

# Convex LPV synthesis of estimators and feedforwards using integral quadratic constraints

Raghu Venkataraman\*<sup>†</sup> and Peter Seiler

*Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, MN 55455, USA*

## SUMMARY

A method is presented for synthesizing output estimators and disturbance feedforward controllers for continuous-time, uncertain, gridded, linear parameter-varying (LPV) systems. Integral quadratic constraints (IQCs) are used to describe the uncertainty. Since gridded LPV systems do not have a valid frequency-domain interpretation, the time-domain, dissipation inequality approach is followed. There are two main contributions. First, a notion of duality is developed for uncertain, gridded LPV systems. Second, convex synthesis conditions are derived for robust output estimators. Together, these two contributions enable the convex synthesis of robust disturbance feedforward controllers. The effectiveness of the proposed method is demonstrated using a numerical example. Copyright © 2017 John Wiley & Sons, Ltd.

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KEY WORDS: robust control; linear parameter-varying systems; integral quadratic constraints

## 1. INTRODUCTION

This paper considers the synthesis of robust output estimators and disturbance feedforward controllers for a certain class of uncertain systems. Robust estimator and feedforward synthesis problems have been widely studied in the literature under various assumptions on the plant and uncertainty. For example, robust estimator synthesis results have been obtained for linear time-invariant (LTI) [1–10], linear time-varying (LTV) [11], and linear parameter-varying (LPV) [12, 13] plants. Previous work has also considered different classes of uncertainties including structured LTI [1, 2], single full block [11], norm-bounded time-varying [3–5], and polytopic [14–16] uncertainties. Moreover, robust estimator synthesis results have been obtained for uncertainties described by static [6, 7] and dynamic [8–10] integral quadratic constraints (IQC). In many of these previous works, convex formulations have been obtained for the synthesis. This is in contrast to more general robust feedback synthesis which is a nonconvex problem thus requiring heuristic approaches such as DK-synthesis [17] or IQC-synthesis [18–20]. The disturbance feedforward problem is structurally the dual of the output estimation problem [17]. As a result, many of the previous results summarized above have parallel results for the robust feedforward synthesis [8, 21–23].

This paper complements the existing literature by deriving convex conditions for the synthesis of output estimators and disturbance feedforward controllers for continuous-time, uncertain LPV systems. The uncertain system is an interconnection of a nominal gridded LPV system and a block structured perturbation that is described using dynamic IQCs. IQCs provide a general framework to characterize the input-output behavior of several different classes of perturbations [24], e.g. LTI uncertainties, static nonlinearities, time delays, etc. A frequency-domain stability theorem was formulated in [24] to analyze a feedback interconnection of a LTI plant and any perturbation that is

\*Correspondence to: R. Venkataraman, Department of Aerospace Engineering and Mechanics, University of Minnesota.

<sup>†</sup>E-mail: venka085@umn.edu

characterizable using IQCs. However, gridded LPV systems are time-varying and hence they do not have a valid frequency response interpretation [25, 26]. Consequently, a theorem was formulated in the time-domain using dissipativity theory for the input-output analysis of uncertain, gridded LPV systems [27, 28] by building on the work of [29]. This paper utilizes the main result of [28] for the convex synthesis of estimators and feedforward controllers for uncertain, gridded LPV systems.

The most closely related works in the literature presented convex solutions for robust LTI synthesis using dynamic IQCs [9, 23]. In particular, convex synthesis conditions were derived for the robust output estimation problem [9]. A frequency-domain duality result was also developed to synthesize feedforward controllers using the (convex) conditions for the estimator synthesis [23]. It was later shown that the estimator and feedforward synthesis problems are special cases of a feedback structure that has no uncertainties in the control channel of the closed-loop [30]. For such feedback structures, [12] provided a general synthesis framework for robust, gain-scheduled controllers. This general synthesis framework is restricted to linear fractional transform (LFT) based LPV plants, whose state matrices are restricted to depend rationally on the scheduling parameters [31–33]. Note that frequency-domain arguments are applicable for LFT-based LPV systems since the nominal plant is LTI. Such frequency-domain arguments are not applicable for gridded LPV systems because, as noted above, these systems do not have a valid frequency response interpretation. This paper instead develops a time-domain duality result for feedforward synthesis.

Before presenting the two main contributions, some background on IQCs and LPV systems is presented in Section 2. This discussion includes previous work on the stability and input-output analysis of uncertain LTI [29, 34] and LPV [27, 28] systems using dissipativity theory. The first main contribution is a time-domain notion of duality for uncertain, gridded LPV systems (Section 3). This is needed to exploit the duality between the estimation and feedforward problems [17]. The second main contribution is a rigorous convex solution for the robust output estimation problem (Section 4.1) for uncertain, gridded LPV systems. Finally, a convex solution is obtained for the robust disturbance feedforward problem by combining the two main contributions (Section 4.2). A numerical example is used to demonstrate feedforward synthesis for a gridded LPV plant that is affected by a sector-constrained nonlinear function (Section 5).

## 2. BACKGROUND

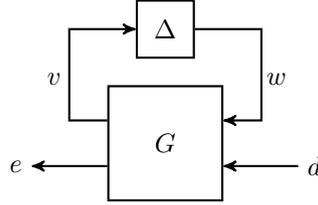
### 2.1. Notation

Most notation used is from [17].  $\mathbb{R}$  and  $\mathbb{C}$  denote the set of real and complex numbers.  $\mathbb{RL}_\infty$  denotes the set of rational functions with real coefficients that are proper and have no poles on the imaginary axis.  $\mathbb{RH}_\infty$  is the subset of functions in  $\mathbb{RL}_\infty$  that are analytic in the closed right half of the complex plane.  $\mathbb{R}^n$  denotes the set of  $n \times 1$  vectors and  $\mathbb{R}^{m \times n}$  denotes the set of  $m \times n$  matrices whose elements are in  $\mathbb{R}$ . Similar notation is used for the sets  $\mathbb{C}$ ,  $\mathbb{RL}_\infty$ , and  $\mathbb{RH}_\infty$ .  $\mathbb{R}^+$  denotes the set of nonnegative real numbers. For a matrix  $M \in \mathbb{C}^{m \times n}$ ,  $M^T$  denotes the transpose and  $M^*$  denotes the Hermitian adjoint.  $\star$  denotes a symmetric block in matrices.  $L_2^n [0, \infty)$  is the space of functions  $v : [0, \infty) \rightarrow \mathbb{R}^n$  satisfying  $\|v\| < \infty$ , where  $\|v\| := \sqrt{\int_0^\infty v(t)^T v(t) dt}$ . For  $v \in L_2^n [0, \infty)$ ,  $v_T$  is the truncated function:  $v_T(t) = v(t)$  for  $t \leq T$  and  $v_T(t) = 0$  otherwise. The extended space, denoted  $L_{2e}$ , is the set of functions  $v$  such that  $v_T \in L_2 \forall T \geq 0$ . The para-Hermitian conjugate of  $H \in \mathbb{RL}_\infty^{m \times n}$  is defined as  $H^\sim(s) := H(-s)^T$ . Finally,  $\mathcal{F}_u(G, \Delta)$  denotes the LFT of  $G$  and  $\Delta$ .

### 2.2. Integral Quadratic Constraints

Figure 1 shows the type of *uncertain* LPV systems considered in this paper.  $G$  is a *nominal* grid-based LPV system, described further in Section 2.3.  $\Delta$  is a block-structured perturbation [17] whose input-output behavior is described using IQCs. IQCs were introduced in [24] and are defined using frequency-domain multipliers  $\Pi : j\mathbb{R} \rightarrow \mathbb{C}^{(n_v+n_w) \times (n_v+n_w)}$  that are measurable Hermitian-valued functions. The signals  $v \in L_2^{n_v} [0, \infty)$  and  $w \in L_2^{n_w} [0, \infty)$  satisfy the IQC defined by  $\Pi$  if:

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0, \quad (1)$$


 Figure 1. Interconnection of gridded LPV plant  $G$  and perturbation  $\Delta$ .

where  $\hat{v}(j\omega)$  and  $\hat{w}(j\omega)$  are the Fourier transforms of  $v$  and  $w$ , respectively. A bounded, causal operator  $\Delta : L_{2e}^{n_v} [0, \infty) \rightarrow L_{2e}^{n_w} [0, \infty)$  satisfies the IQC defined by  $\Pi$  if (1) holds for all  $v \in L_{2e}^{n_v} [0, \infty)$  and  $w = \Delta(v)$ . This is denoted by  $\Delta \in \text{IQC}(\Pi)$ . As such, a set of operators satisfy the IQC defined by  $\Pi$  and  $\Delta$  is a member of this set. Consider the following special class of multipliers.

*Definition 1* ([35])

Let  $\Pi = \Pi^\sim \in \mathbb{R}\mathbb{L}_{\infty}^{(n_v+n_w) \times (n_v+n_w)}$  be partitioned as  $\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\sim & \Pi_{22} \end{bmatrix}$ , where  $\Pi_{11} \in \mathbb{R}\mathbb{L}_{\infty}^{n_v \times n_v}$  and  $\Pi_{22} \in \mathbb{R}\mathbb{L}_{\infty}^{n_w \times n_w}$ .  $\Pi$  is said to be a *strict positive-negative (PN) multiplier* if  $\Pi_{11}(j\omega) > 0$  and  $\Pi_{22}(j\omega) < 0 \forall \omega \in \mathbb{R} \cup \{\infty\}$ .

Since this paper employs IQCs for the robustness analysis of uncertain grid-based LPV systems, IQCs also need to be expressed in the time-domain. A multiplier  $\Pi \in \mathbb{R}\mathbb{L}_{\infty}^{(n_v+n_w) \times (n_v+n_w)}$  can be factorized as  $\Pi = \Psi^T M \Psi$ , where  $M = M^T \in \mathbb{R}^{n_z \times n_z}$  and  $\Psi \in \mathbb{R}\mathbb{H}_{\infty}^{n_z \times (n_v+n_w)}$  [29]. The IQC given in (1) can be rewritten as:

$$\int_0^{\infty} z(t)^T M z(t) dt \geq 0, \quad (2)$$

where  $z := \Psi \begin{bmatrix} v \\ w \end{bmatrix} \in \mathbb{R}^{n_z}$  is the output of the linear system  $\Psi$  driven by the input signals  $v$  and  $w$  starting from zero initial conditions [29]. Let  $\Psi$  have a state-space realization given by:

$$\begin{bmatrix} \dot{x}_{\Psi} \\ z \end{bmatrix} = \begin{bmatrix} A_{\Psi} & B_{\Psi v} & B_{\Psi w} \\ C_{\Psi} & D_{\Psi v} & D_{\Psi w} \end{bmatrix} \begin{bmatrix} x_{\Psi} \\ v \\ w \end{bmatrix}, \quad (3)$$

where  $x_{\Psi} \in \mathbb{R}^{n_{\Psi}}$  and  $x_{\Psi}(0) = 0$ .  $\Delta \in \text{IQC}(\Psi, M)$  indicates that the operator  $\Delta$  satisfies the time-domain IQC given by (2). While there are infinite ways to factorize  $\Pi$ , this paper will use the following special class of factorizations.

*Definition 2* ([36, 37])

$(\hat{\Psi}, J_{n_v, n_w})$  is called a  $J_{n_v, n_w}$ -spectral factor of  $\Pi = \Pi^\sim \in \mathbb{R}\mathbb{L}_{\infty}^{(n_v+n_w) \times (n_v+n_w)}$  if  $\Pi = \hat{\Psi}^\sim J_{n_v, n_w} \hat{\Psi}$ ,  $J_{n_v, n_w} = \begin{bmatrix} I_{n_v} & 0 \\ 0 & -I_{n_w} \end{bmatrix}$ , and  $\hat{\Psi}, \hat{\Psi}^{-1} \in \mathbb{R}\mathbb{H}_{\infty}^{(n_v+n_w) \times (n_v+n_w)}$ .

$J$ -spectral factorizations are special because  $J$  is diagonal and  $\hat{\Psi}$  is square, stable, and stably invertible.  $J$ -spectral factorizations exist for all strict PN multipliers (Lemma 4 in [29]).

### 2.3. Input-Output Analysis of LPV Systems

LPV systems are linear systems whose state-space matrices depend on a time-varying parameter  $\rho : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_{\rho}}$ . Consider the following state-space realization for the LPV system  $G$  (Figure 1):

$$\begin{bmatrix} \dot{x}_G \\ y \end{bmatrix} = \begin{bmatrix} A_G(\rho) & B_G(\rho) \\ C_G(\rho) & D_G(\rho) \end{bmatrix} \begin{bmatrix} x_G \\ u \end{bmatrix}, \quad (4)$$

where  $x_G \in \mathbb{R}^{n_G}$  is the state,  $u = \begin{bmatrix} w \\ d \end{bmatrix} \in \mathbb{R}^{n_w+n_d}$  are the inputs, and  $y = \begin{bmatrix} v \\ e \end{bmatrix} \in \mathbb{R}^{n_v+n_e}$  are the outputs. The matrices in (4) have dimensions compatible with these signals and are continuous functions of  $\rho$ . In the remainder of the paper, the functional dependence of the state matrices on  $\rho$  is occasionally suppressed for brevity. This paper assumes that  $\rho$ : (i) is a continuously differentiable function of time, (ii) is restricted to a known compact set  $\mathcal{P} \subset \mathbb{R}^{n_\rho}$ , and (iii) has infinite bounds on its rate of variation  $\dot{\rho}$ . As per the notation of [28], the set of admissible trajectories is defined as:

$$\mathcal{T} := \left\{ \rho : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_\rho} : \rho \in \mathcal{C}^1, \rho(t) \in \mathcal{P}, \text{ and } \dot{\rho}(t) \in \mathbb{R}^{n_\rho} \forall t \geq 0 \right\}. \quad (5)$$

The results stated in this paper are for the case where  $\dot{\rho}$  is unbounded. However, with additional notation, they can be adapted for the case where  $\dot{\rho}$  is bounded by using parameter-dependent Lyapunov matrices (e.g. [28]). This paper assumes that  $G$  is quadratically stable, as defined next.

