Y}_{22}$ yields the cost of the LQ game $\tilde{J}_{\bar{\Psi}_K,M}$ (Equation 12). As noted previously, the LQ game is obtained by setting $v = 0$ in the min/max game and this can only increase the cost. Hence $\tilde{J}_{\bar{\Psi}_K,M} \geq J_{\bar{\Psi}_K,M}$, i.e. $\bar{Y}_{22} \geq \bar{X}$. Finally, it follows from Lemma 4 (conclusion 1) in the appendix that ARE solutions are not changed by state feedback transformations, i.e. $X = \bar{X}$. Hence $X = \bar{X} \leq \bar{Y}_{22}$. □

Theorem 1 can be generalized to incorporate the feedback gain K.

Theorem 2. *Let $G \in \mathbb{RH}_\infty^{n_v \times (n_w + n_d)}$ be a stable LTI system defined by* (14) *and $\Delta : L_{2e}^{n_v} \to L_{2e}^{n_w}$ be a bounded, causal operator. Further assume $F_u(G, \Delta)$ is well-posed and*

1) *$\Delta \in$ SoftIQC(Ψ, M)*
2) *$\Pi := \Psi^\sim M \Psi$ satisfies $\Pi_{11} > 0$ and $\Pi_{22} < 0$ on $\mathbb{C}_0^\infty$.*
3) *$\exists\ K \in \mathbb{R}^{(n_v+n_w)\times n_\psi}$ such that $A_\psi - B_\psi K$ is Hurwitz and $\bar{\Pi}_K := \bar{\Psi}_K^\sim M \bar{\Psi}_K$ satisfies $\bar{\Pi}_{K,11} > 0$ and $\bar{\Pi}_{K,22} < 0$ on $\mathbb{C}_0^\infty$.*
4) *$\exists\ \bar{Y}_{22} = \bar{Y}_{22}^T$ satisfying $KYP(\bar{A}_K, \bar{B}_{K,2}, \bar{C}_K, \bar{D}_{K,2}, M)$.*
5) *$\exists\ P = P^T$ satisfying the LMI in Equation 18.*
6) *P and $\bar{Y}_{22}$ satisfy the LMI in Equation 19 for some $H = H^T$.*

Then for any $d \in L_2^{n_d}[0,\infty)$ with $\|d\| \leq \alpha$, the trajectories of $F_u(G,\Delta)$ starting from $x(0) = 0$ satisfy $x_G(t) \in \mathcal{E}(H^{-1}, \alpha^2)$ for all $t \geq 0$.

Proof. The proof is similar to the one given for Theorem 2. The only difference is that the soft IQC bound $\int_0^T z(t)^T M z(t)\,dt \geq -\psi(T)^T \bar{Y}_{22} \psi(T)$ can be used based on Lemma 3. □

The third assumption provides additional restrictions on the multiplier required to apply Lemma 3. The benefit is that this introduces the additional degree of freedom K. This can be used to improve the soft IQC lower bound and hence improve the estimate of the invariant set. Unfortunately K and $\bar{Y}_{22}$ enter bilinearly in the KYP constraint (assumption 4). The next section presents an approach to efficiently exploit the freedom in selecting K.

C. Iterative Algorithm

This section introduces an iterative algorithm to compute an invariant set estimate for $F_u(G,\Delta)$ where $G \in \mathbb{RH}_\infty^{n_v \times (n_w+n_d)}$ and $\Delta \in$ SoftIQC(Ψ, M). Assume the soft IQC is parameterized by selecting (fixed) Ψ while M is described by LMI constraints. A summary of the proposed approach is given in Algorithm 1. In the first iteration, $K^0 = 0$ so that $\bar{\Psi}_{K^0} = \Psi$. For this first iteration, the minimization of $trace(H)$ in step 3 is the same as the SDP obtained from the approach in [3] (and restated here as Theorem 1). This SDP, if feasible, yields the optimal variables $(H^1, P^1, \bar{Y}_{22}^1, M^1)$ and optimal cost $\gamma^1 := trace(H^1)$.

In each subsequent iteration ($i > 1$), the soft IQC from the previous solution, i.e. (Ψ, M^{i-1}), is used to construct a stabilizing gain K^{i-1} and filter $\bar{\Psi}_{K^{i-1}}$. The minimization of $trace(H)$ in step 3 is again an SDP for the fixed choices K^{i-1} and $\bar{\Psi}_{K^{i-1}}$. Roughly, step 3 updates the actual soft IQC M^i while steps 5 and 6 update the feedback K^i used to obtain an improved bound. The next theorem demonstrates that this iteration yields a monotonically non-increasing sequence of costs.

