Uncertainty Analysis for Linear Parameter Varying Systems

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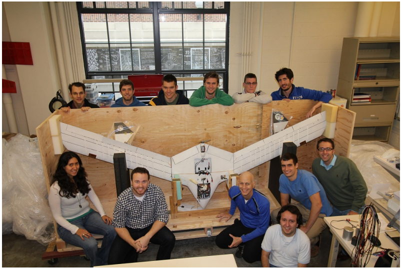
Aeroservoelastic Systems

Objective: Enable lighter, more efficient aircraft by active control of aeroelastic modes.



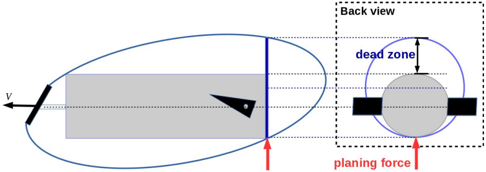
http://www.uav.aem.umn.edu/

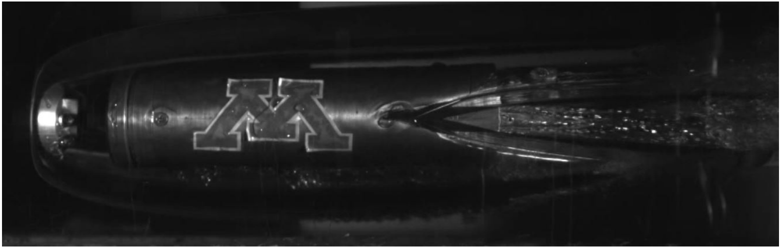
Boeing: 787 Dreamliner

AFLR/Lockheed/NASA: BFF and X56 MUTT

Supercavitating Vehicles

Objective: Increase vehicle speed by traveling within the cavitation bubble.



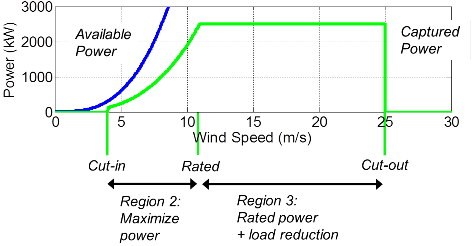


Ref: D. Escobar, G. Balas, and R. Arndt, "Planing Avoidance Control for Supercavitating Vehicles," ACC, 2014.

Wind Turbines

Clipper Turbine at Minnesota Eolos Facility

Objective: Increase power capture, decrease structural loads, and enable wind to provide ancillary services.



http://www.eolos.umn.edu/

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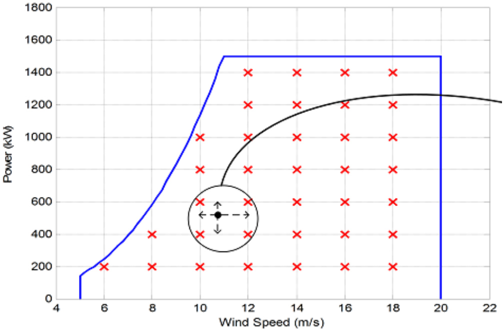

http://www.eolos.umn.edu/

Outline

Goal: Synthesize and analyze controllers for these systems.

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Parameterized Trim Points

These applications can be described by nonlinear models:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), \rho(t)) \\ y(t) &= h(x(t), u(t), \rho(t)) \end{aligned}$$

where ρ is a vector of measurable, exogenous signals.

Assume there are trim points $(\bar{x}(\rho), \bar{u}(\rho), \bar{y}(\rho))$ parameterized by ρ:

$$\begin{aligned} 0 &= f(\bar{x}(\rho), \bar{u}(\rho), \rho) \\ \bar{y}(\rho) &= h(\bar{x}(\rho), \bar{u}(\rho), \rho) \end{aligned}$$

Linearization

Let $(x(t), u(t), y(t), \rho(t))$ denote a solution to the nonlinear system and define perturbed quantities:

$$\begin{aligned}
\delta_x(t) &:= x(t) - \bar{x}(\rho(t)) \\
\delta_u(t) &:= u(t) - \bar{u}(\rho(t)) \\
\delta_y(t) &:= y(t) - \bar{y}(\rho(t))
\end{aligned}$$

Linearize around $(\bar{x}(\rho(t)), \bar{u}(\rho(t)), \bar{y}(\rho(t)), \rho(t))$

$$\begin{aligned}
\dot{\delta}_x &= A(\rho)\delta_x + B(\rho)\delta_u + \Delta_f(\delta_x, \delta_u, \rho) - \dot{\bar{x}}(\rho) \\
\dot{\delta}_y &= C(\rho)\delta_x + D(\rho)\delta_u + \Delta_h(\delta_x, \delta_u, \rho)
\end{aligned}$$

where $A(\rho) := \frac{\partial f}{\partial x}(\bar{x}(\rho), \bar{u}(\rho), \rho)$, etc.