*Definition 3* ([25])

$G$  is quadratically stable if  $\exists P > 0$  such that  $A_G(\rho)^T P + P A_G(\rho) < 0 \forall \rho \in \mathcal{P}$ .

As discussed in Section 1.2 of [25], quadratic stability is a form of internal state stability. In particular, if  $G$  is quadratically stable and autonomous,  $x_G$  exponentially decays to zero for any initial condition  $x_G(0) \in \mathbb{R}^{n_G}$  and any admissible parameter trajectory  $\rho \in \mathcal{T}$ . This is proved after noting that  $x_G^T P x_G$  is a Lyapunov function. In addition to internal state stability, this paper requires some notion of bounded-input, bounded-output (BIBO) stability. Hence,

$$\|G\| := \sup_{\substack{0 \neq u \in L_2^{n_w+n_d} \\ \rho \in \mathcal{T}, x_G(0)=0}} \frac{\|y\|}{\|u\|}. \quad (6)$$

is defined as the induced  $L_2$  norm of  $G$ . BIBO stability is achieved if  $\|G\|$  has a finite upper bound. The following lemma provides sufficient conditions for bounding  $\|G\|$ .

*Lemma 1* ([25])

$G$  is quadratically stable and  $\|G\| < \gamma$  for some  $\gamma \in (0, \infty)$  if  $\exists P > 0$  such that

$$\begin{bmatrix} A_G^T(\rho) P + P A_G(\rho) & \star \\ B_G^T(\rho) P & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C_G^T(\rho) \\ D_G^T(\rho) \end{bmatrix} (\star) < 0 \forall \rho \in \mathcal{P}. \quad (7)$$

This lemma essentially generalizes the Bounded Real Lemma for LPV systems and follows from Theorem 3.3.1 of [25]. By applying the Schur complement lemma on the second term, inequality (7) can be written as a LMI involving the LPV plant  $G$ , a Lyapunov matrix  $P$ , and the gain upper bound  $\gamma$ . In order to find the least upper bound, a semidefinite program can be formulated with  $\gamma$  as the linear cost function to be minimized subject to the LMI constraints  $P > 0$  and inequality (7). Henceforth, LMIs of the form (7) will be referred to using the short form  $LMI_{BR}(G, P, \gamma) < 0$ . The subscript  $BR$  indicates that it is a LMI associated with the Bounded Real Lemma and the arguments  $(G, P, \gamma)$  indicate that the system, Lyapunov matrix, and gain bound are involved.

On a related note, the plant  $G$  can be scaled at the inputs and/or outputs using  $\gamma$  in order to yield a gain bound of 1. For example, if  $B_G$  and  $C_G$  were each scaled by  $\gamma^{-0.5}$  and  $D_G$  was scaled by  $\gamma^{-1}$ , then inequality (7) can be rewritten with a gain bound of 1. Such normalizations will be used in the later sections of this paper.

#### 2.4. Input-Output Analysis of Uncertain LPV Systems

The previous section considered the input-output analysis of *nominal* LPV systems. This section considers the input-output analysis of *uncertain* LPV systems, wherein the uncertainty is described using IQCs. The input-output analysis of  $\mathcal{F}_u(G, \Delta)$ , shown in Figure 1, was considered in [27, 28]. As before, the induced  $L_2$  norm from inputs  $d$  to outputs  $e$  is defined as:

$$\|\mathcal{F}_u(G, \Delta)\| := \sup_{\substack{0 \neq d \in L_2^{n_d} \\ \rho \in \mathcal{T}, x_G(0)=0}} \frac{\|e\|}{\|d\|}. \quad (8)$$

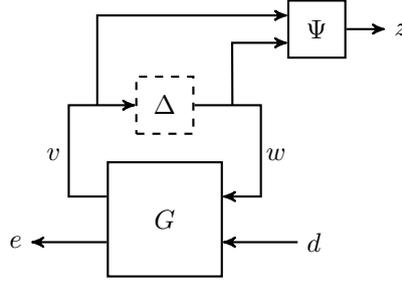


Figure 2. Interconnection of the gridded LPV system  $G$ , perturbation  $\Delta$ , and IQC filter  $\Psi$ .

Since  $\Delta$  contains nonlinearities and uncertainties that are hard to analyze, it is not always possible to compute  $\|\mathcal{F}_u(G, \Delta)\|$  exactly. However, since a *set* of operators satisfy the IQC defined by  $\Pi$ , the worst-case gain over this set can be used in lieu of  $\|\mathcal{F}_u(G, \Delta)\|$ . The *worst-case gain* is defined as:

$$\sup_{\Delta \in \text{IQC}(\Pi)} \|\mathcal{F}_u(G, \Delta)\|. \quad (9)$$

Next, consider Figure 2. In addition to the interconnection of  $G$  and  $\Delta$ , the IQC factor  $\Psi$  is appended such that it is driven by signals  $v$  and  $w$ , and produces signal  $z$ . The extended LPV system, formed by the interconnection of  $G$  and  $\Psi$ , has the following state-space representation:

$$\begin{bmatrix} \dot{x}_e \\ z \\ e \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\rho) & \mathcal{B}_w(\rho) & \mathcal{B}_d(\rho) \\ \mathcal{C}_z(\rho) & \mathcal{D}_{zw}(\rho) & \mathcal{D}_{zd}(\rho) \\ \mathcal{C}_e(\rho) & \mathcal{D}_{ew}(\rho) & \mathcal{D}_{ed}(\rho) \end{bmatrix} \begin{bmatrix} x_e \\ w \\ d \end{bmatrix}, \quad (10)$$

where  $x_e = [x_G^T, x_\Psi^T]^T \in \mathbb{R}^{n_G+n_\Psi}$ . Theorem 2 of [28] provided sufficient conditions for bounding the worst-case gain of  $\mathcal{F}_u(G, \Delta)$  and is paraphrased next.

*Theorem 1* ([28])

Let  $G$  be a quadratically stable LPV system defined by (4) and  $\Delta$  be a bounded, causal operator such that  $\mathcal{F}_u(G, \Delta)$  is well-posed. Assume  $\Delta \in \text{IQC}(\Pi)$  and consider a factorization  $\Pi = \Psi^* M \Psi$  with  $\Psi$  stable. If  $\Pi$  is a strict PN multiplier and  $\exists P = P^T$  such that

$$\begin{bmatrix} \mathcal{A}^T(\rho)P + P\mathcal{A}(\rho) & \star & \star \\ \mathcal{B}_w^T(\rho)P & 0 & \star \\ \mathcal{B}_d^T(\rho)P & 0 & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} \mathcal{C}_e^T(\rho) \\ \mathcal{D}_{ew}^T(\rho) \\ \mathcal{D}_{ed}^T(\rho) \end{bmatrix} (\star) + \begin{bmatrix} \mathcal{C}_z^T(\rho) \\ \mathcal{D}_{zw}^T(\rho) \\ \mathcal{D}_{zd}^T(\rho) \end{bmatrix} M(\star) < 0 \quad \forall \rho \in \mathcal{P}, \quad (11)$$

for some  $\gamma \in (0, \infty)$  then,

- (1)  $\lim_{T \rightarrow \infty} x_e(T) = 0 \quad \forall x_e(0) \in \mathbb{R}^{n_G+n_\Psi}, \forall d \in L_2^{n_d}$ , and  $\forall \rho \in \mathcal{T}$ , and
- (2)  $\sup_{\Delta \in \text{IQC}(\Pi)} \|\mathcal{F}_u(G, \Delta)\| \leq \gamma$ .

This theorem provides sufficient conditions for  $\mathcal{F}_u(G, \Delta)$  to have bounded worst-case gain. By applying the Schur complement lemma on the second term, inequality (11) can be written as a LMI involving the LPV plant  $G$ , a Lyapunov matrix  $P$ , the gain upper bound  $\gamma$ , and the IQC factorization  $(\Psi, M)$ . Henceforth, LMIs of the form (11) will be referred to using the short form  $\text{LMI}_{WC}(G, P, \gamma, \Psi, M) < 0$ . The subscript *WC* indicates that it is a LMI associated with a worst-case gain problem and the arguments  $(G, P, \gamma, \Psi, M)$  indicate that the system, Lyapunov matrix, gain bound, and IQC factorization are involved.

Finally, Theorem 1 does not require the matrix  $P$  to be positive definite. This is in contrast to Lemma 1 where  $P > 0$  was required. The IQCs used in Theorem 1 contain hidden energy. Arguments from game theory can be used to define a new Lyapunov function that includes this hidden energy, and which is indeed positive definite. Section 3 of [28] provides the full proof.

### 3. DUAL INPUT-OUTPUT ANALYSIS

A convex formulation for the disturbance feedforward problem requires certain duality results. Thus, this section precisely defines the required notion of input-output duality and proves some intermediate results that hold for gridded LPV systems. A technical issue is that gridded LPV systems are time-varying and hence they do not have a valid frequency response interpretation. Hence the duality results contained in this section make use of time-domain arguments. The reader who is only interested in the convex synthesis results may skip ahead to Section 4.

#### 3.1. Dual LPV Systems

The concept of duality is well developed for LTI systems, e.g. duality between the concepts of controllability and observability [17]. For LPV systems, duality is defined as follows.

##### Definition 4

If  $G \stackrel{s}{=} \left[ \begin{array}{c|c} A_G(\rho) & B_G(\rho) \\ \hline C_G(\rho) & D_G(\rho) \end{array} \right]$  is a (primal) LPV system then  $G^T \stackrel{s}{=} \left[ \begin{array}{c|c} A_G^T(\rho) & C_G^T(\rho) \\ \hline B_G^T(\rho) & D_G^T(\rho) \end{array} \right]$  is the corresponding dual system.

Lemma 1 proves that the existence of  $P > 0$  such that  $LMI_{BR}(G, P, \gamma) < 0$  is sufficient for  $\|G\| < \gamma$ . In a similar manner, the existence of  $Q > 0$  such that  $LMI_{BR}(G^T, Q, \gamma) < 0$  is sufficient for  $\|G^T\| < \gamma$ . The next lemma relates the two sets of sufficient conditions.

##### Lemma 2

$LMI_{BR}(G, P, \gamma) < 0$  for  $P > 0$  if and only if  $LMI_{BR}(G^T, Q, \gamma) < 0$  for  $Q := P^{-1} > 0$ .

##### Proof

It follows from linear algebra that  $P > 0$  if and only if  $Q := P^{-1} > 0$ . Apply the Schur complement lemma to show that  $LMI_{BR}(G, P, \gamma) < 0$  is equivalent to

$$\begin{bmatrix} A_G^T(\rho)P + PA_G(\rho) & \star & \star \\ B_G^T(\rho)P & -\gamma I & \star \\ C_G(\rho) & D_G(\rho) & -\gamma I \end{bmatrix} < 0 \quad \forall \rho \in \mathcal{P}. \quad (12)$$

Next, apply the congruence transformation  $\text{diag}(P^{-1}, I, I)$  on the left and right of LMI (12). Finally, apply the Schur complement lemma to the (2, 2) block of the resulting LMI to show that it is equivalent to  $LMI_{BR}(G^T, Q, \gamma) < 0$ .  $\square$

Therefore, Lemma 2 effectively shows that the sufficient conditions for bounding the induced  $L_2$  norms of the primal and dual forms of a nominal LPV system are equivalent.

#### 3.2. Dual IQCs

The notion of duality for uncertain LPV systems requires a specific notion of duality for IQCs. Dual IQCs were previously introduced in [23]. These dual IQCs were defined in the frequency-domain for the stability analysis and feedforward control of LTI systems [23]. The results in [23] are briefly summarized in this subsection as this will ultimately lead to a related time-domain definition for a dual IQC. To begin, consider the uncertain system shown in Figure 1 with the following assumptions: (i)  $G$  is LTI and (ii)  $\Delta \in \text{IQC}(\Pi)$ . The main IQC theorem in [24] roughly states that the following frequency domain inequality is a sufficient condition for the stability of  $\mathcal{F}_u(G, \Delta)$ :

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}. \quad (13)$$

Inequality (13) is the primal form involving  $G$  and  $\Pi$ . It is shown in Section 2.1 of [23] that inequality (13) is equivalent to

$$\begin{bmatrix} I \\ -G(j\omega)^* \end{bmatrix}^* \Pi(j\omega)^{-1} \begin{bmatrix} I \\ -G(j\omega)^* \end{bmatrix} > 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}. \quad (14)$$

Inequality (14) is a dual inequality and this gives rise to dual multipliers  $\Pi^{-1}$  as defined in [23]. However, the following slightly different, but equivalent, definition is used in this paper in order to bring (14) into a standard form.