Theorem 3. *The optimal costs at each iteration of Algorithm 1 satisfy $\gamma^{i+1} \leq \gamma^i$.*

Algorithm 1 Iterative Invariant Set Estimation

1: **Initialize:** $K^0 = 0$
2: **for** $i = 1 : N_{iter}$ **do**
3: Minimize $trace(H)$ subject to the conditions in Theorem 2 using K^{i-1}. This yields $(H^i, P^i, \bar{Y}_{22}^i, M^i)$ and optimal cost $\gamma^i := trace(H^i)$.
4: **if** $i < N_{iter}$ **then**
5: Solve $ARE(A_\psi, B_\psi, C_\psi, D_\psi, M^i)$ for X^i.
6: Compute the stabilizing gain $K^i = R^{-1}\left(X^i B + S\right)$ where $R := D_\psi^T M^i D_\psi$ and $S := C_\psi^T M^i C_\psi$.
7: **end if**
8: **end for**

Proof. Let $(H^i, P^i, \bar{Y}_{22}^i, M^i)$ and γ^i be the optimal variables and cost from the i^{th} iteration. These variables are feasible for the conditions in Theorem 2 with K^{i-1}. This implies that (Ψ, M^i) and K^{i-1} satisfy the conditions of Lemma 3. As a result, X^i in Step 5 exists with stabilizing gain K^i. Moreover, $X^i \leq \bar{Y}_{22}^i$.

Define the following point for the next iteration $i+1$:

$$(H^{i+1}, P^{i+1}, \bar{Y}_{22}^{i+1}, M^{i+1}) = (H^i, P^i, X^i, M^i) \tag{26}$$

It is shown that this point satisfies assumptions 1-6 in Theorem 2 with K^i and hence is feasible for iteration $i+1$. First, the feasibility of iteration i implies $\Delta \in$ SoftIQC(Ψ, M^{i+1}) (Assumption 1) and P^{i+1} satisfies the LMI in Equation 18 (Assumption 4). In addition, P^{i+1}, $\bar{Y}_{22}^{i+1}$, and H^{i+1} satisfy the LMI in Equation 19 because $\bar{Y}_{22}^{i+1} := X^i \leq \bar{Y}_{22}^i$ (Assumption 6). Next note that X^i and its stabilizing solution K^i are feasible for $ARE(\bar{A}_{K^i}, \bar{B}_{K^i,2}, \bar{C}_{K^i}, \bar{D}_{K^i,2}, M^{i+1})$ by conclusion 2 of Lemma 4 in the appendix. Thus, by the Schur complement lemma [2], X^i satisfies $KYP(\bar{A}_{K^i}, \bar{B}_{K^i,2}, \bar{C}_{K^i}, \bar{D}_{K^i,2}, M^{i+1})$ with the non-strict inequality $\leq$ (Assumption 5). Finally, define $\bar{\Pi}_{K^i} := \bar{\Psi}_{K^i}^\sim M^{i+1} \bar{\Psi}_{K^i}$ This satisfies $\bar{\Pi}_{K^i,22} < 0$ because $D_{\psi,2}^T M^{i+1} D_{\psi,2} < 0$ and, as noted above, X^i is the stabilizing solution for the ARE corresponding to the (2,2) block of $\bar{\Pi}_{K^i}$ [17]. It can similarly be shown that $\bar{\Pi}_{K^i,11} > 0$. Thus assumption 3 holds. In summary, this demonstrates that the variables in Equation 26 are marginally feasible for all conditions in Theorem 2 using K^i. The marginal feasibility is because X^i, in general, satisfies the non-strict KYP constraint. The cost of this marginally feasible point is $trace(H^{i+1}) = \gamma^i$. Strictly feasible points can be generated with cost arbitrarily close to this value[2] and hence the optimal cost of iteration $i+1$ must satisfy $\gamma^{i+1} \leq \gamma^i$. □

As noted above, the first iteration of Algorithm 1 corresponds to the approach in [3]. By Theorem 3, the cost can only improve on subsequent iterations. In particular, if $v = 0$ fails to be optimal for the min/max game then the ARE solution provides a strictly better soft IQC bound than the KYP LMI, i.e. $X^i < \bar{Y}_{22}^i$. This is often the case. As a consequence the point in Equation 26 typically allows for some room to modify H^{i+1} in the LMI 19 thus enabling a strict decrease in the cost. Also note that Algorithm 1 is written to terminate after a fixed number N_{iter} of iterations. It can be easily modified to incorporate alternative stopping conditions, e.g. terminate if $\gamma^i - \gamma^{i+1}$ is less than a specified absolute or relative tolerance. In fact γ^i will converge since it is a monotone sequence bounded below by zero.