LPV Systems

This yields a linear parameter-varying (LPV) model:

$$\dot{\delta}_x = A(\rho)\delta_x + B(\rho)\delta_u + \Delta_f(\delta_x, \delta_u, \rho) - \dot{\bar{x}}(\rho)$$
$$\dot{\delta}_y = C(\rho)\delta_x + D(\rho)\delta_u + \Delta_h(\delta_x, \delta_u, \rho)$$

Comments:

- LPV theory a extension of classical gain-scheduling used in industry, e.g. flight controls.
- Large body of literature in 90's: Shamma, Rugh, Athans, Leith, Leithead, Packard, Scherer, Wu, Gahinet, Apkarian, and many others.
- $-\dot{\bar{x}}(\rho)$ can be retained as a measurable disturbance.
- Higher order terms Δ_f and Δ_h can be treated as memoryless, nonlinear uncertainties.

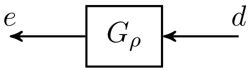
Grid-based LPV Systems

$$\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))d(t)$$
$$e(t) = C(\rho(t))x(t) + D(\rho(t))d(t)$$

Parameter vector ρ lies within a set of admissible trajectories

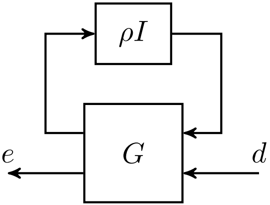
$$\mathcal{A} := \{\rho : \mathbb{R}^+ \to \mathbb{R}^{n_\rho} \ : \ \rho(t) \in \mathcal{P}, \ \dot{\rho}(t) \in \dot{\mathcal{P}} \ \forall t \geq 0\}$$

Grid based LPV systems



(Pfifer, Seiler, ACC, 2014)

LFT based LPV systems



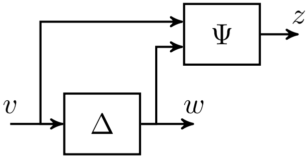
(Scherer, Kose, TAC, 2012)

Outline

Goal: Synthesize and analyze controllers for these systems.

1. Linear Parameter Varying (LPV) Systems
2. **Uncertainty Modeling with IQCs**
3. Robustness Analysis for LPV Systems
4. Connection between Time and Frequency Domain
5. Summary

Integral Quadratic Constraints (IQCs)



Let Ψ be a stable, LTI system and M a constant matrix.

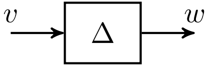
Def.: Δ satisfies IQC defined by Ψ and M if

$$\int_0^T z(t)^T M z(t) dt \geq 0$$

for all $v \in L_2[0, \infty)$, $w = \Delta(v)$, and $T \geq 0$.

(Megretski, Rantzer, TAC, 1997)

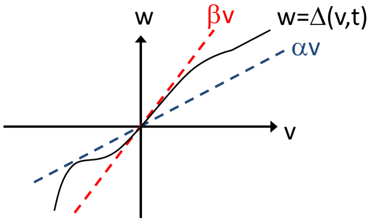
Example: Memoryless Nonlinearity



$w = \Delta(v,t)$ is a memoryless nonlinearity in the sector $[\alpha, \beta]$.

$$\Downarrow$$

$$2(\beta v(t) - w(t))(w(t) - \alpha v(t)) \geq 0 \; \forall t$$



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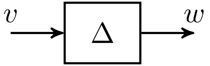


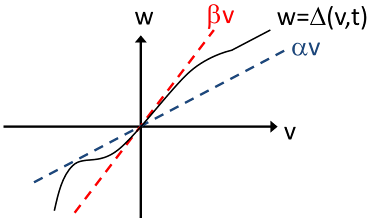
$w = \Delta(v,t)$ is a memoryless nonlinearity in the sector $[\alpha, \beta]$.

$$\Downarrow$$

$$2(\beta v(t) - w(t))(w(t) - \alpha v(t)) \geq 0 \ \forall t$$

$$\Downarrow$$

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^* \begin{bmatrix} -2\alpha\beta & \alpha+\beta \\ \alpha+\beta & -2 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} \geq 0 \ \forall t$$



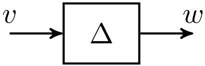
Pointwise quadratic constraint

Example: Norm Bounded Uncertainty

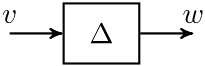
Δ is a causal, SISO operator with $\|\Delta\| \leq 1$.

$$\Downarrow$$

$$\|w\| \leq \|v\|$$



Example: Norm Bounded Uncertainty



Δ is a causal, SISO operator with $\|\Delta\| \leq 1$.

$$\Downarrow$$

$$\|w\| \leq \|v\|$$

$$\Downarrow$$

$$\int_0^\infty \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0$$

for all $v \in L_2[0,\infty)$ and $w = \Delta(v)$.