*Definition 5*

Given the primal IQC multiplier  $\Pi \in \mathbb{RL}_{\infty}^{(n_v+n_w) \times (n_v+n_w)}$ , the dual IQC multiplier is denoted by  $D(\Pi) \in \mathbb{RL}_{\infty}^{(n_w+n_v) \times (n_w+n_v)}$  and is defined as:

$$D(\Pi) := \begin{bmatrix} 0 & -I_{n_w} \\ I_{n_v} & 0 \end{bmatrix} \Pi^{-T} \begin{bmatrix} 0 & -I_{n_v} \\ I_{n_w} & 0 \end{bmatrix}. \quad (15)$$

Using this definition, inequality (14) is equivalent to

$$\begin{bmatrix} G(j\omega)^T \\ I \end{bmatrix}^* D(\Pi(j\omega)) \begin{bmatrix} G(j\omega)^T \\ I \end{bmatrix} < 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}. \quad (16)$$

$G^T$  in inequality (16) is the dual of  $G$  in the sense of Definition 4 (albeit with no parameter dependence). Note that the form of inequality (16) is similar to that of (13), except it involves the dual LTI plant  $G^T$  and the dual IQC multiplier  $D(\Pi)$ .

It is worth noting some subtle points about the dual IQC multiplier. In the standard IQC analysis problem, once the perturbation  $\Delta$  is specified, the multiplier  $\Pi$  is chosen from a library [24] such that  $\Delta \in \text{IQC}(\Pi)$ . The dual multiplier  $D(\Pi)$  is different because, rather than being chosen from a library, it is derived from the primal multiplier  $\Pi$  using Definition 5. As with any IQC multiplier, a set of operators satisfy the IQC defined by  $D(\Pi)$ . Let  $\Delta_D$  denote a member of this set. Although  $\Delta_D$  appears to be an uncertainty, it is actually an artificial construct that simply satisfies the IQC defined by  $D(\Pi)$ . To summarize, the main result in [24] is that the primal frequency domain inequality (13) is sufficient for the stability of  $\mathcal{F}_u(G, \Delta)$ . This is equivalent to stating that the dual frequency domain inequality (16) is sufficient for the stability of  $\mathcal{F}_u(G^T, \Delta_D)$ . This result enables certain problems, e.g. feedforward synthesis, to be convexified by converting from primal to dual form.

The goal of this section is to extend these results for gridded LPV systems  $G$  for which there is no frequency response interpretation. This extension requires a time-domain definition for dual IQCs. In particular, dual  $J$ -spectral factorizations are considered. Let  $(\hat{\Psi}, J)$  be a  $J$ -spectral factorization of  $\Pi$ . Using Definition 5, the dual IQC multiplier can be expressed as

$$D(\Pi) = \begin{bmatrix} 0 & -I_{n_w} \\ I_{n_v} & 0 \end{bmatrix} \hat{\Psi}^{-T} J^{-1} \hat{\Psi}^{-T} \begin{bmatrix} 0 & -I_{n_v} \\ I_{n_w} & 0 \end{bmatrix}. \quad (17)$$

With a few more steps, it follows that

$$D(\Pi) = D(\hat{\Psi}) \tilde{J} D(\hat{\Psi}), \quad \text{where } D(\hat{\Psi}) := \begin{bmatrix} 0 & -I_{n_w} \\ I_{n_v} & 0 \end{bmatrix} \hat{\Psi}^{-T} \begin{bmatrix} 0 & -I_{n_v} \\ I_{n_w} & 0 \end{bmatrix}. \quad (18)$$

Further,  $\hat{\Psi}, \hat{\Psi}^{-1} \in \mathbb{RH}_{\infty}^{(n_v+n_w) \times (n_v+n_w)}$  if and only if  $D(\hat{\Psi}), D(\hat{\Psi})^{-1} \in \mathbb{RH}_{\infty}^{(n_w+n_v) \times (n_w+n_v)}$ .

Therefore,  $(D(\hat{\Psi}), J)$  is a  $J$ -spectral factorization of  $D(\Pi)$ .

### 3.3. Relation Between Nominal and Uncertain Input-Output Analyses

Extending duality to uncertain LPV systems requires not only dual LPV systems and dual IQCs, but also technical insight into the uncertain input-output analysis problem. To begin, consider the roles played by Lemmas 1 and 2 in the input-output analysis of nominal LPV plants. The Bounded Real Lemma, as stated in Lemma 1, provides a sufficient LMI condition to bound the induced  $L_2$  norm for a nominal LPV system. Lemma 2 demonstrates an equivalence between the primal and dual forms of this LMI condition. This section derives a similar set of results for uncertain LPV systems. Theorem 1 already establishes sufficient conditions to bound the induced  $L_2$  norm for an

uncertain LPV system. Hence, the missing piece in the puzzle is a lemma for uncertain LPV systems that demonstrates equivalence between the primal and dual forms, analogous to the result in Lemma 2 for nominal LPV systems. This subsection will cover some intermediate steps leading up to this result and the next subsection will formally state this dualization lemma for uncertain LPV systems.

First note that the dualization of  $LMI_{BR}(G, P, \gamma) < 0$  for nominal LPV systems in Lemma 2 is straightforward because the Schur complement lemma can be applied to blocks involving  $I$  or  $-I$ . However, a similar procedure cannot be followed for uncertain LPV systems because of the presence of an additional IQC term in  $LMI_{WC}(G, P, \gamma, \Psi, M) < 0$ . To demonstrate the effect of the IQC term, consider the  $J$ -spectral factor  $(\hat{\Psi}, J)$  of  $\Pi$ . Since  $\hat{\Psi}$  is square, its output can be partitioned as  $z = \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix}$ , where  $\tilde{v}$  and  $\tilde{w}$  have the same sizes as  $v$  and  $w$ , respectively. In addition, the state-space matrices associated with  $z$  (see equation (10)) can be partitioned as:

$$\begin{bmatrix} \mathcal{C}_z(\rho) & \mathcal{D}_{zw}(\rho) & \mathcal{D}_{zd}(\rho) \end{bmatrix} = \begin{bmatrix} \mathcal{C}_{\tilde{v}}(\rho) & \mathcal{D}_{\tilde{v}w}(\rho) & \mathcal{D}_{\tilde{v}d}(\rho) \\ \mathcal{C}_{\tilde{w}}(\rho) & \mathcal{D}_{\tilde{w}w}(\rho) & \mathcal{D}_{\tilde{w}d}(\rho) \end{bmatrix}. \quad (19)$$

Using the above matrix partitions, the full form of  $LMI_{WC}(G, P, \gamma, \hat{\Psi}, J) < 0$  is:

$$\begin{bmatrix} \mathcal{A}^T(\rho)P + \star & \star & \star \\ \mathcal{B}_w^T(\rho)P & 0 & \star \\ \mathcal{B}_d^T(\rho)P & 0 & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} \mathcal{C}_e^T(\rho) \\ \mathcal{D}_{ew}^T(\rho) \\ \mathcal{D}_{ed}^T(\rho) \end{bmatrix} (\star) + \begin{bmatrix} \mathcal{C}_{\tilde{v}}^T(\rho) & \mathcal{C}_{\tilde{w}}^T(\rho) \\ \mathcal{D}_{\tilde{v}w}^T(\rho) & \mathcal{D}_{\tilde{w}w}^T(\rho) \\ \mathcal{D}_{\tilde{v}d}^T(\rho) & \mathcal{D}_{\tilde{w}d}^T(\rho) \end{bmatrix} J(\star) < 0 \quad \forall \rho \in \mathcal{P}. \quad (20)$$

In this inequality, the third term is the IQC term involving  $\hat{\Psi}$  and  $J$ . Since  $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$  is sign indefinite, the Schur complement lemma cannot be applied to the entire IQC term. Rather, it can only be applied to the positive definite sub-block of the IQC term.

Since the Schur complement lemma cannot be applied to the entire IQC term, an alternative approach is followed wherein inequality (20) is simplified. First, note that the (2, 2) block of (20) is  $\gamma^{-1} \mathcal{D}_{ew}^T(\rho) \mathcal{D}_{ew}(\rho) + \mathcal{D}_{\tilde{v}w}^T(\rho) \mathcal{D}_{\tilde{v}w}(\rho) - \mathcal{D}_{\tilde{w}w}^T(\rho) \mathcal{D}_{\tilde{w}w}(\rho) < 0$ . This can be rearranged to show that  $\mathcal{D}_{\tilde{w}w}^T(\rho) \mathcal{D}_{\tilde{w}w}(\rho) > 0$ , i.e.  $\mathcal{D}_{\tilde{w}w}(\rho)$  is nonsingular for all  $\rho \in \mathcal{P}$ . Next, define the following parameter-dependent congruence transformation matrix:

$$T(\rho) := \begin{bmatrix} I & 0 & 0 \\ -\mathcal{D}_{\tilde{w}w}^{-1}(\rho) \mathcal{C}_{\tilde{w}}(\rho) & \mathcal{D}_{\tilde{w}w}^{-1}(\rho) & -\gamma^{-0.5} \mathcal{D}_{\tilde{w}w}^{-1}(\rho) \mathcal{D}_{\tilde{w}d}(\rho) \\ 0 & 0 & \gamma^{-0.5} I \end{bmatrix}. \quad (21)$$

Multiplying inequality (20) on the left and right by  $T^T(\rho)$  and  $T(\rho)$ , respectively, results in:

$$\begin{bmatrix} \bar{\mathcal{A}}^T(\rho)P + P\bar{\mathcal{A}}(\rho) & \star \\ \bar{\mathcal{B}}^T(\rho)P & -I \end{bmatrix} + \begin{bmatrix} \bar{\mathcal{C}}^T(\rho) \\ \bar{\mathcal{D}}^T(\rho) \end{bmatrix} (\star) < 0 \quad \forall \rho \in \mathcal{P}, \quad (22)$$

where,

$$\bar{\mathcal{A}}(\rho) := \mathcal{A}(\rho) - \mathcal{B}_w(\rho) \mathcal{D}_{\tilde{w}w}^{-1}(\rho) \mathcal{C}_{\tilde{w}}(\rho), \quad (23)$$

$$\bar{\mathcal{B}}(\rho) := [\mathcal{B}_w(\rho) \mathcal{D}_{\tilde{w}w}^{-1}(\rho) \quad \gamma^{-0.5} (-\mathcal{B}_w(\rho) \mathcal{D}_{\tilde{w}w}^{-1}(\rho) \mathcal{D}_{\tilde{w}d}(\rho) + \mathcal{B}_d(\rho))], \quad (24)$$

$$\bar{\mathcal{C}}(\rho) := \begin{bmatrix} \mathcal{C}_{\tilde{v}}(\rho) - \mathcal{D}_{\tilde{v}w}(\rho) \mathcal{D}_{\tilde{w}w}^{-1}(\rho) \mathcal{C}_{\tilde{w}}(\rho) \\ \gamma^{-0.5} (\mathcal{C}_e(\rho) - \mathcal{D}_{ew}(\rho) \mathcal{D}_{\tilde{w}w}^{-1}(\rho) \mathcal{C}_{\tilde{w}}(\rho)) \end{bmatrix}, \quad (25)$$

$$\bar{\mathcal{D}}(\rho) := \begin{bmatrix} \mathcal{D}_{\tilde{v}w}(\rho) \mathcal{D}_{\tilde{w}w}^{-1}(\rho) & \gamma^{-0.5} (-\mathcal{D}_{\tilde{v}w}(\rho) \mathcal{D}_{\tilde{w}w}^{-1}(\rho) \mathcal{D}_{\tilde{w}d}(\rho) + \mathcal{D}_{\tilde{v}d}(\rho)) \\ \gamma^{-0.5} (\mathcal{D}_{ew}(\rho) \mathcal{D}_{\tilde{w}w}^{-1}(\rho)) & \gamma^{-1} (-\mathcal{D}_{ew}(\rho) \mathcal{D}_{\tilde{w}w}^{-1}(\rho) \mathcal{D}_{\tilde{w}d}(\rho) + \mathcal{D}_{ed}(\rho)) \end{bmatrix}. \quad (26)$$

Note that inequality (22) is similar to LMI (7) in the Bounded Real Lemma, except that it involves transformed state-space matrices. Consistent with the notation introduced earlier, inequality (22) can be shortened to  $LMI_{BR}(\bar{G}, P, 1) < 0$ , where  $\bar{G} \stackrel{s}{=} \left[ \begin{array}{c|c} \bar{\mathcal{A}}(\rho) & \bar{\mathcal{B}}(\rho) \\ \hline \bar{\mathcal{C}}(\rho) & \bar{\mathcal{D}}(\rho) \end{array} \right]$  depends on  $G$ ,  $\gamma$ , and

$\hat{\Psi}$ . From Equations (23) through (26), it can be inferred that the second input and second output of  $\bar{G}$  are scaled by  $\gamma^{-0.5}$  each. This results in a gain bound of 1 in  $LMI_{BR}(\bar{G}, P, 1) < 0$ . The equivalence of  $LMI_{WC}(G, P, \gamma, \hat{\Psi}, J) < 0$  and  $LMI_{BR}(\bar{G}, P, 1) < 0$  was previously reported as Lemma 2 in [20] and is rephrased below.