[2] This follows because X^i is feasible for $ARE(\bar{A}_{K^i}, \bar{B}_{K^i,2}, \bar{C}_{K^i}, \bar{D}_{K^i,2}, M^{i+1})$. This implies feasibility of the corresponding strict KYP constraint. Moreover, X^i is minimal in that it lower bounds any solution to the KYP constraint.

ν	0	1	2	4	8
Iter #1	285.30	15.08	14.10	14.03	14.03
Iter #2	285.30	12.68	12.19	12.19	12.18
Iter #3	285.30	12.51	11.87	11.85	11.83
Iter #4	285.30	12.49	11.85	11.83	11.81
Iter #5	**285.30**	**12.49**	**11.85**	**11.82**	**11.80**
Time (sec)	1.73	1.80	1.95	3.37	9.22

TABLE I

BOUNDS ON $trace(H)$ AND TOTAL COMPUTATION TIMES FOR $\kappa = 0.4$

ν	0	1	2	4	8
Iter #1	∞	32.74	27.13	26.07	25.98
Iter #2	∞	20.78	20.12	20.00	19.84
Iter #3	∞	20.16	18.50	18.53	18.52
Iter #4	∞	20.09	18.48	18.46	18.46
Iter #5	∞	**20.08**	**18.48**	**18.45**	**18.45**
Time (sec)	N/A	1.81	2.11	3.39	9.84

TABLE II

BOUNDS ON $trace(H)$ AND TOTAL COMPUTATION TIMES FOR $\kappa = 0.8$

IV. NUMERICAL EXAMPLE

Example 1 in [3] consists of the uncertain system $F_u(G, \Delta)$ with real parametric uncertainty. Specifically, the state matrices of G are:

$$A_G = \begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & 0.1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B_G = \begin{bmatrix} 1 & 2 \\ 0 & -0.1 \\ -0.1 & 0.2 \end{bmatrix}$$
$$C_G = \begin{bmatrix} -1.1 & 0.5 & 0.1 \end{bmatrix} \qquad D_G = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The uncertainty is $\Delta(z) = \delta z$ where $\delta \in \mathbb{R}$ and $|\delta| \leq \kappa$. The stability margin for $F_u(G, \Delta)$, computed using `robstab` in Matlab, is $\kappa_{SM} = 1.073$. The proposed algorithm is used to estimate invariant sets for two values of $\kappa < \kappa_{SM}$.

The IQCs for Δ are specified as $\Pi := \Psi^{\sim} M \Psi$ where

$$\Psi_\nu := \begin{bmatrix} \kappa H_\nu & 0 \\ 0 & H_\nu \end{bmatrix}, \; M := \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & -M_{11} \end{bmatrix} \tag{27}$$

Here $H_\nu := \left[1, \frac{1}{s+1}, \ldots, \frac{1}{(s+1)^\nu}\right]^T$ is a (fixed) vector of stable basis functions. M_{11} and M_{12} are (matrix) variables to be selected in the optimization. If $M_{12} = -M_{12}^T$ and $H_\nu^{\sim} M_{11} H_\nu > 0$ on C_0^∞ then $\Delta \in$ SoftIQC(Ψ_ν, M) [7], [13]. Define $\Pi_\nu := \Psi_\nu^{\sim} M \Psi_\nu$. The constraint $H_\nu^{\sim} M_{11} H_\nu > 0$ can be enforced by a KYP LMI [9] and ensures both $\Pi_{\nu,11} > 0$ and $\Pi_{\nu,22} < 0$. The choices for ν and the pole of H_ν can, in general, affect the analysis result [13].

Algorithm 1 was run for $N_{iter} = 5$ iterations with $\kappa = 0.4$ and $\kappa = 0.8$. Tables I and II show the optimal cost at each iteration for several values of ν. Total computation time for the five iterations is also shown for each value of ν. The computation was performed on a laptop with an Intel core i5 processor. The optimal costs for Iteration #1 agree with the results reported in [3]. Note that the optimal costs are non-increasing thus confirming Theorem 3. Moreover, the final cost after five iterations shows improvement in each case over the results in [3]. The results for $\nu = 0$ and $\kappa = 0.8$ were infeasible at the first iteration and are reported as $\gamma^i = \infty$.