Infinite time horizon constraint

Example: Norm Bounded Uncertainty

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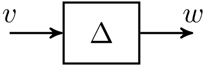
$$\Downarrow$$

$$\|w\| \leq \|v\|$$

$$\Downarrow$$

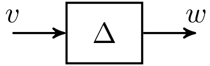


$$\int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0$$

for all $v \in L_2[0, \infty)$, $w = \Delta(v)$, and $T \geq 0$

Causality implies finite-time constraint.

Example: Norm Bounded Uncertainty



Δ causal with $\|\Delta\| \leq 1$

$$\Downarrow$$

$$\int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \geq 0$$

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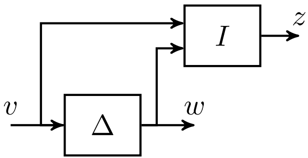
Example: Norm Bounded Uncertainty



Δ causal with $\|\Delta\| \leq 1$

$$\Downarrow$$

$$\int_0^T z(t)^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} z(t)\, dt \geq 0$$

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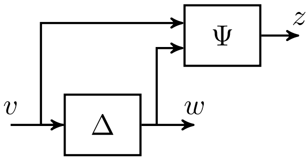
Example: Norm Bounded Uncertainty



Δ causal with $\|\Delta\| \leq 1$

$\Downarrow$

$$\int_0^T z(t)^T M z(t)\, dt \geq 0$$

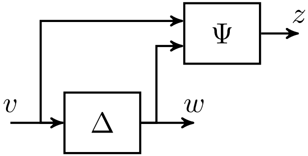
$\forall v \in L_2[0, \infty)$ and $w = \Delta(v)$.

$\Downarrow$

Δ satisfies IQC defined by

$$\Psi = I_2 \text{ and } M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Example: Norm Bounded LTI Uncertainty



$$\int_0^T z(t)^T M z(t)\, dt \geq 0$$

Δ is LTI and $\|\Delta\| \leq 1$

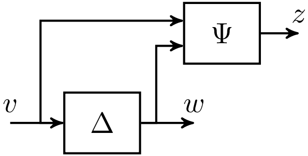
$\Downarrow$

For any stable system D, Δ satisfies IQC defined by

$$\Psi = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Equivalent to D-scales in μ-analysis

IQCs in the Time Domain



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(Megretski, Rantzer, TAC, 1997)

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Background

Nominal Performance of LPV Systems

Induced L_2 gain:

$$\|G_\rho\| = \sup_{d \neq 0, d \in L_2, \rho \in \mathcal{A}, x(0)=0} \frac{\|e\|}{\|d\|}$$

Bounded Real Lemma like condition to compute upper bound

(Wu, Packard, ACC 1995)

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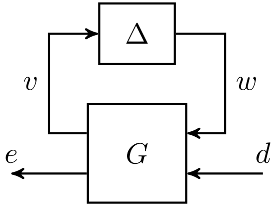
Bounded Real Lemma like condition to compute upper bound

(Wu, Packard, ACC 1995)

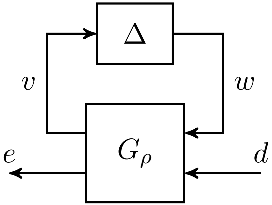
Integral Quadratic Constraints

- general framework for robustness analysis
- originally in the frequency domain
- known **LTI** system under perturbations



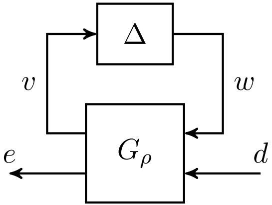
(Megretski, Rantzer, TAC, 1997)

Worst-case Gain



- **Goal:** Assess stability and performance for the interconnection of known **LPV** system G_ρ and "perturbation" Δ.

Worst-case Gain



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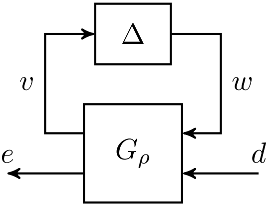
- **Approach:** Use IQCs to specify a **finite time horizon** constraint on the input/output behavior of Δ.

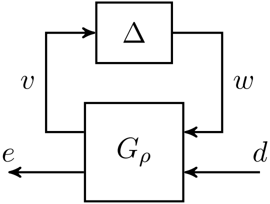
Worst-case Gain



- **Goal:** Assess stability and performance for the interconnection of known **LPV** system G_ρ and "perturbation" Δ.
- **Approach:** Use IQCs to specify a **finite time horizon** constraint on the input/output behavior of Δ.
- **Metric:** Worst case gain

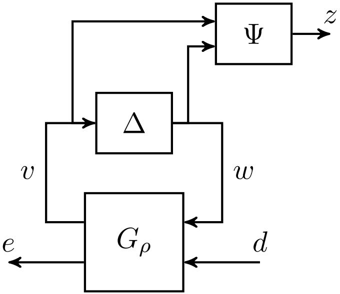
$$\sup_{\Delta \in \mathrm{IQC}(\Psi, M)} \; \sup_{d \neq 0, d \in L_2, \rho \in \mathcal{A}, x(0)=0} \frac{\|e\|}{\|d\|}$$

Worst-case Gain Analysis with IQCs



Approach: Replace "precise" behavior of Δ with IQC on I/O signals.