*Lemma 3* ([20])

Let  $G$  be the LPV system defined in (4) and  $(\hat{\Psi}, J)$  be a  $J$ -spectral factor of  $\Pi$ .  $P = P^T$  satisfies  $LMI_{WC}(G, P, \gamma, \hat{\Psi}, J) < 0$  if and only if it satisfies  $LMI_{BR}(\bar{G}, P, 1) < 0$ , where the state-space matrices of  $\bar{G}$  are defined in Equations (23) through (26).

As per Theorem 1,  $LMI_{WC}(G, P, \gamma, \hat{\Psi}, J) < 0$  is a sufficient condition for the worst-case gain of  $\mathcal{F}_u(G, \Delta)$  to be bounded by  $\gamma$ . As per the Bounded Real Lemma,  $LMI_{BR}(\bar{G}, P, 1) < 0$  is a sufficient condition for  $\|\bar{G}\|$  to be bounded by 1. Lemma 3 states that the sufficient condition for bounding the worst-case gain of  $\mathcal{F}_u(G, \Delta)$  by  $\gamma$  is equivalent to the sufficient condition for bounding the induced  $L_2$  norm of  $\bar{G}$  by 1. The next subsection explains the role played by Lemma 3 in duality.

### 3.4. Technical Results

This subsection brings together all the components discussed thus far, including dual LPV systems, dual IQCs, and the relation between the nominal and uncertain input-output analysis problems. In particular, dual uncertain LPV plants are considered along with sufficient conditions for bounding their worst-case gain. First, however, consider the sufficient condition  $LMI_{WC}(G, P, \gamma, \Psi, M) < 0$  that was presented in Theorem 1 for bounding the worst-case gain of an uncertain LPV system. Since  $LMI_{WC}(G, P, \gamma, \Psi, M) < 0$  can be composed using any stable factorization of  $\Pi$ , it is important to understand how its feasibility depends on the factorization. To do this, the next lemma relates the state-space realizations of two stable factorizations of  $\Pi$ .

*Lemma 4*

Let a frequency-domain IQC multiplier have the following two factorizations,

$$\Pi(s) = \Pi^\sim(s) = \Psi_1^\sim(s) M_1 \Psi_1(s) = \Psi_2^\sim(s) M_2 \Psi_2(s), \quad (27)$$

where  $\Psi_1(s) \stackrel{s}{=} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}$  and  $\Psi_2(s) \stackrel{s}{=} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}$  are stable, minimal realizations with state dimension  $n$ . Define  $\begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} = \begin{bmatrix} C_i^T \\ D_i^T \end{bmatrix} M_i \begin{bmatrix} C_i & D_i \end{bmatrix}$  for  $i = 1, 2$ . Then  $\exists T_1 \in \mathbb{R}^{n \times n}$  such that:

- (1)  $A_2 = T_1 A_1 T_1^{-1}$ ,
- (2)  $B_2 = T_1 B_1$ , and
- (3)  $\begin{bmatrix} Q_2 & S_2 \\ S_2^T & R_2 \end{bmatrix} = \begin{bmatrix} T_1^{-T} & 0 \\ 0 & I \end{bmatrix} \left( \begin{bmatrix} Q_1 & S_1 \\ S_1^T & R_1 \end{bmatrix} - \begin{bmatrix} A_1^T \bar{X} + \bar{X} A_1 & \bar{X} B_1 \\ B_1^T \bar{X} & 0 \end{bmatrix} \right) \begin{bmatrix} T_1^{-1} & 0 \\ 0 & I \end{bmatrix}$ , where  $\bar{X} = \bar{X}^T$  is the unique solution to the Lyapunov Equation  $A_1^T \bar{X} + \bar{X} A_1 = Q_1 - T_1^T Q_2 T_1$ .

*Proof*

The proof mainly relies on standard facts regarding Lyapunov equalities. See Appendix A.  $\square$

The next lemma relates the feasibility of two worst-case gain LMIs using Lemma 4.

*Lemma 5*

Consider two factorizations  $(\Psi_1, M_1)$  and  $(\Psi_2, M_2)$  of  $\Pi$  such that  $\Psi_1$  and  $\Psi_2$  are stable and have minimal state-space realizations. There exists  $P_1 = P_1^T$  satisfying  $LMI_{WC}(G, P_1, \gamma, \Psi_1, M_1) < 0$  if and only if there exists  $P_2 = P_2^T$  satisfying  $LMI_{WC}(G, P_2, \gamma, \Psi_2, M_2) < 0$ .

*Proof*

By assumption, the two factorizations of  $\Pi$  are stable and have minimal state-space realizations. Hence, there exist  $T_1$  and  $\bar{X}$  satisfying conclusions (1)-(3) in Lemma 4. Next, define

$T_\beta := \text{diag}(I, T_1^{-1}, I, I)$ . To prove necessity, assume that there exists  $P_2 = P_2^T$  satisfying  $LMI_{WC}(G, P_2, \gamma, \Psi_2, M_2) < 0$ . Then, multiply  $LMI_{WC}(G, P_2, \gamma, \Psi_2, M_2) < 0$  on the left and right by  $T_\beta^{-T}$  and  $T_\beta^{-1}$ , respectively. Finally, use statements (1) and (2) of Lemma 4 to show that  $P_1$  satisfies  $LMI_{WC}(G, P_1, \gamma, \Psi_1, M_1) < 0$ , where  $P_1$  and  $P_2$  are related as:

$$P_1 = \begin{bmatrix} I_{n_G} & 0 \\ 0 & T_1^T \end{bmatrix} P_2 \begin{bmatrix} I_{n_G} & 0 \\ 0 & T_1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \bar{X} \end{bmatrix}. \quad (28)$$

To prove sufficiency, reverse the algebraic steps.  $\square$

Lemma 5 essentially proves that the feasibility of  $LMI_{WC}(G, P, \gamma, \Psi, M) < 0$  is independent of the factorization  $(\Psi, M)$ , as long as  $\Psi$  is stable and has a minimal state-space realization. In other words, the feasibility of  $LMI_{WC}(G, P, \gamma, \Psi, M) < 0$  only depends on  $G$  and  $\Pi$ . Note that  $LMI_{WC}(G, P, \gamma, \Psi, M) < 0$  is one of the sufficient conditions presented in Theorem 1 for bounding the worst-case gain of  $\mathcal{F}_u(G, \Delta)$ . The main technical lemma is presented next.

#### Lemma 6

Given  $G$  and  $\Pi$ , the following statements hold.

- (1)  $G$  is quadratically stable if and only if  $G^T$  is quadratically stable.
- (2)  $\Pi$  is a strict PN multiplier if and only if  $\mathbf{D}(\Pi)$  is a strict PN multiplier.
- (3) Let  $(\Psi, M)$  be any stable factorization of  $\Pi$  and  $(\Gamma, N)$  be any stable factorization of  $\mathbf{D}(\Pi)$ . Then  $\exists P = P^T$  satisfying  $LMI_{WC}(G, P, \gamma, \Psi, M) < 0$  if and only if  $\exists Q = Q^T$  satisfying  $LMI_{WC}(G^T, Q, \gamma, \Gamma, N) < 0$ .

#### Proof

Statement (1) follows from Lemmas 1 and 2. Statement (2) is proved as follows. For sufficiency, assume that  $\Pi$  is a strict PN multiplier. First, from Definition 1,  $\Pi_{11}(j\omega) > 0$  and  $\Pi_{22}(j\omega) < 0 \forall \omega \in \mathbb{R} \cup \{\infty\}$ . This implies that  $\Pi^{-1}(j\omega)$  exists  $\forall \omega \in \mathbb{R} \cup \{\infty\}$  and can be partitioned as  $\begin{bmatrix} W_{11}(j\omega) & W_{12}(j\omega) \\ W_{12}^T(j\omega) & W_{22}(j\omega) \end{bmatrix}$ . In the next few steps, the argument  $(j\omega)$  is dropped for brevity.

From the matrix inversion lemma, it follows that  $W_{11} = (\Pi_{11} - \Pi_{12}\Pi_{22}^{-1}\Pi_{21})^{-1}$  and  $W_{22} = (\Pi_{22} - \Pi_{21}\Pi_{11}^{-1}\Pi_{12})^{-1}$ . Note that  $\Pi_{11} > 0$  and  $\Pi_{22} < 0$  together imply  $W_{11} > 0$  and  $W_{22} < 0$ . Next, partition  $\mathbf{D}(\Pi)$  as  $\begin{bmatrix} \Pi_{D11} & \Pi_{D12} \\ \Pi_{D12}^T & \Pi_{D22} \end{bmatrix}$ , where  $\Pi_{D11} \in \mathbb{R}\mathbb{L}_\infty^{n_w \times n_w}$  and  $\Pi_{D22} \in \mathbb{R}\mathbb{L}_\infty^{n_v \times n_v}$ . Finally, using Definition 5, it can be shown that  $\Pi_{D11} = -W_{11}^T > 0$  and  $\Pi_{D22} = -W_{22}^T < 0$ . Hence,  $\mathbf{D}(\Pi)$  is a strict PN multiplier. For necessity, note that  $\mathbf{D}(\mathbf{D}(\Pi)) = \Pi$  and use similar arguments.

Statement (3) is proved by invoking multiple previous lemmas. First, consider the statement:  $\exists P = P^T$  satisfying  $LMI_{WC}(G, P, \gamma, \Psi, M) < 0$  for some stable factorization  $(\Psi, M)$  of  $\Pi$ . Let  $(\hat{\Psi}, J)$  denote a  $J$ -spectral factorization of  $\Pi$ . Since  $\hat{\Psi}$  is stable by definition, it is inferred from Lemma 5 that  $\exists \hat{P} = \hat{P}^T$  satisfying  $LMI_{WC}(G, \hat{P}, \gamma, \hat{\Psi}, J) < 0$ . Further, from Lemma 3,  $\hat{P} = \hat{P}^T$  satisfies  $LMI_{WC}(G, \hat{P}, \gamma, \hat{\Psi}, J) < 0$  if and only if it satisfies  $LMI_{BR}(\bar{G}, \hat{P}, 1) < 0$ , where  $\bar{G}$  depends on  $G, \gamma$ , and  $\hat{\Psi}$  through Equations (23) through (26).

Next, consider the statement:  $\exists Q = Q^T$  satisfying  $LMI_{WC}(G^T, Q, \gamma, \Gamma, N) < 0$  for some stable factorization  $(\Gamma, N)$  of  $\mathbf{D}(\Pi)$ . Let  $(\mathbf{D}(\hat{\Psi}), J)$  denote a  $J$ -spectral factorization of  $\mathbf{D}(\Pi)$ . Since  $\mathbf{D}(\hat{\Psi})$  is stable by definition, it is inferred from Lemma 5 that  $\exists \hat{Q} = \hat{Q}^T$  satisfying  $LMI_{WC}(G^T, \hat{Q}, \gamma, \mathbf{D}(\hat{\Psi}), J) < 0$ . Further, from Lemma 3,  $\hat{Q} = \hat{Q}^T$  satisfies  $LMI_{WC}(G^T, \hat{Q}, \gamma, \mathbf{D}(\hat{\Psi}), J) < 0$  if and only if it satisfies  $LMI_{BR}(\underline{G}, \hat{Q}, 1) < 0$ , where  $\underline{G}$  depends on  $G^T, \gamma$ , and  $\mathbf{D}(\hat{\Psi})$  through equations that are similar to Equations (23) through (26).

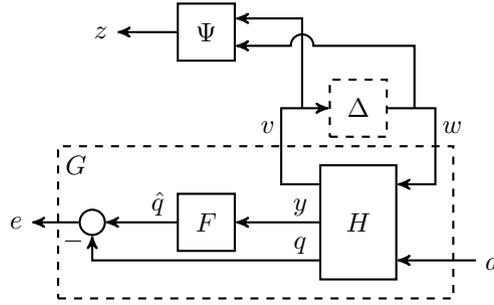


Figure 3. The output estimation problem.

It can be verified, with a significant amount of algebra, that  $\underline{G} = \bar{G}^T$ , i.e.  $\bar{G}$  and  $\underline{G}$  are dual systems. Finally, from Lemma 2,  $\hat{P} = \hat{P}^T$  satisfies  $LMI_{BR}(\bar{G}, \hat{P}, 1) < 0$  if and only if  $\hat{Q} := \hat{P}^{-1}$  satisfies  $LMI_{BR}(\underline{G}, \hat{Q}, 1) < 0$ .  $\square$

Lemma 6 can be better understood in the context of two related worst-case gain problems. The primal problem involves bounding the worst-case gain of  $\mathcal{F}_u(G, \Delta)$  over the set of uncertainties that satisfy the IQC defined by  $\Pi$ . The dual problem involves bounding the worst-case gain of  $\mathcal{F}_u(G^T, \Delta_D)$  over the set of uncertainties that satisfy the IQC defined by  $D(\Pi)$ . Both problems have separate (but similar) sets of sufficient conditions (Theorem 1) for bounding their respective worst-case gains. Lemma 6 essentially states that the two sets of sufficient conditions are equivalent. Statement (1) establishes equivalence between the primal and dual nominal LPV systems, in the sense of quadratic stability. Statement (2) establishes equivalence between the primal and dual IQC multipliers, in the sense of the strict PN property. Statement (3) establishes equivalence between the primal and dual worst-case gain LMI conditions.