V. CONCLUSIONS

This paper proposed a method to estimate invariant sets for uncertain systems. The main technical contribution of the paper is a new finite-horizon bound for soft IQCs based on a state-feedback transformation. An iterative algorithm to estimate invariant sets is proposed using this bound. The soft IQC bounds allows for a wide class of IQC multipliers to be used for estimating invariant sets.

REFERENCES

[1] V. Balakrishnan. Lyapunov functionals in complex μ analysis. *IEEE Transactions on Automatic Control*, 47(9):1466–1479, 2002.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, 1994.

[3] M. Fetzer, C.W. Scherer, and J. Veenman. Invariance with dynamic multipliers. *submitted to the IEEE Trans. on Automatic Control*, 2017.

[4] M. Fu, S. Dasgupta, and Y.C. Soh. Integral quadratic constraint approach vs. multiplier approach. *Automatica*, 41:281–287, 2005.

[5] I. Gohberg, P. Lancaster, and L. Rodman. On hermitian solutions of the symmetric algebraic Riccati equation. *SIAM Journal of Control and Optimization*, 24(6):1323–1334, 1986.

[6] A. Helmersson. An IQC-based stability criterion for systems with slowly varying parameters. Technical Report LiTH-ISY-R-1979, Linköping University, 1997.

[7] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. on Automatic Control*, 42:819–830, 1997.

[8] A. Packard and J. Doyle. The complex structured singular value. *Automatica*, 29:71–109, 1993.

[9] A. Rantzer. On the Kalman-Yakubovich-Popov lemma. *Systems and Control Letters*, 28(1):7–10, 1996.

[10] C. Scherer and S. Wieland. Linear matrix inequalities in control. Lecture notes for a course of the Dutch institute of systems and control, Delft University of Technology, 2004.

[11] C.W. Scherer. *The Riccati inequality and state-space H_∞-optimal control*. PhD thesis, Bayerische Julius-Maximilians U. W., 1991.

[12] P. Seiler. Stability analysis with dissipation inequalities and integral quadratic constraints. *IEEE TAC*, 60(6):1704–1709, 2015.

[13] J. Veenman, C.W Scherer, and H. Köroğlu. Robust stability and performance analysis based on integral quadratic constraints. *European Journal of Control*, 31:1–32, 2016.

[14] J.C. Willems. Least squares stationary optimal control and the algebraic Riccati equation. *IEEE TAC*, 16:621–634, 1971.

[15] J.C. Willems. Dissipative dynamical systems part i: General theory. *Archive for Rational Mech. and Analysis*, 45(5):321–351, 1972.

[16] J.C. Willems. Dissipative dynamical systems part ii: Linear systems with quadratic supply rates. *Archive for Rational Mech. and Analysis*, 45(5):352–393, 1972.

[17] K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, 1996.

APPENDIX

Lemma 4. *Let (A, B, C, D, M) be matrices of appropriate dimension. Let $B = [B_1, B_2]$ and $D = [D_1, D_2]$ be any partitioning.*

1) *For any matrix K of appropriate dimension, X is a stabilizing solution of $ARE(A, B, C, D, M)$ if and only if X is also a stabilizing solution of $ARE(A - BK, B, C - DK, D, M)$.*
2) *Let X be the stabilizing solution of $ARE(A, B, C, D, M)$ with stabilizing gain $K_s := R^{-1}(XB + S)^T$ where $R := D^T M D$ and $S := C^T M D$. Then X is the stabilizing solution of $ARE(A - BK_s, B_2, C - DK_s, D_2, M)$.*

Proof. (Conclusion 1) Let X be a stabilizing solution to $ARE(A - BK, B, C - DK, D, M)$ with stabilizing gain $\bar{K}_s = R^{-1}(XB + \bar{S})^T$ where $R := D^T M D$ and $\bar{S} = (C - DK)^T DM$. Hence $(A - BK) - B\bar{K}_s$ is Hurwitz. It can be directly verified that X is also a solution to $ARE(A, B, C, D, M)$ and the corresponding gain is $K_s = R^{-1}(XB + S)^T$ with $S := C^T MD$. Finally, $(A - BK) - B\bar{K}_s$ can be rewritten as $(A - BK_s)$ and hence K_s is also stabilizing. The reverse direction follows similarly.

(Conclusion 2) Verify by direct substitution using $(XB + S) - K_s^T R^T = 0$ which implies $XB_2 + (C - DK_s)^T M D_2 = 0$. $\square$