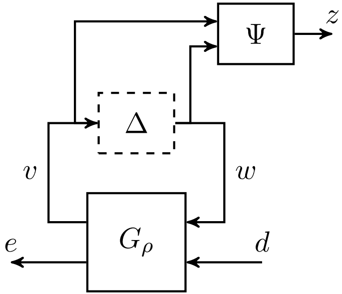
Worst-case Gain Analysis with IQCs



Approach: Replace "precise" behavior of Δ with IQC on I/O signals.

- Append system Ψ to Δ.

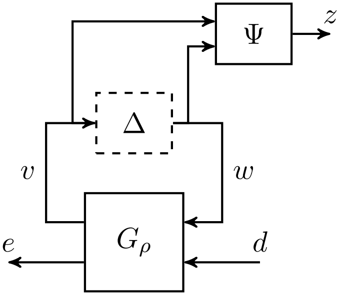
Worst-case Gain Analysis with IQCs



Approach: Replace "precise" behavior of Δ with IQC on I/O signals.

- Append system Ψ to Δ.
- Treat w as external signal subject to IQC.

Worst-case Gain Analysis with IQCs

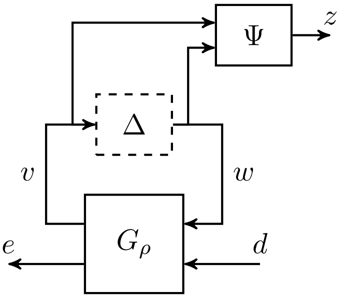


Approach: Replace "precise" behavior of Δ with IQC on I/O signals.

- Append system Ψ to Δ.
- Treat w as external signal subject to IQC.
- Denote extended dynamics by

$$\dot{x} = F(x, w, d, \rho)$$
$$\begin{bmatrix} z \\ e \end{bmatrix} = H(x, w, d, \rho)$$

Dissipation Inequality Condition



Theorem: Assume:

1. Interconnection is well-posed.
2. Δ satisfies IQC(Ψ, M)
3. $\exists\, V \geq 0$ and $\gamma > 0$ such that

$$\nabla V \cdot F(x, w, d, \rho) + z^T M z < d^T d - \gamma^{-2} e^T e$$

for all $x \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$, $d \in \mathbb{R}^{n_d}$.

Then gain from d to e is $\leq \gamma$.

Proof Sketch

Let $d \in L[0, \infty)$ be any input signal and $x(0) = 0$:

$$\nabla V \cdot F(x, w, d) + z^T M z < d^T d - \gamma^{-2} e^T e$$

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Let $d \in L[0, \infty)$ be any input signal and $x(0) = 0$:

$$\nabla V \cdot F(x, w, d) + z^T M z < d^T d - \gamma^{-2} e^T e$$

$\Downarrow$ Integrate from $t = 0$ to $t = T$

$$V(x(T)) - V(x(0)) + \int_0^T z(t)^T M z(t) dt < \int_0^T d(t)^T d(t) dt - \gamma^{-2} \int_0^T e(t)^T e(t) dt$$

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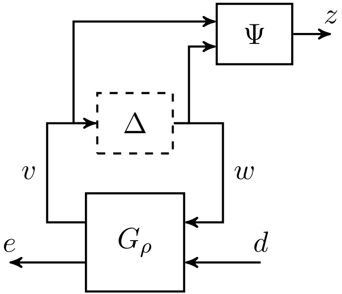
$\Downarrow$ IQC constraint, V nonnegative

$$\int_0^T e(t)^T e(t) dt < \gamma^2 \int_0^T d(t)^T d(t) dt$$

Hence $\|e\| \leq \gamma \|d\|$

Linear Matrix Inequality Condition



Extended System Dynamics:

$$\begin{aligned} \dot{x} &= A(\rho)x + B_1(\rho)w + B_2(\rho)d \\ z &= C_1(\rho)x + D_{11}(\rho)w + D_{12}(\rho)d \\ e &= C_2(\rho)x + D_{21}(\rho)w + D_{22}(\rho)d, \end{aligned}$$

What is the “best” bound on the worst-case gain?