#### 4. CONVEX SYNTHESIS FOR UNCERTAIN LPV SYSTEMS

##### 4.1. Output Estimation

While the output estimation problem was previously considered in [13], the derivation of the synthesis conditions provided in this section is more rigorous in three specific ways. First, the state-space matrices of the estimator are completely eliminated from the LMI conditions given in the synthesis theorem. Second, a matrix dilation lemma is used to complete the sign indefinite Lyapunov matrix. Third, an explicit method is provided to reconstruct the state-space matrices of the estimator.

The output estimation problem is formulated using the interconnection shown in Figure 3.  $H$  is a nominal LPV plant with parameters  $\rho \in \mathcal{P}$ , states  $x_H \in \mathbb{R}^{n_H}$ , disturbance inputs  $d \in \mathbb{R}^{n_d}$ , measurable outputs  $y \in \mathbb{R}^{n_y}$ , and unmeasurable outputs  $q \in \mathbb{R}^{n_q}$ . The LPV plant  $H$  is connected with an uncertainty  $\Delta$  via signals  $v \in \mathbb{R}^{n_v}$  and  $w \in \mathbb{R}^{n_w}$ . This creates an uncertain LPV system  $\mathcal{F}_u(H, \Delta)$  from the input disturbance  $d$  to the outputs  $y$  and  $q$ . The problem is to synthesize an estimator  $F$  that uses the measurements  $y$  to generate an estimate of  $q$ . Let  $\hat{q}$  denote the estimate of  $q$  and  $e := \hat{q} - q$  denote the estimation error. The synthesis objective is to bound the worst-case induced  $L_2$  norm from  $d$  to  $e$  over the set of uncertainties that satisfy the IQC defined by  $\Pi$ .

In addition to the LFT of  $H$  and  $\Delta$ , the IQC filter  $\Psi$  is appended such that it is driven by signals  $v$  and  $w$ , and produces signal  $z$ . The interconnection of  $H$  and  $\Psi$  has the state-space representation

$$\begin{bmatrix} \dot{x} \\ z \\ y \\ q \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \\ C_3(\rho) & D_{31}(\rho) & D_{32}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ d \end{bmatrix}, \quad (29)$$

where  $x = [x_H^T, x_\Psi^T]^T \in \mathbb{R}^{n_H+n_\Psi}$  are the combined states of  $H$  and  $\Psi$ . It is assumed that  $D_{22}(\rho) \in \mathbb{R}^{n_y \times n_d}$  has full row rank  $\forall \rho \in \mathcal{P}$ . This assumption is used to ensure that all components of the measurement  $y$  are affected by some component of the disturbance  $d$ .

The estimator  $F$  to be synthesized has the state-space representation:

$$\begin{bmatrix} \dot{x}_F \\ \hat{q} \end{bmatrix} = \begin{bmatrix} A_F(\rho) & B_F(\rho) \\ C_F(\rho) & D_F(\rho) \end{bmatrix} \begin{bmatrix} x_F \\ y \end{bmatrix}, \quad (30)$$

where  $x_F \in \mathbb{R}^{n_F}$  are the estimator states,  $y$  is the input to the estimator, and  $\hat{q}$  is the output from the estimator. As shown by the large dashed box in Figure 3, the closed-loop formed by the interconnection of  $H$  and  $F$  is denoted by  $G$ , with states  $x_G = [x_H^T, x_F^T]^T$ . In the remainder of this section, the notation  $G(H, F)$  will be used in some cases to make explicit the dependence of  $G$  on  $H$  and  $F$ . Theorem 1 provides conditions to bound the worst-case gain of  $\mathcal{F}_u(G(H, F), \Delta)$ . The objective is to synthesize  $F$  which minimizes this bound.

To formulate the synthesis theorem, consider the extended LPV system formed by the interconnection of  $G(H, F)$  and  $\Psi$ . This extended system has the state-space realization:

$$\begin{bmatrix} \dot{x}_e \\ z \\ e \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\rho) & \mathcal{B}_w(\rho) & \mathcal{B}_d(\rho) \\ \mathcal{C}_z(\rho) & \mathcal{D}_{zw}(\rho) & \mathcal{D}_{zd}(\rho) \\ \mathcal{C}_e(\rho) & \mathcal{D}_{ew}(\rho) & \mathcal{D}_{ed}(\rho) \end{bmatrix} \begin{bmatrix} x_e \\ w \\ d \end{bmatrix}, \quad (31)$$

where  $x_e = [x_H^T, x_\Psi^T, x_F^T]^T \in \mathbb{R}^{n_H+n_\Psi+n_F}$  are the combined states of  $H$ ,  $\Psi$ , and  $F$ . These state-space matrices are expressed in terms of the matrices appearing in Equations (29) and (30) as:

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_w & \mathcal{B}_d \\ \mathcal{C}_z & \mathcal{D}_{zw} & \mathcal{D}_{zd} \\ \mathcal{C}_e & \mathcal{D}_{ew} & \mathcal{D}_{ed} \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 & B_2 \\ 0 & 0 & 0 & 0 \\ -C_1 & 0 & -D_{11} & -D_{12} \\ -C_3 & 0 & -D_{31} & -D_{32} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \begin{bmatrix} 0 & I & 0 & 0 \\ C_2 & 0 & D_{21} & D_{22} \end{bmatrix}, \quad (32)$$

where the dependence of the matrices on  $\rho$  is suppressed for brevity.

According to Theorem 1, the worst-case gain of  $\mathcal{F}_u(G(H, F), \Delta)$  is bounded by  $\gamma$  if there exists  $P = P^T$  satisfying

$$\begin{bmatrix} \mathcal{A}^T(\rho)P + P\mathcal{A}(\rho) & \star & \star \\ \mathcal{B}_w^T(\rho)P & 0 & \star \\ \mathcal{B}_d^T(\rho)P & 0 & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} \mathcal{C}_e^T(\rho) \\ \mathcal{D}_{ew}^T(\rho) \\ \mathcal{D}_{ed}^T(\rho) \end{bmatrix} (\star) + \begin{bmatrix} \mathcal{C}_z^T(\rho) \\ \mathcal{D}_{zw}^T(\rho) \\ \mathcal{D}_{zd}^T(\rho) \end{bmatrix} M (\star) < 0 \quad \forall \rho \in \mathcal{P}. \quad (33)$$

However, inequality (33) is not a LMI because of the presence of bilinear terms involving  $P$  and the state-space matrices of  $F$ . For example, the term  $\mathcal{A}^T(\rho)P$  involves the product of  $A_F(\rho)$  and  $P$ , both of which are variables to be selected. Since (33) is a bilinear matrix inequality (BMI), it will be referred to using the short form  $BMI_{WC}(G(H, F), P, \gamma, \Psi, M) < 0$ . The subscript  $WC$  indicates that it is a BMI associated with a worst-case gain problem and the arguments  $(G(H, F), P, \gamma, \Psi, M)$  indicate that the plant  $H$ , estimator  $F$ , Lyapunov matrix  $P$ , gain bound  $\gamma$ , and IQC factorization  $(\Psi, M)$  are involved. Although  $BMI_{WC}(G(H, F), P, \gamma, \Psi, M) < 0$  is a sufficient condition for  $\mathcal{F}_u(G(H, F), \Delta)$  to have bounded worst-case gain, it does not admit a convex solution. The next theorem is adapted from [13] and provides convex LMI conditions that are equivalent to  $BMI_{WC}(G(H, F), P, \gamma, \Psi, M) < 0$ .

### Theorem 2

Let  $H$  be a quadratically stable LPV system,  $\Pi$  be a strict PN multiplier, and  $(\Psi, M)$  be a stable factorization of  $\Pi$ . Let the interconnection of  $H$  and  $\Psi$  have the state-space realization given in (29). Let the columns of  $N(\rho)$  form bases for the null space of  $[C_2(\rho) \ D_{21}(\rho) \ D_{22}(\rho)]$ , where  $D_{22}(\rho)$  has full row rank  $\forall \rho \in \mathcal{P}$ . Denote  $\bar{N} := \text{diag}(N(\rho), I)$ . There exists a quadratically stable estimator  $F$  of order  $n_F$  and some matrix  $P = P^T$  satisfying  $BMI_{WC}(G(H, F), P, \gamma, \Psi, M) < 0$  if and only if there exist symmetric matrices  $X$  and  $Z$  satisfying

$$X - Z \geq 0, \quad \text{rank}(X - Z) \leq n_F, \quad (34)$$

$$\begin{bmatrix} A^T(\rho)Z + ZA(\rho) & \star & \star \\ B_1^T(\rho)Z & 0 & \star \\ B_2^T(\rho)Z & 0 & -\gamma I \end{bmatrix} + \begin{bmatrix} C_1^T(\rho) \\ D_{11}^T(\rho) \\ D_{12}^T(\rho) \end{bmatrix} M(\star) < 0 \quad \forall \rho \in \mathcal{P}, \text{ and} \quad (35)$$

$$\bar{N}^T \left( \begin{bmatrix} A^T(\rho)X + \star & \star & \star & \star \\ B_1^T(\rho)X & 0 & \star & \star \\ B_2^T(\rho)X & 0 & -\gamma I & \star \\ -C_3(\rho) & -D_{31}(\rho) & -D_{32}(\rho) & -\gamma I \end{bmatrix} + \begin{bmatrix} C_1^T(\rho) \\ D_{11}^T(\rho) \\ D_{12}^T(\rho) \\ 0 \end{bmatrix} M(\star) \right) \bar{N} < 0 \quad \forall \rho \in \mathcal{P}. \quad (36)$$

Further, feasibility of conditions (34), (35), and (36) implies that  $\mathcal{F}_u(G(H, F), \Delta)$  satisfies

- (1)  $\lim_{T \rightarrow \infty} x_e(T) = 0 \quad \forall x_e(0) \in \mathbb{R}^{n_H + n_\Psi + n_F}, \forall d \in L_2^{n_d}, \forall \Delta \in \text{IQC}(\Pi)$ , and  $\forall \rho \in \mathcal{T}$ , and
- (2)  $\sup_{\Delta \in \text{IQC}(\Pi)} \|\mathcal{F}_u(G(H, F), \Delta)\| \leq \gamma$ .

*Proof*

The proof of sufficiency adapts the proof of Lemma 3.1 in [38]. Assume there exists a quadratically stable estimator  $F$  of order  $n_F$  (where  $n_F$  is any positive integer) and some matrix  $P = P^T$  satisfying  $BMI_{WC}(G(H, F), P, \gamma, \Psi, M) < 0$ . Apply the Schur complement lemma to show that  $BMI_{WC}(G(H, F), P, \gamma, \Psi, M) < 0$  is equivalent to

$$\begin{bmatrix} \mathcal{A}^T(\rho)P + P\mathcal{A}(\rho) & \star & \star & \star \\ \mathcal{B}_w^T(\rho)P & 0 & \star & \star \\ \mathcal{B}_d^T(\rho)P & 0 & -\gamma I & \star \\ C_e(\rho) & \mathcal{D}_{ew}(\rho) & \mathcal{D}_{ed}(\rho) & -\gamma I \end{bmatrix} + \begin{bmatrix} C_z^T(\rho) \\ \mathcal{D}_{zw}^T(\rho) \\ \mathcal{D}_{zd}^T(\rho) \\ 0 \end{bmatrix} M(\star) < 0 \quad \forall \rho \in \mathcal{P}. \quad (37)$$

Inequality (37) can be rewritten, using the matrix expressions given in (32), as

$$L(\rho) + Q^T \Theta(\rho) R(\rho) + R^T(\rho) \Theta^T(\rho) Q < 0 \quad \forall \rho \in \mathcal{P}, \quad (38)$$

where  $\Theta(\rho) := \begin{bmatrix} A_F(\rho) & B_F(\rho) \\ C_F(\rho) & D_F(\rho) \end{bmatrix}$ ,

$$L(\rho) := \begin{bmatrix} \begin{bmatrix} A(\rho) & 0 \\ 0 & 0 \end{bmatrix}^T P + \star & \star & \star & \star \\ \begin{bmatrix} B_1^T(\rho) \\ B_2^T(\rho) \end{bmatrix} P & 0 & \star & \star \\ \begin{bmatrix} B_2^T(\rho) \\ -C_3(\rho) \end{bmatrix} P & 0 & -\gamma I & \star \\ -D_{31}(\rho) & -D_{32}(\rho) & -\gamma I & \star \end{bmatrix} + \begin{bmatrix} C_1^T(\rho) \\ 0 \\ D_{11}^T(\rho) \\ D_{12}^T(\rho) \\ 0 \end{bmatrix} M(\star), \quad (39)$$

$$Q := \begin{bmatrix} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} P & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \text{ and } R(\rho) := \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ C_2(\rho) & 0 & D_{21}(\rho) & D_{22}(\rho) & 0 \end{bmatrix}. \quad (40)$$