Linear Matrix Inequality Condition

Theorem

The gain of $F_u(G_\rho, \Delta)$ is $< \gamma$ if there exists a matrix $P \in \mathbb{R}^{n_x \times n_x}$ and a scalar $\lambda > 0$ such that $P > 0$ and $\forall \rho \in \mathcal{P}$

$$\begin{bmatrix} PA(\rho) + A(\rho)^T P & PB_1(\rho) & PB_2(\rho) \\ B_1(\rho)^T P & 0 & 0 \\ B_2(\rho)^T P & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} C_1(\rho)^T \\ D_{11}(\rho)^T \\ D_{12}(\rho)^T \end{bmatrix} M \begin{bmatrix} C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \end{bmatrix}$$
$$+ \frac{1}{\gamma^2} \begin{bmatrix} C_2(\rho)^T \\ D_{21}(\rho)^T \\ D_{22}(\rho)^T \end{bmatrix} \begin{bmatrix} C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} < 0$$

Linear Matrix Inequality Condition

Theorem

The gain of $F_u(G_\rho, \Delta)$ *is* $< \gamma$ *if there exists a matrix* $P \in \mathbb{R}^{n_x \times n_x}$ *and a scalar* $\lambda > 0$ *such that* $P > 0$ *and* $\forall \rho \in \mathcal{P}$

$$\begin{bmatrix} PA(\rho) + A(\rho)^T P & PB_1(\rho) & PB_2(\rho) \\ B_1(\rho)^T P & 0 & 0 \\ B_2(\rho)^T P & 0 & -I \end{bmatrix} + \lambda \begin{bmatrix} C_1(\rho)^T \\ D_{11}(\rho)^T \\ D_{12}(\rho)^T \end{bmatrix} M \begin{bmatrix} C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \end{bmatrix}$$

$$+ \frac{1}{\gamma^2} \begin{bmatrix} C_2(\rho)^T \\ D_{21}(\rho)^T \\ D_{22}(\rho)^T \end{bmatrix} \begin{bmatrix} C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} < 0$$

Proof:

- Left/right multiplying by $[x^T, w^T, d^T]$ and $[x^T, w^T, d^T]^T$
- $V(x) := x^T P x$ satisfies dissipation inequality

$$\dot{V} + \lambda z^T M z \leq d^T d - \gamma^{-2} e^T e$$

Numerical Issues

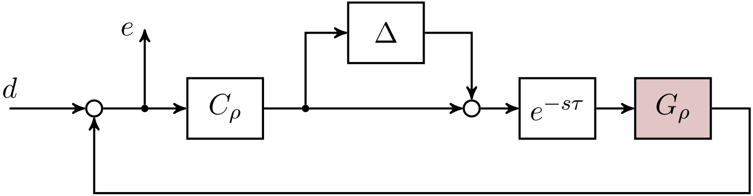
Parameter dependent LMIs depending on decision variable $P(\rho)$

Approximations on the test conditions:

- grid over parameter space
- basis function for $P(\rho)$
- rational functions for Ψ

`LPVTools` toolbox developed to support LPV objects, analysis and synthesis.

(Simple) Numerical Example



Plant:

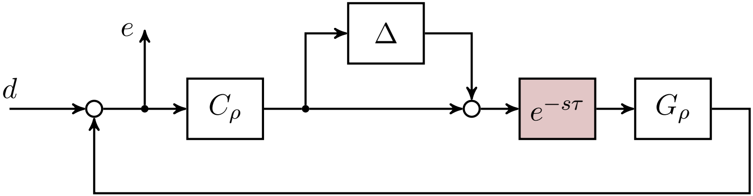
- First order LPV system G_ρ

$$\dot{x} = -\frac{1}{\tau(\rho)}x + \frac{1}{\tau(\rho)}u \quad \tau(\rho) = \sqrt{133.6 - 16.8\rho}$$

$$y = K(\rho)x \qquad K(\rho) = \sqrt{4.8\rho - 8.6} \qquad \rho \in [2, 7]$$

More complex example: Hjartarson, Seiler, Balas, "LPV Analysis of a Gain Scheduled Control for an Aeroelastic Aircraft", ACC, 2014.

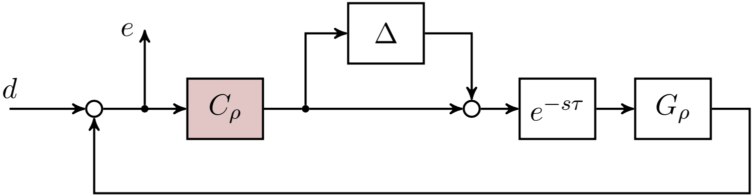
(Simple) Numerical Example



Time delay:

- 0.5 seconds
- 2nd order Pade approximation

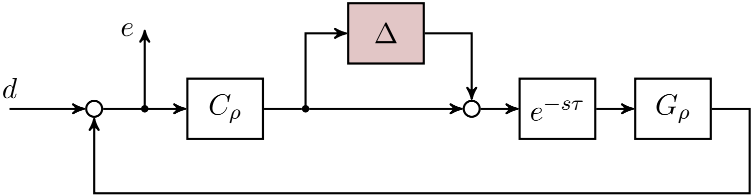
(Simple) Numerical Example



Controller:

- Gain-scheduled PI controller C_ρ
- Gains are chosen such that at each frozen value ρ
  - Closed loop damping $= 0.7$
  - Closed loop frequency $= 0.25$

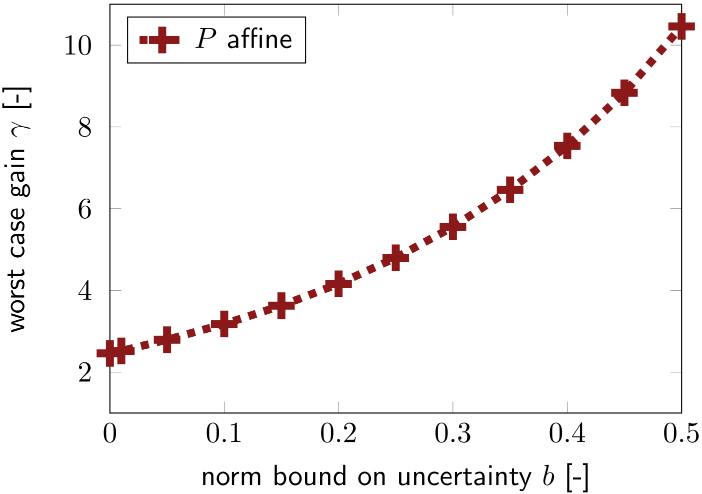
(Simple) Numerical Example



Uncertainty:

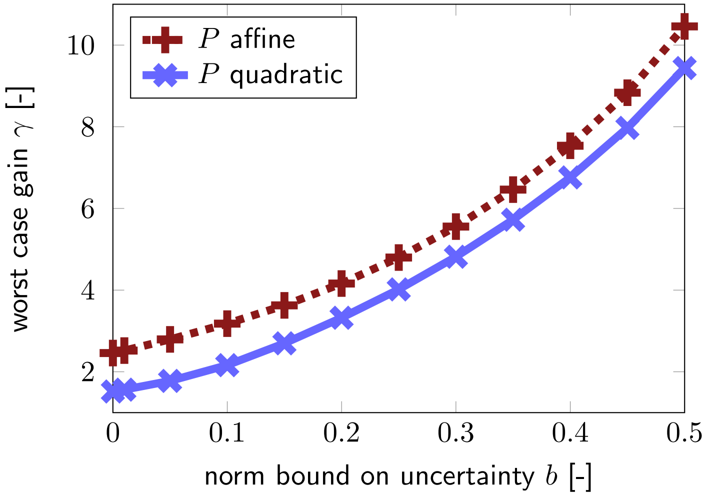
- Causal, norm-bounded operator Δ
- $\|\Delta\| \leq b$

Numerical Example



Rate-bounded analysis for $|\dot{\rho}| \leq 0.1$.

Numerical Example

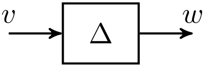


Rate-bounded analysis for $|\dot{\rho}| \leq 0.1$.

Outline

Goal: Synthesize and analyze controllers for these systems.

IQCs in the Frequency Domain



Let $\Pi : j\mathbb{R} \to \mathbb{C}^{m \times m}$ be Hermitian-valued.

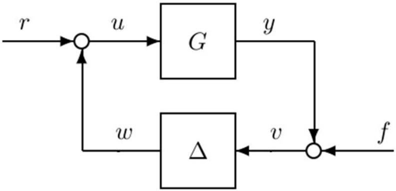
Def.: Δ satisfies IQC defined by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0$$

for all $v \in L_2[0, \infty)$ and $w = \Delta(v)$.

(Megretski, Rantzer, TAC, 1997)

Frequency Domain Stability Condition



Thm: Assume:

1. Interconnection of G and $\tau\Delta$ is well-posed $\forall \tau \in [0,1]$
2. $\tau\Delta \in$ IQC(Π) $\forall \tau \in [0,1]$.
3. $\exists\ \epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le -\epsilon I \ \forall \omega$$

Then interconnection is stable.

Connection between Time and Frequency Domain

1. Time Domain IQC (TD IQC) defined by (Ψ, M):

$$\int_0^T z(t)^T M z(t)\, dt \geq 0 \;\; \forall T \geq 0$$

where $z = \Psi \left[\begin{smallmatrix} v \\ w \end{smallmatrix}\right]$.

2. Frequency Domain IQC (FD IQC) defined by Π:

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0$$

A non-unique factorization $\Pi = \Psi^{\sim} M \Psi$ connects the approaches but there are two issues.