Let the columns of  $N(\rho)$  form bases for the null space of  $\begin{bmatrix} C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix}$ . Define the matrices  $N_Q$  and  $N_R(\rho)$  as:

$$N_Q := \begin{bmatrix} P^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \text{ and } N_R(\rho) := \begin{bmatrix} N_1(\rho) & 0 \\ 0 & 0 \\ N_2(\rho) & 0 \\ N_3(\rho) & 0 \\ 0 & I \end{bmatrix}, \quad (41)$$

where  $N_1(\rho)$ ,  $N_2(\rho)$ , and  $N_3(\rho)$  correspond to a block partition of the rows of  $N(\rho)$  consistent with the dimensions of  $C_2(\rho)$ ,  $D_{21}(\rho)$ , and  $D_{22}(\rho)$ , respectively. The columns of  $N_Q$  and  $N_R(\rho)$  form bases for the null spaces of  $Q$  and  $R(\rho)$ , respectively. From the matrix elimination lemma (Lemma 3.1 in [38]), there exists a matrix  $\Theta(\rho)$  of compatible dimensions satisfying inequality (38) if and only if

$$N_Q^T L(\rho) N_Q < 0 \quad \forall \rho \in \mathcal{P} \text{ and} \quad (42)$$

$$N_R^T(\rho) L(\rho) N_R(\rho) < 0 \quad \forall \rho \in \mathcal{P}. \quad (43)$$

Next, define  $n := n_H + n_\Psi$  and partition  $P$  as  $\begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}$ , where  $X = X^T \in \mathbb{R}^{n \times n}$ ,  $X_3 = X_3^T \in \mathbb{R}^{n_F \times n_F}$ , and  $X_2 \in \mathbb{R}^{n \times n_F}$ . Further, partition  $P^{-1}$  as  $\begin{bmatrix} Y & \bullet \\ \bullet & \bullet \end{bmatrix}$ , where  $Y = Y^T \in \mathbb{R}^{n \times n}$  and  $\bullet$  denotes terms that are not relevant here. Using these Lyapunov matrix partitions, inequality (43) is shown to be equivalent to inequality (36). Further, inequality (42) is shown to be equivalent to

$$\begin{bmatrix} YA^T(\rho) + A(\rho)Y & \star & \star \\ B_1^T(\rho) & 0 & \star \\ B_2^T(\rho) & 0 & -\gamma I \end{bmatrix} + \begin{bmatrix} YC_1^T(\rho) \\ D_{11}^T(\rho) \\ D_{12}^T(\rho) \end{bmatrix} M(\star) < 0 \quad \forall \rho \in \mathcal{P}. \quad (44)$$

Setting  $Z := Y^{-1}$  and multiplying inequality (44) on the left and right by  $\text{diag}(Z, I, I)$  yields (35).

Using the partition for  $P$  given above, the (1, 1) block of  $BMI_{WC}(G(H, F), P, \gamma, \Psi, M) < 0$  yields  $A_F^T(\rho)X_3 + X_3A_F(\rho) < 0 \quad \forall \rho \in \mathcal{P}$ . Since  $F$  is assumed to be quadratically stable, Definition 3 implies that  $X_3 > 0$ . A variation of the matrix dilation lemma (Lemma 7.9 in [39]) is stated and proved as Lemma 7 in Appendix B. From Lemma 7, it is concluded that  $X - Z \geq 0$  and  $\text{rank}(X - Z) \leq n_F$ .

For necessity, assume that there exist symmetric matrices  $X, Z \in \mathbb{R}^{n \times n}$  satisfying conditions (34), (35), and (36). By a variation of the matrix dilation lemma (Lemma 7 in Appendix B), there exist  $X_2 \in \mathbb{R}^{n \times n_F}$  and  $X_3 = X_3^T \in \mathbb{R}^{n_F \times n_F}$  such that  $X_3 > 0$  and  $\begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Z^{-1} & \bullet \\ \bullet & \bullet \end{bmatrix}$ . The algebraic steps used in the proof of sufficiency are now reversed. Specifically, from the matrix elimination lemma,  $X$  and  $Z$  satisfy LMIs (35) and (36) if and only if there exists a matrix  $\Theta(\rho)$  of compatible dimensions satisfying inequality (38). Partition  $\Theta(\rho)$  as given before, where  $A_F \in \mathbb{R}^{n_F \times n_F}$ . Note that  $F \stackrel{s}{=} \begin{bmatrix} A_F(\rho) & B_F(\rho) \\ C_F(\rho) & D_F(\rho) \end{bmatrix}$  and  $P := \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}$  satisfy  $BMI_{WC}(G(H, F), P, \gamma, \Psi, M) < 0$ . Using this partition for  $P$ , the (1, 1) block of  $BMI_{WC}(G(H, F), P, \gamma, \Psi, M) < 0$  yields  $A_F^T(\rho)X_3 + X_3A_F(\rho) < 0 \quad \forall \rho \in \mathcal{P}$ . From Definition 3,  $X_3 > 0$  implies that  $F$  is quadratically stable.

Finally, since  $H$  was already assumed to be quadratically stable, the quadratic stability of  $F$  implies the quadratic stability of  $G(H, F)$ . From Theorem 1, if there exists  $P = P^T$  satisfying  $BMI_{WC}(G(H, F), P, \gamma, \Psi, M) < 0$ , then  $\mathcal{F}_u(G(H, F), \Delta)$  satisfies statements (1) and (2).  $\square$

Note that conditions (34), (35), and (36) are LMIs in the variables  $X, Z, M$ , and  $\gamma$ . Hence, Theorem 2 circumvents the non-convexity of  $BMI_{WC}(G(H, F), P, \gamma, \Psi, M) < 0$  by providing equivalent LMI conditions. In implementation, a semidefinite program is formulated with  $\gamma$  as the linear cost function to be minimized while subjected to these LMIs. Further, the parameter space is discretized into a finite number of grid points and the LMIs are enforced at each grid point. All the LMIs share a common closed-loop Lyapunov matrix, making this approach significantly different from a pointwise synthesis. Theorem 2 results in no additional conservatism over the sufficient conditions of Theorem 1. Moreover, Theorem 2 is different from the existing results because it allows for grid-based LPV plants whose state matrices are arbitrary functions of the parameters.

The rank constraint on  $X - Z$  given in (34) is not convex when  $n_F < n$ . However, by choosing  $n_F \geq n$  one can ensure that the rank constraint is automatically satisfied. In practice, it suffices to choose  $n_F = n$ , yielding an estimator whose order equals the combined order of  $H$  and  $\Psi$ .

The optimal values of the decision variables  $X$  and  $Z$  obtained from the semidefinite program are used to complete  $P$  as  $\begin{bmatrix} X & Z - X \\ Z - X & X - Z \end{bmatrix}$ . Finally, the estimator  $F$  is reconstructed via an explicit procedure that only relies on the state-space matrices of  $H$  and  $\Psi$ , and the optimal values of  $X$  and  $Z$ . This entails deriving explicit expressions for  $N_Q$  and  $N_R(\rho)$ , and then forming a matrix  $T$  that spans the union of the null spaces of  $Q$  and  $R(\rho)$ . Upon observing that  $T$  is nonsingular, it is used in a congruence transformation of inequality (38). The remainder of the reconstruction procedure is omitted here, since it closely follows the proof of Lemma 3.1 given in Appendix A of [38].

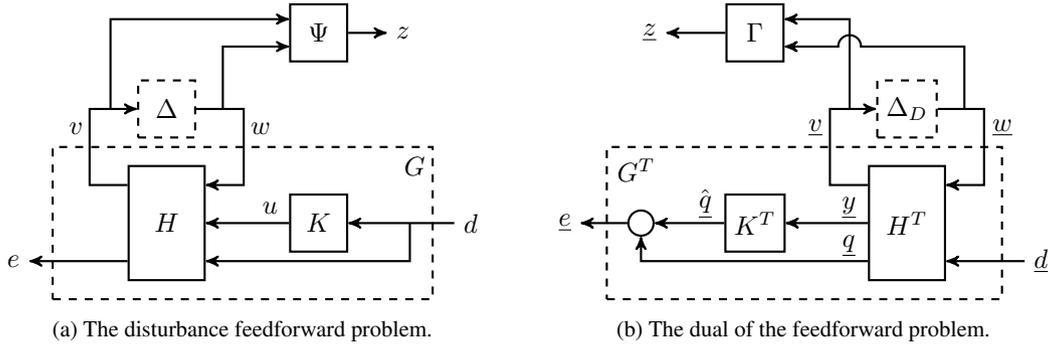


Figure 4. The dual of the disturbance feedforward problem is the output estimation problem.

#### 4.2. Disturbance Feedforward

The disturbance feedforward problem is formulated using the interconnection shown in Figure 4a.  $H$  is a nominal LPV plant with parameters  $\rho \in \mathcal{P}$ , states  $x_H \in \mathbb{R}^{n_H}$ , control inputs  $u \in \mathbb{R}^{n_u}$ , disturbance inputs  $d \in \mathbb{R}^{n_d}$ , and generalized errors  $e \in \mathbb{R}^{n_e}$ . The LPV plant  $H$  is connected with an uncertainty  $\Delta$  via signals  $v \in \mathbb{R}^{n_v}$  and  $w \in \mathbb{R}^{n_w}$ . This creates an uncertain LPV system  $\mathcal{F}_u(H, \Delta)$  from the inputs  $u$  and  $d$  to the output  $e$ . The problem is to synthesize a feedforward  $K$  that uses the disturbances  $d$  to generate control inputs  $u$ . The synthesis objective is to bound the worst-case induced  $L_2$  norm from  $d$  to  $e$  over the set of uncertainties that satisfy the IQC defined by  $\Pi$ .

In addition to the LFT of  $H$  and  $\Delta$ , the IQC filter  $\Psi$  is appended such that it is driven by signals  $v$  and  $w$ , and produces signal  $z$ . The interconnection of  $H$  and  $\Psi$  has the state-space representation

$$\begin{bmatrix} \dot{x} \\ z \\ e \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) & B_3(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) & D_{13}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) & D_{23}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \\ d \end{bmatrix}, \quad (45)$$

where  $x = [x_H^T, x_\Psi^T]^T \in \mathbb{R}^{n_H+n_\Psi}$  are the combined states of  $H$  and  $\Psi$ . It is assumed that  $D_{22}(\rho) \in \mathbb{R}^{n_e \times n_u}$  has full column rank  $\forall \rho \in \mathcal{P}$ . This assumption is used to ensure that all components of the control input  $u$  affect some component of the generalized error  $e$ .

The feedforward  $K$  to be synthesized has the state-space representation:

$$\begin{bmatrix} \dot{x}_K \\ u \end{bmatrix} = \begin{bmatrix} A_K(\rho) & B_K(\rho) \\ C_K(\rho) & D_K(\rho) \end{bmatrix} \begin{bmatrix} x_K \\ d \end{bmatrix}, \quad (46)$$

where  $x_K \in \mathbb{R}^{n_K}$  is the state,  $d$  is the input, and  $u$  is the output of the feedforward controller. As shown by the large dashed box in Figure 4a, the closed-loop formed by the interconnection of  $H$  and  $K$  is denoted by  $G$ , with states  $x_G = [x_H^T, x_K^T]^T$ . In the remainder of this section, the notation  $G(H, K)$  will be used in some cases to make explicit the dependence of  $G$  on  $H$  and  $K$ . Theorem 1 provides conditions to bound the worst-case gain of  $\mathcal{F}_u(G(H, K), \Delta)$ . The objective is to synthesize  $K$  which minimizes this bound.

Next consider the extended LPV system formed by the interconnection of  $G(H, K)$  and  $\Psi$ . This extended system has the state-space realization:

$$\begin{bmatrix} \dot{x}_e \\ z \\ e \end{bmatrix} = \begin{bmatrix} \mathcal{A}(\rho) & \mathcal{B}_w(\rho) & \mathcal{B}_d(\rho) \\ \mathcal{C}_z(\rho) & \mathcal{D}_{zw}(\rho) & \mathcal{D}_{zd}(\rho) \\ \mathcal{C}_e(\rho) & \mathcal{D}_{ew}(\rho) & \mathcal{D}_{ed}(\rho) \end{bmatrix} \begin{bmatrix} x_e \\ w \\ d \end{bmatrix}, \quad (47)$$

where  $x_e = [x_H^T, x_\Psi^T, x_K^T]^T \in \mathbb{R}^{n_H+n_\Psi+n_K}$  are the combined states of  $H$ ,  $\Psi$ , and  $K$ . These state-space matrices are expressed in terms of the matrices appearing in Equations (45) and (46) as:

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_w & \mathcal{B}_d \\ \mathcal{C}_z & \mathcal{D}_{zw} & \mathcal{D}_{zd} \\ \mathcal{C}_e & \mathcal{D}_{ew} & \mathcal{D}_{ed} \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 & B_3 \\ 0 & 0 & 0 & 0 \\ -\bar{C}_1 & -\bar{0} & \bar{D}_{11} & \bar{D}_{13} \\ -\bar{C}_2 & -\bar{0} & \bar{D}_{21} & \bar{D}_{23} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ -I & 0 \\ -\bar{0} & \bar{D}_{12} \\ -\bar{0} & \bar{D}_{22} \end{bmatrix} \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad (48)$$

where the dependence of the matrices on  $\rho$  is suppressed for brevity.

According to Theorem 1, the worst-case gain of  $\mathcal{F}_u(G(H, K), \Delta)$  is bounded by  $\gamma$  if there exists  $P = P^T$  satisfying  $BMI_{WC}(G(H, K), P, \gamma, \Psi, M) < 0$ . Applying the Schur complement lemma,  $BMI_{WC}(G(H, K), P, \gamma, \Psi, M) < 0$  is equivalent to the inequality

$$\begin{bmatrix} \mathcal{A}^T(\rho)P + P\mathcal{A}(\rho) & \star & \star & \star \\ \mathcal{B}_w^T(\rho)P & 0 & \star & \star \\ \mathcal{B}_d^T(\rho)P & 0 & -\gamma I & \star \\ \mathcal{C}_e(\rho) & \mathcal{D}_{ew}(\rho) & \mathcal{D}_{ed}(\rho) & -\gamma I \end{bmatrix} + \begin{bmatrix} \mathcal{C}_z^T(\rho) \\ \mathcal{D}_{zw}^T(\rho) \\ \mathcal{D}_{zd}^T(\rho) \\ 0 \end{bmatrix} M(\star) < 0 \quad \forall \rho \in \mathcal{P}. \quad (49)$$

As before, inequality (49) is not a LMI because of the presence of bilinear terms involving  $P$  and the state-space matrices of  $K$ . For example, the term  $\mathcal{A}^T(\rho)P$  involves the product of  $A_K(\rho)$  and  $P$ , similar to that appearing in the output estimation problem. However, the disturbance feedforward problem has the additional complication that  $\mathcal{C}_z(\rho)$ ,  $\mathcal{D}_{zw}(\rho)$ , and  $\mathcal{D}_{zd}(\rho)$  depend on the state-space matrices of the feedforward to be synthesized. Thus, the second term in inequality (49) involves quadratic products of the state-space matrices of  $K$ . As a consequence, it does not seem possible to convert inequality (49) into equivalent LMI conditions via the matrix elimination lemma. Hence an alternative approach is followed wherein the dual of the disturbance feedforward problem is considered. This alternative approach provides convex (LMI) synthesis conditions for  $K$ .

Before considering the dual problem, recall the main implication of Lemma 6. Lemma 6 proved that the sufficient conditions for bounding the worst-case gain of  $\mathcal{F}_u(G, \Delta)$  over the uncertainty set IQC( $\Pi$ ) are equivalent to the sufficient conditions for bounding the worst-case gain of  $\mathcal{F}_u(G^T, \Delta_D)$  over the uncertainty set IQC( $\mathcal{D}(\Pi)$ ). Now, denote  $\mathcal{F}_u(G(H, K), \Delta)$  shown in Figure 4a as the *primal* uncertain LPV system. The corresponding *dual* uncertain LPV system is  $\mathcal{F}_u(G^T(H^T, K^T), \Delta_D)$  as shown in Figure 4b. Here,  $G^T$  is the dual of  $G$  in the sense of Definition 4 and  $\Delta_D$  is an artificial construct that simply satisfies the IQC defined by  $\mathcal{D}(\Pi)$ . It is verified from algebra that  $G^T$  is the interconnection of  $H^T$  and  $K^T$  that is shown in Figure 4b. As before, the notation  $G^T(H^T, K^T)$  is used sometimes to make explicit the dependence of  $G^T$  on  $H^T$  and  $K^T$ .

$H^T$  is the dual of  $H$  with state  $\underline{x}_H \in \mathbb{R}^{n_H}$ , inputs  $\underline{w} \in \mathbb{R}^{n_w}$  and  $\underline{d} \in \mathbb{R}^{n_e}$ , and outputs  $\underline{v} \in \mathbb{R}^{n_w}$ ,  $\underline{y} \in \mathbb{R}^{n_u}$ , and  $\underline{q} \in \mathbb{R}^{n_d}$ . The inputs of  $H^T$  are partitioned conformably with the outputs of  $H$ . For example, in the preceding discussion, the outputs of  $H$  were partitioned as  $v \in \mathbb{R}^{n_w}$  and  $e \in \mathbb{R}^{n_e}$ . Consequently, the inputs of  $H^T$  are partitioned as  $\underline{w} \in \mathbb{R}^{n_w}$  and  $\underline{d} \in \mathbb{R}^{n_e}$ . Similarly, the outputs of  $H^T$  are partitioned conformably with the inputs of  $H$ .  $K^T$  is the dual of  $K$  with state  $\underline{x}_K \in \mathbb{R}^{n_K}$ , input  $\underline{y} \in \mathbb{R}^{n_u}$ , and output  $\hat{q} \in \mathbb{R}^{n_d}$ . On comparing Figure 4b with Figure 3, it is inferred that  $K^T$  is effectively an output estimator for  $H^T$ . The output  $\hat{q}$  of  $K^T$  is effectively an estimate of the output  $q$  of  $H^T$ . Because of the way the feedforward problem is formulated in Figure 4a, note that the estimation error  $\underline{e}$  equals  $\hat{q} + \underline{q}$  rather than  $\hat{q} - \underline{q}$  as was the case in Section 4.1. The next theorem proves that synthesizing an output estimator  $K^T$  to bound the worst-case gain of  $\mathcal{F}_u(G^T(H^T, K^T), \Delta_D)$  is equivalent to synthesizing a disturbance feedforward to bound the worst-case gain of  $\mathcal{F}_u(G(H, K), \Delta)$ .

### Theorem 3

Let  $H$  be a quadratically stable LPV system and  $\Pi$  be a strict PN multiplier. Let  $(\Psi, M)$  be any stable factorization of  $\Pi$  and  $(\Gamma, N)$  be any stable factorization of  $\mathcal{D}(\Pi)$ . Let  $G(H, K)$  and  $G^T(H^T, K^T)$  denote the closed-loop primal (Figure 4a) and dual (Figure 4b) systems for a given  $K$ , respectively. Then  $K$  is a quadratically stable feedforward that satisfies  $BMI_{WC}(G(H, K), P, \gamma, \Psi, M) < 0$  for some symmetric matrix  $P$  if and only if  $K^T$  is a quadratically stable estimator that satisfies

$BMI_{WC}(G^T(H^T, K^T), Q, \gamma, \Gamma, N) < 0$  for some symmetric matrix  $Q$ . Further, the feasibility of  $BMI_{WC}(G(H, K), P, \gamma, \Psi, M) < 0$  implies that  $\mathcal{F}_u(G(H, K), \Delta)$  satisfies:

- (1)  $\lim_{T \rightarrow \infty} x_e(T) = 0 \forall x_e(0) \in \mathbb{R}^{n_H + n_\Psi + n_K}, \forall d \in L_2^{n_d}, \forall \Delta \in \text{IQC}(\Pi)$ , and  $\forall \rho \in \mathcal{T}$ , and
- (2)  $\sup_{\Delta \in \text{IQC}(\Pi)} \|\mathcal{F}_u(G(H, K), \Delta)\| \leq \gamma$ .

*Proof*

From statement (1) of Lemma 6,  $H$  is quadratically stable if and only if  $H^T$  is quadratically stable. Similarly,  $K$  is quadratically stable if and only if  $K^T$  is quadratically stable. From statement (2) of Lemma 6,  $\Pi$  is a strict PN multiplier if and only if  $\mathbf{D}(\Pi)$  is a strict PN multiplier. For sufficiency, assume  $K$  is a quadratically stable feedforward that satisfies  $BMI_{WC}(G(H, K), P, \gamma, \Psi, M) < 0$  for some  $P = P^T$ . It is verified that  $G(H, K)$  is quadratically stable because  $H$  and  $K$  are both quadratically stable. From statement (1) of Lemma 6, it follows that  $G(H, K)^T = G^T(H^T, K^T)$  is also quadratically stable. From statement (3) of Lemma 6,  $\exists Q = Q^T$  satisfying  $BMI_{WC}(G^T(H^T, K^T), Q, \gamma, \Gamma, N) < 0$ . For necessity, use similar arguments. Finally, from Theorem 1, if there exists  $P = P^T$  satisfying  $BMI_{WC}(G(H, K), P, \gamma, \Psi, M) < 0$ , then  $\mathcal{F}_u(G(H, K), \Delta)$  satisfies statements (1) and (2).  $\square$

Theorem 3 shows that  $K^T$  is an output estimator that satisfies the sufficient conditions for bounding the worst-case gain of  $\mathcal{F}_u(G^T(H^T, K^T), \Delta_D)$  by  $\gamma$  if and only if  $K$  is a disturbance feedforward that satisfies the sufficient conditions for bounding the worst-case gain of  $\mathcal{F}_u(G(H, K), \Delta)$  by  $\gamma$ . Hence, the disturbance feedforward problem is solved by implementing its corresponding dual form. In particular, when a feedforward synthesis problem is specified using  $H$  and  $\Pi$ , Theorem 2 is invoked on  $H^T$  and  $\mathbf{D}(\Pi)$  so that the estimator  $K^T$  is synthesized instead. Theorem 2 is implemented as a semidefinite program as explained in Section 4.1.

One technical issue is that the solution of the disturbance feedforward problem by the dual semidefinite program requires an appropriate parametrization of the IQC multiplier. For example, if  $\Delta$  is defined by multiplication in the time-domain with a norm-bounded, time-varying real scalar, it satisfies all IQCs defined by multipliers  $\Pi$  of the form  $\begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$ , where  $X = X^T \geq 0$  and  $Y = -Y^T$  [24]. In this example,  $\Pi$  is parametrized by the real symmetric matrix  $X$  and the real skew-symmetric matrix  $Y$ . In general,  $\Pi$  is parametrized by several variables. While parametrizations aid in the enlargement of the set of feasible IQC multipliers, only those that preserve the linearity of the matrix inequalities can be implemented in semidefinite programs. For several perturbations, suitable parametrizations of  $\Pi$  are available in Section 4.2 of [19].

However, the dual multipliers (Definition 5) involve matrix inversion. As a result, even if the primal multiplier has a convex parametrization, the dual multiplier may not. Hence, suitable parametrizations of the dual multiplier should be found independently and on a case-by-case basis. However, for linear perturbations, if  $\Delta$  satisfies the IQCs defined by several primal multipliers  $\Pi_i$ , then  $\Delta_D$  satisfies the IQCs defined by every one of the corresponding dual multipliers  $\mathbf{D}(\Pi_i)$  [23]. Consequently, affine parametrizations of  $\Pi_i$  and  $\mathbf{D}(\Pi_i)$  can be used for the primal and dual worst-case gain problems, respectively. More details can be found in Section 2.1 of [23].

## 5. NUMERICAL EXAMPLE

The following numerical example illustrates convex feedforward synthesis for a grid-based LPV plant that is affected by a sector-constrained nonlinearity. Figure 5a depicts a spring-mass-damper system consisting of two springs, two masses, and two dampers. The masses are  $m_1 = 1\text{kg}$  and  $m_2 = 0.5\text{kg}$ . The spring connecting the wall and mass  $m_1$  is linear and has a spring constant  $k_1 = 1\text{N m}^{-1}$ . The spring connecting the two masses is nonlinear, where  $f : \mathbb{R} \rightarrow \mathbb{R}$  denotes the nonlinear function mapping the spring deformation to the spring force. For a spring deformation  $v \in \mathbb{R}$ , the spring force is  $f(v) := k_2 v + \Delta(v)$ . Here,  $k_2 = 1\text{N m}^{-1}$  denotes the linear spring constant and  $\Delta : \mathbb{R} \rightarrow \mathbb{R}$  denotes a sector-constrained nonlinear function. The damping coefficient  $c_1$  is certain, but depends on a time-varying scheduling parameter  $\rho(t)$  as  $c_1 = |\sin(\rho(t))|$ . Admissible parameter

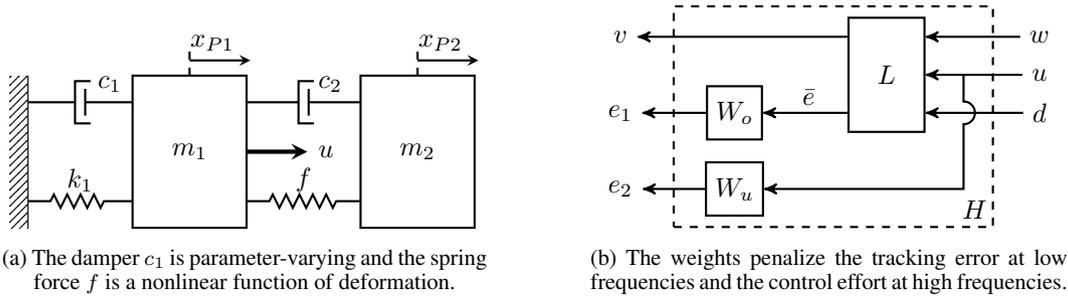


Figure 5. Left: Spring-mass-damper system. Right: Interconnection showing frequency-dependent weights.

trajectories satisfy  $\rho(t) \in \mathcal{P} = [0, \frac{\pi}{3}]$  and  $\dot{\rho}(t) \in \mathbb{R} \forall t \geq 0$ . Since  $c_1$  is a transcendental function of  $\rho$ , this problem is not directly solvable by the LFT-LPV approach [12]. Following the grid-based LPV approach, the parameter space is gridded into three points  $\{0, \frac{\pi}{6}, \frac{\pi}{3}\}$ . These three points are simply chosen for the purpose of demonstration and the grid may be made as dense as needed [25]. The damping coefficient  $c_2 = 2 \text{ N s m}^{-1}$  is certain and time-invariant.