"Soft" Infinite Horizon Constraint

Freq. Dom. IQC: $$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0$$

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$\Updownarrow$ **Factorization** $\Pi = \Psi^{\sim} M \Psi$

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Psi(j\omega)^* M \Psi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega = \int_{-\infty}^{\infty} \hat{z}^*(j\omega) M \hat{z}(j\omega) \geq 0$$

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$\Updownarrow$ Parseval's Theorem

"Soft" IQC: $$\int_0^{\infty} z(t)^T M z(t) dt \geq 0$$

Issue # 1: DI stability test requires "hard" finite-horizon IQC

Sign-Indefinite Quadratic Storage

Factorize $\Pi = \Psi^{\sim} M \Psi$ and define $\Psi \begin{bmatrix} G \\ I \end{bmatrix} := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$.

Sign-Indefinite Quadratic Storage

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(*) KYP LMI: $\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} C^T \\ D^T \end{bmatrix} M \begin{bmatrix} C & D \end{bmatrix} < 0$

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KYP Lemma: $\exists \epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I$$

iff $\exists$ $P = P^T$ satisfying the KYP LMI (*).

Lemma: $V = x^T P x$ satisfies

$$\nabla V \cdot F(x, w, d) + z^T M z < \gamma^2 d^T d - e^T e$$

for some finite $\gamma > 0$ iff $\exists$ $P \geq 0$ satisfying the KYP LMI (*).

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Issue # 2: DI stability test requires $P \geq 0$

Equivalence of Approaches (Seiler, 2014)

Def.: $\Pi = \Psi^{\sim} M \Psi$ is a **J-Spectral factorization** if $M = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ and Ψ, Ψ^{-1} are stable.

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Thm.: If $\Pi = \Psi^{\sim} M \Psi$ is a J-spectral factorization then:

1. If $\Delta \in$IQC(Π) then $\Delta \in$ IQC(Ψ, M)
   (FD IQC $\Leftrightarrow$ Finite Horizon Time-Domain IQC)
2. All solutions of KYP LMI satisfy $P \geq 0$.

Proof: 1. follows from Megretski (Arxiv, 2010)
2. use results in Willems (TAC, 1972) and Engwerda (2005). ■

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Proof: 1. follows from Megretski (Arxiv, 2010)
2. use results in Willems (TAC, 1972) and Engwerda (2005). ■

Thm.: Partition $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{21}^* \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$. Π has a J-spectral factorization if $\Pi_{11}(j\omega) > 0$ and $\Pi_{22}(j\omega) < 0$ $\forall \omega \in \mathbb{R} \cup \{+\infty\}$.
Proof: Use equalizing vectors thm. of Meinsma (SCL, 1995) ■.

Outline

Goal: Synthesize and analyze controllers for these systems.

Summary

Conclusions:

- Developed conditions to assess the stability and performance of uncertain (gridded) LPV systems.
- Provided connection between time and frequency domain IQC conditions.

Future Work:

1. Robust synthesis for grid-based LPV models (Shu, Pfifer, Seiler, submitted to CDC 2014)
2. Lower bounds for (Nominal) LPV analysis: Can we efficiently construct "bad" allowable parameter trajectories? (Peni, Seiler, submitted to CDC 2014)
3. Demonstrate utility of analysis tools to compute classical margins for gain-scheduled and/or LPV controllers.

Acknowledgements

1. National Science Foundation under Grant No. NSF-CMMI-1254129 entitled "CAREER: Probabilistic Tools for High Reliability Monitoring and Control of Wind Farms," Program Manager: George Chiu.
2. NASA Langley NRA NNX12AM55A: "Analytical Validation Tools for Safety Critical Systems Under Loss-of-Control Conditions," Technical Monitor: Dr. Christine Belcastro.
3. Air Force Office of Scientific Research: Grant No. FA9550-12-0339, "A Merged IQC/SOS Theory for Analysis of Nonlinear Control Systems," Technical Monitor: Dr. Fariba Fahroo.

Brief Summary of LPV Lower Bound Algorithm

There are many exact results and computational algorithms for LTV and periodic systems (Colaneri, Varga, Cantoni/Sandberg, many others)

The basic idea for computing a lower bound on $\|G_\rho\|$ is to search over periodic parameter trajectories and apply known results for periodic systems.

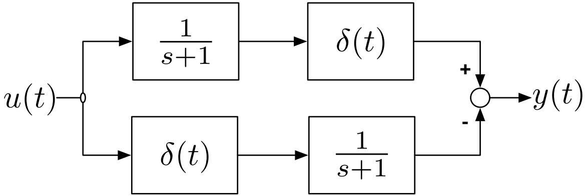
$$\|G_\rho\| := \sup_{\rho\in\mathcal{A}} \sup_{u\neq 0, u\in\mathcal{L}_2} \frac{\|G_\rho u\|}{\|u\|} \geq \sup_{\rho\in\mathcal{A}_h} \sup_{u\neq 0, u\in\mathcal{L}_2} \frac{\|G_\rho u\|}{\|u\|}$$

where $\mathcal{A}_p \subset \mathcal{A}$ denotes the set of admissible *periodic* trajectories.