A command tracking problem is formulated as follows. Mass  $m_1$  is externally forced through the control input  $u$ . The positions of  $m_1$  and  $m_2$  relative to their respective equilibrium positions are denoted by  $x_{P1}$  and  $x_{P2}$ . The commanded position of mass  $m_2$  relative to its equilibrium position is denoted by  $d$ . The objective is to design a feedforward controller  $K$  that uses  $d$  to generate  $u$  such that  $x_{P2}$  tracks the reference command. The design should ensure that large tracking errors are avoided at low frequencies and large control inputs are avoided at high frequencies. The feedforward controller should be scheduled with the parameter  $\rho(t)$  and should be robust to the sector-constrained nonlinearity  $\Delta$ . In order to describe the equations of motion of the spring-mass-damper system, consider a LPV system  $L$  with the state-space representation

$$\begin{bmatrix} \dot{x}_{P1} \\ \dot{x}_{P2} \\ \dot{x}_{P3} \\ \dot{x}_{P4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & \frac{-(c_1+c_2)}{m_1} & \frac{c_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & \frac{c_2}{m_2} & \frac{-c_2}{m_2} \end{bmatrix} \begin{bmatrix} x_{P1} \\ x_{P2} \\ x_{P3} \\ x_{P4} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{m_1} & \frac{1}{m_1} & 0 \\ \frac{-1}{m_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ u \\ d \end{bmatrix}, \quad (50)$$

$$v = x_{P2} - x_{P1}, \text{ and } \bar{e} = d - x_{P2}. \quad (51)$$

where  $x_{P3} = \dot{x}_{P1}$  and  $x_{P4} = \dot{x}_{P2}$  are the velocities of the two masses and  $w = \Delta(v)$ .

The output  $v$  captures the net deformation of the spring connecting the two masses. As per the definition of the function  $f$  given previously, the resulting spring force is  $f(v) = k_2 x_{P2} - k_2 x_{P1} + w$ , where  $w = \Delta(v)$  is the nonlinear component of the spring force. The output  $\bar{e}$  captures the error between the commanded and actual positions of mass  $m_2$ . The equations of motion for the entire spring-mass-damper system, including the nonlinearity, is given by  $\mathcal{F}_u(L, \Delta)$ . Figure 5b shows the interconnection of  $L$  and two weighting functions  $W_o$  and  $W_u$ . The weight  $W_o = \frac{0.1}{(s+0.1)(s+0.01)}$  penalizes the tracking error  $\bar{e}$  at low frequencies. The weighting function  $W_u = \frac{100(s+0.1)}{s+1000}$  penalizes the control effort  $u$  at high frequencies. The generalized error vector is denoted by  $e$  and has two components:  $e_1 := W_o \bar{e}$  and  $e_2 := W_u u$ . As shown in Figure 5b, the interconnection of  $L$ ,  $W_o$ , and  $W_u$  is denoted by  $H$  in order to relate back to the notation used Figure 4a.

Since  $\Delta$  is a sector-constrained nonlinearity, it satisfies all IQCs defined by multipliers  $\Pi$  of the form  $\begin{bmatrix} -2\alpha\beta & \alpha + \beta \\ \alpha + \beta & -2 \end{bmatrix}$ , where  $\alpha$  and  $\beta$  define the slopes of the sector [24]. In this example, the sector is defined by lines of slope  $\alpha = -0.9$  and  $\beta = 1.5$ . These values of  $\alpha$  and  $\beta$  ensure that  $\Pi$  is a strict PN multiplier. The choice of the sector is important in ensuring the applicability of Theorem 3. For example,  $f$  is a nonlinear function in a rotated sector defined by lines of slope 0.1 and 2.5. However, the multiplier for this sector is not strict PN and cannot be used in Theorem 3. The dual multiplier

$D(\Pi) = (\alpha - \beta)^{-2} \Pi$  is simply a scaled version of the primal. The  $J$ -spectral factor of  $D(\Pi)$  is used in the analysis and synthesis LMI conditions.

First, a pure analysis problem is considered, wherein  $K = 0$ . The upper bound on the worst-case gain of  $\mathcal{F}_u(G, \Delta)$  is made a cost function in a semidefinite program. Theorem 1 is applied to obtain a worst-case gain bound of 100. Next, the synthesis problem is considered and Theorem 3 is applied to obtain a worst-case gain bound of 5.48, demonstrating that the command tracking for  $m_2$  is significantly better with an optimally designed LPV feedforward controller.

## 6. CONCLUSIONS

This paper considered the twin problems of synthesizing output estimators and disturbance feedforward controllers for continuous-time, uncertain, gridded, linear parameter-varying (LPV) systems. Integral quadratic constraints (IQCs) were used to describe the uncertainty. While convex conditions are readily obtained for the output estimation problem, it does not seem possible to directly obtain convex conditions for the disturbance feedforward problem. Hence, notions of duality were developed for LPV systems and IQCs in the time-domain. These were used to show that the two synthesis problems are duals of each other. Consequently, a convex synthesis of feedforward controllers is possible by solving the dual estimation problem. The duality result has no loss in conservatism. A numerical example illustrated convex feedforward synthesis for a gridded LPV plant that was affected by a sector-constrained nonlinear function.

### APPENDIX A: PROOF OF LEMMA 4

First, note that  $\Pi$  has the following frequency-domain representations for  $i = 1, 2$ :

$$\Pi(s) = \Psi_i^\sim(s) M_i \Psi_i(s) = \begin{bmatrix} (sI - A_i)^{-1} B_i \\ I \end{bmatrix} \sim \begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix} \begin{bmatrix} (sI - A_i)^{-1} B_i \\ I \end{bmatrix}. \quad (52)$$

This yields the following two state-space realizations for  $\Pi$ :

$$\Pi \stackrel{s}{=} \left[ \begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline C_i & D_i \end{array} \right] := \left[ \begin{array}{cc|c} A_i & 0 & B_i \\ -Q_i & -A_i^T & -S_i \\ \hline S_i^T & B_i^T & R_i \end{array} \right] \text{ for } i = 1, 2. \quad (53)$$

These two realizations of  $\Pi$  are minimal since the  $A_i$  are the state matrices of the (assumed) minimal realizations for  $\Psi_i$ . Hence  $\bar{A}_1$  and  $\bar{A}_2$  share the same eigenvalues and hence are similar. Consequently,  $A_1$  and  $A_2$  share the same eigenvalues and hence are similar matrices. This proves the existence of a similarity transformation matrix  $T_1 \in \mathbb{R}^{n \times n}$  such that:

$$A_2 = T_1 A_1 T_1^{-1}. \quad (54)$$

Moreover, the two minimal realizations of  $\Pi$  are also related by a similarity transformation:

$$\exists T \in \mathbb{R}^{2n \times 2n} : \left[ \begin{array}{c|c} T\bar{A}_1 T^{-1} & T\bar{B}_1 \\ \hline C_1 T^{-1} & D_1 \end{array} \right] = \left[ \begin{array}{c|c} \bar{A}_2 & \bar{B}_2 \\ \hline C_2 & D_2 \end{array} \right]. \quad (55)$$

Equating the (1,1) blocks of (55) yields  $T\bar{A}_1 = \bar{A}_2 T$ . Using the partition  $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ , this is:

$$\begin{bmatrix} T_{11} A_1 - T_{12} Q_1 & -T_{12} A_1^T \\ T_{21} A_1 - T_{22} Q_1 & -T_{22} A_1^T \end{bmatrix} = \begin{bmatrix} A_2 T_{11} & A_2 T_{12} \\ -Q_2 T_{11} - A_2^T T_{21} & -Q_2 T_{12} - A_2^T T_{22} \end{bmatrix}. \quad (56)$$

Equating the (1,2) blocks of (56) yields the relation  $-T_{12} A_1^T = A_2 T_{12}$ . However,  $A_1$  and  $A_2$  are also related by Equation (54). These two relations together yield the relation  $-T_{11}^{-1} T_{12} A_1^T = A_1 T_{11}^{-1} T_{12}$ . This can be rewritten as the Lyapunov Equation  $A_1 \bar{Z} + \bar{Z} A_1^T = 0$  where  $\bar{Z} := T_{11}^{-1} T_{12}$ . Since  $A_1$  is Hurwitz it follows that  $\bar{Z} = 0$  is the unique solution to this Lyapunov Equation.

Moreover,  $\bar{Z} = 0$  implies  $T_{12} = 0$ , i.e.  $T$  is block lower triangular. Equating the (1,1) and (2,2) blocks of (56) then implies  $T_{11} = T_1$  and  $T_{22} = T_1^{-T}$ . Finally, denoting  $\bar{X} := T_1^T T_{21}$  yields the block partitions  $T = \begin{bmatrix} T_1 & 0 \\ T_1^{-T} \bar{X} & T_1^{-T} \end{bmatrix}$  and  $T^{-1} = \begin{bmatrix} T_1^{-1} & 0 \\ -\bar{X} T_1^{-1} & T_1^T \end{bmatrix}$ .

Equating the (2,1) blocks of (56) yields the Lyapunov Equation  $A_1^T \bar{X} + \bar{X} A_1 = Q_1 - T_1^T Q_2 T_1$ . The solution  $\bar{X} = \bar{X}^T$  to this Lyapunov Equation exists and is unique because  $A_1$  is Hurwitz.

Equating the (1,2) blocks of (55) yields:

$$B_2 = T_1 B_1 \text{ and } T_1^{-T} \bar{X} B_1 = T_1^{-T} S_1 - S_2. \quad (57)$$

Equating the (2,2) blocks of (55) yields  $\bar{D}_1 = \bar{D}_2$  which further implies  $R_1 = R_2$ . Finally, the following expressions are obtained for  $Q_2$ ,  $S_2$ , and  $R_2$ :

$$Q_2 = T_1^{-T} (Q_1 - \bar{X} A_1 - A_1^T \bar{X}) T_1^{-1}, \quad S_2 = T_1^{-T} (S_1 - \bar{X} B_1), \text{ and } R_2 = R_1. \quad (58)$$

Equations (54), (57), and (58) prove statements (1), (2) and (3), respectively.  $\square$

## APPENDIX B: MATRIX DILATION RESULT

### Lemma 7

Let  $X = X^T \in \mathbb{R}^{n \times n}$ ,  $Y = Y^T \in \mathbb{R}^{n \times n}$ , and a positive integer  $n_F$  be given. Then there exist matrices  $X_2, Y_2 \in \mathbb{R}^{n \times n_F}$  and symmetric matrices  $X_3, Y_3 \in \mathbb{R}^{n_F \times n_F}$ , satisfying

$$X_3 > 0 \text{ and } \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \quad (59)$$

if and only if

$$X - Y^{-1} \geq 0 \text{ and } \text{rank}(X - Y^{-1}) \leq n_F. \quad (60)$$

### Proof

For sufficiency, assume that the conditions given in (59) hold. By the matrix inversion lemma,

$$\begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} (X - X_2 X_3^{-1} X_2^T)^{-1} & -(X - X_2 X_3^{-1} X_2^T)^{-1} X_2 X_3^{-1} \\ -(X_3 - X_2^T X^{-1} X_2)^{-1} X_2^T X^{-1} & (X_3 - X_2^T X^{-1} X_2)^{-1} \end{bmatrix}. \quad (61)$$

Comparing the expressions in (59) with (61) yields  $Y = (X - X_2 X_3^{-1} X_2^T)^{-1}$ . Applying the matrix inverse to both sides of this relation and rearranging terms yields  $X - Y^{-1} = X_2 X_3^{-1} X_2^T$ . Thus the assumption  $X_3 > 0$  implies  $X - Y^{-1} \geq 0$ . Further, since  $X_2 \in \mathbb{R}^{n \times n_F}$ ,  $\text{rank}(X - Y^{-1}) \leq n_F$ .

For necessity, assume that the conditions given in (60) hold. Since  $\text{rank}(X - Y^{-1}) \leq n_F$ ,  $\exists X_2 \in \mathbb{R}^{n \times n_F}$  so that  $X - Y^{-1} = X_2 X_2^T \geq 0$ . This relation can be rearranged to obtain  $Y = (X - X_2 X_2^T)^{-1}$ . By the matrix inversion lemma,  $\begin{bmatrix} X & X_2 \\ X_2^T & I \end{bmatrix}^{-1} = \begin{bmatrix} Y & -Y X_2 \\ -X_2^T Y & X_2^T Y X_2 + I \end{bmatrix}$ . Hence, set  $X_3 = I$ ,  $Y_2 = -Y X_2$ , and  $Y_3 = I + X_2^T Y X_2$ .  $\square$

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