Ref: T. Peni and P. Seiler, Computation of lower bounds for the induced $\mathcal{L}_2$ norm of LPV systems, submitted to the 2015 CDC.

Numerical example

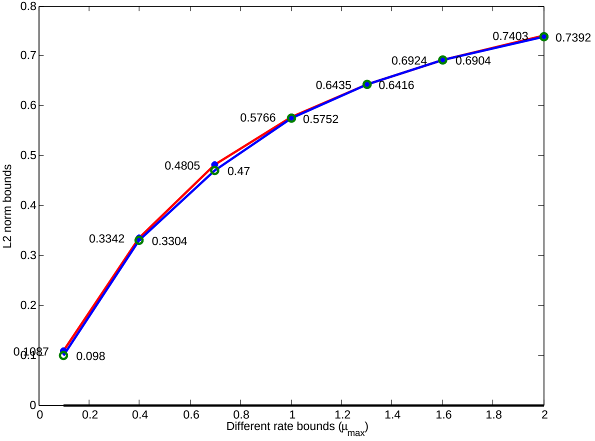
Simple, 1-parameter LPV system:



with $-1 \leq \delta(t) \leq 1$, and $-\overline{\mu} \leq \dot{\delta}(t) \leq \overline{\mu}$

The upper bound was computed by searching for a polynomial storage function.

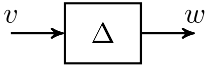
Upper and Lower Bounds

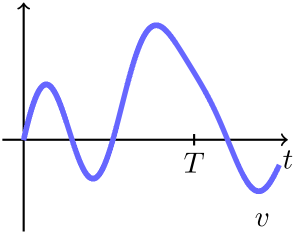


Question: Can this approach be extended to compute lower bounds for uncertain LPV systems?

Example: Norm Bounded Uncertainty

Truncated signal $\tilde{v}(t) = \begin{cases} v(t) & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases}$ and $\tilde{w} = \Delta(\tilde{v})$





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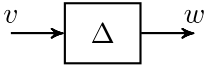

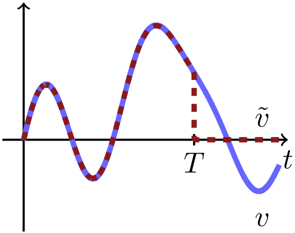

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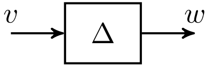
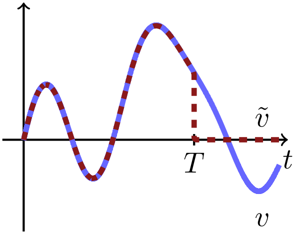
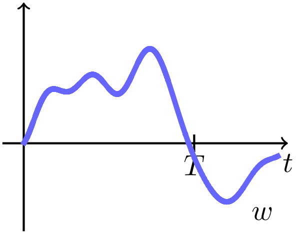


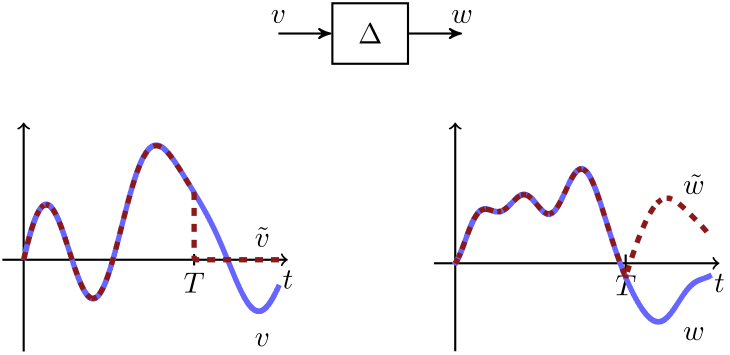




$\tilde{w}(t) = w(t)$ for $t \leq T$

Example: Norm Bounded Uncertainty

Truncated signal $\tilde{v}(t) = \begin{cases} v(t) & \text{for } t \le T \\ 0 & \text{for } t > T \end{cases}$ and $\tilde{w} = \Delta(\tilde{v})$



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$$0 \le \int_0^\infty \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix} dt$$

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Truncation of v:

$$\leq \int_0^T \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix} dt$$

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$$\leq \int_0^T \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix} dt$$

Causality of Δ:

$$\leq \int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt$$

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$$\begin{aligned} 0 &\leq \int_0^\infty \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix} dt \\ &\leq \int_0^T \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{v}(t) \\ \tilde{w}(t) \end{bmatrix} dt \\ &\leq \int_0^T \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v(t) \\ w(t) \end{bmatrix} dt \end{aligned}$$

Finite time horizon constraint